Mathematical and Analytical Techniques with Applications to Engineering

P. P. Teodorescu

Mechanical Systems, Classical Models

Volume 2
Mechanics of Discrete and Continuous Systems

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Mechanical Systems, Classical Models

Volume II: Mechanics of Discrete and Continuous Systems

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Preface

As it was already seen in the first volume of the present book, its guideline is precisely the mathematical model of mechanics. The classical models which we refer to are in fact models based on the Newtonian model of mechanics, on its five principles, i.e.: the inertia, the forces action, the action and reaction, the parallelogram and the initial conditions principle, respectively. Other models, e.g., the model of attraction forces between the particles of a discrete mechanical system, are part of the considered Newtonian model. Kepler's laws brilliantly verify this model in case of velocities much smaller than the light velocity in vacuum. The non-classical models are relativistic and quantic.

Mechanics has as object of study mechanical systems. The first volume of this book dealt with particle dynamics. The present one deals with discrete mechanical systems for particles in a number greater than the unity, as well as with continuous mechanical systems. We put in evidence the difference between these models, as well as the specificity of the corresponding studies; the generality of the proofs and of the corresponding computations yields a common form of the obtained mechanical results for both discrete and continuous systems. We mention the thoroughness by which the dynamics of the rigid solid with a fixed point has been presented. The discrete or continuous mechanical systems can be non-deformable (e.g., rigid solids) or deformable (deformable particle systems or deformable continuous media); for instance, the condition of equilibrium and motion, expressed by means of the "torsor", are necessary and sufficient in case of non-deformable systems and only necessary in case of deformable ones.

Passing by non-significant details, one presents some applications connected to important phenomena of the nature and one gives also the possibility to solve problems presenting interest from technical, engineering point of view. In this form, the book becomes – we dare say – a unique outline of the literature in the field; the author wishes to present the most important aspects connected with the study of mechanical systems, mechanics being regarded as a science of nature, as well as its links to other sciences of nature. Implications in technical sciences are not neglected.

Concerning the mathematical tool, the five appendices contained in the first volume give the book an autonomy with respect to other works, special previous mathematical knowledge being not necessary. The numeration of the chapters follows that of the first volume, to which one makes reference for various results (theorems, formulae etc.).

The book covers a wide number of problems (classical or new ones), as one can see from its contents. It uses the known literature, as well as the original results of the author and his more than fifty years experience as a Professor of Mechanics at the

University of Bucharest. It is devoted to a large circle of readers: mathematicians (especially those involved in applied mathematics), physicists (particularly, those interested in mechanics and its connections), chemists, biologists, astronomers, engineers of various specialities (civil, mechanical engineers etc., who are scientific researchers or designers), students in various domains etc.

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P.P. Teodorescu

Chapter 11

Dynamics of Discrete Mechanical Systems

In dynamics one studies the motion of *open mechanical systems*, which are subjected to the action of given external forces (input) and which exert certain actions upon other systems (output); *closed mechanical systems* (without input and output, which are moving only due to the interaction of the component particles) will be considered too. We begin the study by the motion with respect to an inertial (Galilean) frame of reference, passing then to the case of a non-inertial (non-Galilean) frame; we consider thus discrete mechanical systems, free or subjected to constraints. We present also the differential principles of mechanics.

11.1 Dynamics of Discrete Mechanical Systems with Respect to an Inertial Frame of Reference

First of all, some introductory notions are given: general theorems and conservation theorems of discrete mechanical systems, free or subjected to constraints, with respect to an inertial frame of reference, which lead to first integrals; the motion with respect to a non-inertial frame is taken into consideration too. We mention also applications to various important problems, e.g. the problem of n particles.

11.1.1 Introductory Notions

In what follows we introduce the notions of momentum, moment of momentum, kinetic and potential energy, work and power, with respect to an inertial frame, in case of a discrete mechanical system, extending the notions corresponding to a particle (see Chap. 6, Sect. 1.1). We mention also the formulation of the problem of mechanical systems of free particles.

11.1.1.1 Moment. Moment of Momentum. Torsor of Momentum

Let be a mechanical system $\mathscr{S}$ of geometric support Ω. *The momentum* (*linear momentum*) of the system with respect to a given fixed frame of reference (considered to be inertial) is given by

$$\mathbf{H} = \int_\Omega \mathbf{v}\,\mathrm{d}m = \int_\Omega \dot{\mathbf{r}}\,\mathrm{d}m\,, \tag{11.1.1}$$

the integral being a Stieltjes one, while the mass $m = m(\mathbf{r})$ is a distribution. Introducing the unit mass (1.1.71–1.1.71"), we may write

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$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{H}_i = \sum_{i=1}^{n} m_i \mathbf{v}_i = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \tag{11.1.1'}$$

for a discrete mechanical system $\mathscr{S}$ of n particles P_i of masses m_i and of position vectors $\mathbf{r}_i$, $i = 1,2,...,n$ (the momentum of a single particle P_i is $\mathbf{H}_i = m_i \mathbf{v}_i = m_i \dot{\mathbf{r}}_i$).

The moment of the momentum $\mathbf{H}_i$ with respect to the pole O of the considered frame is the moment of momentum of the particle P_i, with respect to that pole, and is given by

$$\mathbf{K}_{Oi} = \mathbf{r}_i \times \mathbf{H}_i = m_i \mathbf{r}_i \times \mathbf{v}_i = m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = 2 m_i \mathbf{\Omega}_{Oi} ,$$

where we have introduced the areal velocity (5.1.16) too. *The moment of momentum* (*angular momentum*) of the mechanical system $\mathscr{S}$ with respect to the pole O (which can be a given fixed point, immovable with respect to an inertial frame) is given by

$$\mathbf{K}_O = \int_{\Omega} \mathbf{r} \times \mathbf{v} \mathrm{d}m = \int_{\Omega} \mathbf{r} \times \dot{\mathbf{r}} \mathrm{d}m = 2 \int_{\Omega} \mathbf{\Omega}_O \mathrm{d}m ; \tag{11.1.2}$$

in case of a discrete mechanical system, we obtain

$$\mathbf{K}_O = \sum_{i=1}^{n} \mathbf{K}_{Oi} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{H}_i = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = 2 \sum_{i=1}^{n} m_i \mathbf{\Omega}_{Oi} . \tag{11.1.2'}$$

In components, we have

$$H_j = \sum_{i=1}^{n} m_i \dot{x}_j^{(i)} , \quad K_{Oj} = \in_{jkl} \sum_{i=1}^{n} m_i x_k^{(i)} \dot{x}_l^{(i)} = 2 \sum_{i=1}^{n} m_i \Omega_{Oj}^{(i)} , \quad j = 1,2,3 . \tag{11.1.3}$$

The notion of torsor, introduced in Chap. 6, Sect. 1.1.1 for a single particle, allows us to write

$$\tau_O \{ \mathbf{H}_i \} = \{ \mathbf{H}, \mathbf{K}_O \} ; \tag{11.1.4}$$

hence, the set formed by the linear and the angular momenta of a mechanical system $\mathscr{S}$ represents *the torsor* of the system $\{ \mathbf{H}_i \}$, formed by the momenta of the component particles of the mechanical system with respect to the considered pole. For the problems of dynamics, the torsor plays thus a rôle analogous to that played in statics.

11.1.1.2 Work. Kinetic and Potential Energy. Conservative Forces

The notion of *work* has been introduced in Chap. 3, Sect. 2.1.2, in the form of *elementary work* (3.2.3) (we omit the adjective “real”) of a system of given forces $\mathbf{F}_i$, applied at the points P_i of position vectors $\mathbf{r}_i$, $i = 1,2,...,n$. In general, we denote by $\mathbf{F}_i$ the given external forces; the given internal forces $\mathbf{F}_{ik}$, which verify a relation of the form (1.1.81) (inclusive a relation of the form (2.2.50)), put in evidence the influence of the particle (point) P_k upon the particle (point) P_i. *The elementary work* $\mathrm{d}W_{\mathrm{int}}$ of the given internal forces is expressed in the form (3.2.4); we notice that $\mathrm{d}W_{\mathrm{int}}$ is a non-negative quantity, vanishing if and only if the mechanical system $\mathcal{S}$ is non-deformable.

Analogously, *the elementary work of the constraint external forces* $\mathbf{R}_i$, $i = 1,2,...,n$, is given by

$$\mathrm{d}W_R = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathrm{d}\mathbf{r}_i \ , \tag{11.1.5}$$

while *the elementary work of the constraint internal forces* $\mathbf{R}_{ik} = R_{ik}\mathbf{u}$ ($R_{ik} = R_{ki}$ are positive quantities in case of repulsive constraint forces and negative ones in case of attractive such forces) has the remarkable expression

$$\begin{aligned} \mathrm{d}W_{R\mathrm{int}} &= \sum_{i=k+1}^{n} \sum_{k=1}^{n-1} (\mathbf{R}_{ik} \cdot \mathrm{d}\mathbf{r}_i + \mathbf{R}_{ki} \cdot \mathrm{d}\mathbf{r}_k) = \sum_{i=k+1}^{n} \sum_{k=1}^{n-1} R_{ik} \mathrm{d}r_{ik} \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} R_{ik} \mathrm{d}r_{ik} \ , \quad i \neq k \ , \end{aligned} \tag{11.1.5'}$$

where $\mathbf{r}_{ik} = r_{ik}\mathbf{u} = \mathbf{r}_k - \mathbf{r}_i$, $\mathbf{u} = \mathrm{vers}\,\mathbf{r}_{ik}$, is the vector which links two particles (points) of the mechanical system $\mathcal{S}$.

We can introduce *the kinetic energy* $T_i = m_i v_i^2 / 2 = m_i \dot{\mathbf{r}}_i^2 / 2$, which is a quantity of state of the particle P_i. *The kinetic energy of the mechanical system* $\mathcal{S}$ is, in this case, given by

$$T = \frac{1}{2} \int_{\Omega} v^2 \mathrm{d}m = \frac{1}{2} \int_{\Omega} \dot{\mathbf{r}}^2 \mathrm{d}m \,, \tag{11.1.6}$$

and if this system is discrete, then it results

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i^2 \ . \tag{11.1.6'}$$

If there exists a function U_{ik} so that $\mathrm{d}U_{ik} = F_{ik}\mathrm{d}r_{ik}$, then we obtain a potential U of the form (3.2.6'); in this case, the internal forces are conservative forces (the mechanical system is conservative) and we can introduce the potential energy $V = -U$, being thus led to the mechanical energy (6.1.15).

Immaterial whether the forces are external or internal ones, we introduce also *the virtual work of the given forces* in the form (3.2.3'), while *the virtual work of the constraint forces* is given by (3.2.7').

As in Chap. 6, Sect. 1.1.3, one can introduce the notions of *power* and *mechanical efficiency* for a mechanical system $\mathcal{S}$. Thus, starting from the elementary work (3.2.3), we get *the power of the given external forces* in the form

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i \; ; \tag{11.1.7}$$

as well, *the power of the internal forces* is

$$P_{\mathrm{int}} = \frac{\mathrm{d}W_{\mathrm{int}}}{\mathrm{d}t} = \sum_{i=1}^{n}\sum_{k=1}^{n} \mathbf{F}_{ik} \cdot \mathbf{v}_i = \sum_{i=k+1}^{n}\sum_{k=1}^{n-1} F_{ik} \frac{\mathrm{d}r_{ik}}{\mathrm{d}t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} F_{ik} \frac{\mathrm{d}r_{ik}}{\mathrm{d}t} , \quad i \neq k \, . \tag{11.1.7'}$$

In case of conservative internal forces, which derive from a potential U, we obtain

$$P_{\mathrm{int}} = \frac{\mathrm{d}U}{\mathrm{d}t} ; \tag{11.1.7''}$$

we can make analogous observations concerning *the power P of the external forces*. Analogously, we may introduce *the power of the constraint forces* too.

11.1.1.3 Formulation of Problems of Mechanical Systems in Motion

Let us consider a free discrete mechanical system $\mathcal{S}$, formed by n particles P_i, of masses m_i and position vectors $\mathbf{r}_i$, acted upon by the given external forces $\mathbf{F}_i$ (eventually, resultants of given forces, applied at the points P_i) and by the given internal forces $\mathbf{F}_{ik}$, $i \neq k$, $i,k = 1,2,...,n$. According to the principle of action of forces, Euler showed that the motion of the mechanical system $\mathcal{S}$ is governed by the system of vector differential equations (to simplify the notation, the sums of terms with two indices will be denoted by "prime" if the case of equal indices is excluded)

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} , \quad i = 1,2,...,n \; ; \tag{11.1.8}$$

in components, we may write

$$m_i \ddot{x}_j^{(i)} = F_j^{(i)} + \sum_{k=1}^{n}{}' F_j^{(ik)} , \quad i = 1,2,...,n , \quad j = 1,2,3 . \tag{11.1.8'}$$

The position of the free discrete mechanical system $\mathcal{S}$ at a given moment t can be determined if one knows the functions $x_j^{(i)} = x_j^{(i)}(t)$, $i = 1,2,...,n$, $j = 1,2,3$; hence, this mechanical system has $3n$ degrees of freedom. In general,

$$\mathbf{F}_i = \mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, ..., \dot{\mathbf{r}}_n; t) \equiv \mathbf{F}_i(\mathbf{r}_l, \dot{\mathbf{r}}_l; t) , \quad \mathbf{F}_{ik} = \mathbf{F}_{ik}(\mathbf{r}_l, \dot{\mathbf{r}}_l; t) ,$$
$$i \neq k , \quad i,k,l = 1,2,...,n ,$$

the motion of each particle depending on the motions of all other particles. As well, the given internal forces can be determined (that is, may be given) only taking into account some hypotheses concerning the structure of the mechanical system (rigidity, deformability – elasticity, plasticity, viscosity etc.), hence if a mathematical model of it is set up; in fact, in case of deformable mechanical systems, the motion of the particles and of the internal forces may be determined only simultaneously.

In *the first fundamental problem* (*the direct problem*) which is put the forces $\mathbf{F}_i$ and $\mathbf{F}_{ik}$, $i \neq k$, $i,k = 1,2,...,n$, are given, and one must determine the trajectories of the particles, hence the vector functions $\mathbf{r}_i = \mathbf{r}_i(t)$, and the velocities $\mathbf{v}_i = \mathbf{v}_i(t) = \dot{\mathbf{r}}_i(t)$. The problem is solved by integrating the system of n vector equations (11.1.8) or the system of $3n$ scalar equations of second order (11.1.8') with certain boundary conditions. For the most part, initial conditions (at the initial moment $t = t_0$) of the form

$$\mathbf{r}_i(t_0) = \mathbf{r}_i^0, \quad \mathbf{v}_i(t_0) = \mathbf{v}_i^0, \quad i = 1,2,...,n, \tag{11.1.9}$$

are put; one may put also other boundary conditions (e.g., bilocal conditions at the moments $t = t_0$ and $t = t_1$). In certain conditions, sufficiently large, the solution of the problem is unique.

In *the second fundamental problem* (*the inverse problem*) the motion of the particles P_i is known (hence, the position vectors $\mathbf{r}_i$ are given) and the forces $\mathbf{F}_i$ and $\mathbf{F}_{ik}$, $i \neq k$, $i,k = 1,2,...,n$, which provoke this motion, have to be determined. In general, the solution of the problem is not unique.

We mention also *the mixed fundamental problem* in which some elements which characterize the motion and some elements which characterize the forces are given; one must determine the other elements remained unknown, to can specify entirely the motion of the particles and the forces applied upon them. This problem has also not a unique solution, in general, but only in certain conditions.

As in the case of a single particle (see Chap. 6, Sect. 1.1.4 too), we must impose other conditions to the law which expresses the forces, so that the two last problems be determined.

Using the results in Chap. 6, Sect. 1.1.5, we may express the equations of motion in other systems of co-ordinates, as it is more convenient from the point of view of the computation.

The system of equations of motion (11.1.8) of the discrete mechanical system $\mathscr{S}$ is written in an inertial frame of reference $\mathscr{R}$, with respect to which the principles which are at the basis of the Newtonian model of mechanics are supposed to be verified. They remain, further, in the same form with respect to any other inertial frame which is deduced from the first one by a rectilinear and uniform motion of translation, hence by a transformation of co-ordinates belonging to the Galileo-Newton group (see Chap. 6, Sect. 1.2.3 too).

11.1.1.4 Theorems of Existence and Uniqueness

By means of the notations (see Chap. 3, Sect. 2.2.2 too)

$$X_{3(i-1)+j} = x_j^{(i)}, \quad V_{3(i-1)+j} = v_j^{(i)}, \quad i = 1,2,...,n, \quad j = 1,2,3, \tag{11.1.10}$$

we can pass from the geometric support Ω of the discrete mechanical system $\mathscr{S}$ in the space E_3 (formed by the geometric points P_i) to a representative geometric point P (of co-ordinates X_k, $k = 1,2,...,3n$) in the representative space E_{3n}; we also introduce the notations

$$Q_{3(i-1)+j} = \frac{1}{m_i}\left(F_j^{(i)} + \sum_{k=1}^{n}{}' F_j^{(ik)} \right), \quad i = 1,2,...,n, \quad j = 1,2,3. \tag{11.1.10'}$$

Therefore, we replace the study of the motion of the discrete mechanical system $\mathscr{S}$ by the study of the motion of the representative point P in the representative space E_{3n}; the motion of the point P is governed by the system of differential equations

$$\ddot{X}_k = Q_k, \quad k = 1,2,...,3n. \tag{11.1.11}$$

We replace this system of $3n$ differential equations of second order by a system of $6n$ differential equations of first order, written in the normal form

$$\dot{X}_k = V_k, \quad \dot{V}_k = Q_k, \quad k = 1,2,...,3n, \tag{11.1.11'}$$

where $V_k = V_k(X_l;t)$, $Q_k = Q_k(X_l, V_l;t)$, $k,l = 1,2,...,3n$. Such a system is *non-autonomous*; if the time does not intervene explicitly in V_k and Q_k, then the system is *autonomous* (or *dynamic*). We associate the initial conditions (corresponding to the conditions (11.1.9))

$$X_k(t_0) = X_k^0, \quad V_k(t_0) = V_k^0, \quad k = 1,2,...,3n, \tag{11.1.11''}$$

to this system, so that the boundary value problem (11.1.11'), (11.1.11") becomes a problem of Cauchy type. The boundary value problem (11.1.8), (11.1.9) is equivalent to the boundary value problem (11.1.11'), (11.1.11"); for the latter problem, one may state

Theorem 11.1.1 (*of existence and uniqueness; Cauchy-Lipschitz*). *If the functions* V_k *and* Q_k, $k = 1,2,...,3n$, *are continuous on the interval* $(6n+1)$ – *dimensional* $\mathscr{D}$, *specified by* $X_k^0 - X_{k0} \le X_k \le X_k^0 + X_{k0}$, $V_k^0 - V_{k0} \le V_k \le V_k^0 + V_{k0}$, $t_0 - t^0 \le t \le t_0 + t^0$, $X_{k0}, V_{k0}, t^0 = \text{const}$, $k = 1,2,...,3n$, *and defined in the space Cartesian product of the phase space* (*of canonical co-ordinates* $X_1, X_2, ..., X_{3n}$, $V_1, V_2, ..., V_{3n}$) *by the time space* (*of co-ordinate* t), *and if the Lipschitz conditions*

$$\left| V_k(X_l;t) - V_k(\bar{X}_l;t) \right| \le \frac{1}{\mathscr{T}} \sum_{l=1}^{3n} \left| X_l - \bar{X}_l \right|,$$

$$\left| Q_k(X_l, V_l;t) - Q_k(\bar{X}_l, \bar{V}_l;t) \right| \le \frac{1}{\mathscr{T}} \sum_{l=1}^{3n} \left(\frac{1}{\tau} \left| X_l - \bar{X}_l \right| + \left| V_l - \bar{V}_l \right| \right),$$

for $k = 1,2,...,3n$, *where* $\mathscr{T}$ *is an independent of* V_k *and t time constant, while* τ *is a time constant equal to unity, are satisfied, then it exists a unique solution* $X_k = X_k(t)$, $V_k = V_k(t)$ *of the system* (11.1.11'), *which satisfies the initial conditions* (11.1.11") *and is defined on the interval* $t_0 - T \leq t \leq t_0 + T$, *where*

$$T \leq \min\left(t_0, \frac{X_{k0}}{\mathscr{V}}, \tau\frac{V_{k0}}{\mathscr{V}}, \mathscr{T} \right), \quad \mathscr{V} = \max(V_k, \tau Q_k) \text{ in } \mathscr{D}.$$

According to *Peano's theorem*, *the existence of the solution* is ensured by the continuity of the functions V_k and Q_k on the interval $\mathscr{D}$. For *the uniqueness of the solution*, Lipschitz's conditions must be fulfilled too; the latter conditions can be replaced by other more restrictive ones according to which the partial derivatives of first order of the functions V_k and Q_k, $k = 1,2,...,3n$, must exist and be bounded in absolute value on the interval $\mathscr{D}$, as it was shown by Picard, using a *method of successive approximations*. As a matter of fact, the Theorem 11.1.1 can be demonstrated by an analogous method. We must mention that the conditions in Theorem 11.1.1 are *sufficient conditions* of existence and uniqueness, which are not necessary too.

The existence and the uniqueness of the solution have been put in evidence only on the interval $[t_0 - T, t_0 + T]$, in the vicinity of the initial moment t_0 (in fact, t_0 can be an arbitrary chosen moment, not necessarily the initial one); taking, for instance, $t_0 + T$ as initial moment, by repeating the above reasoning, it is possible to extend the solution on an interval $2T_1$, $T_1 > T$ (obviously, if the sufficient conditions of existence and uniqueness of the Theorem 11.1.1 are fulfilled in the vicinity of this new initial moment). We can obtain thus a *prolongation* of the solution for $t \in [t_1, t_2]$, corresponding to an arbitrary interval of time in which the considered mechanical phenomenon takes place or even for $t \in (-\infty, +\infty)$.

As in the case of a single particle (see Chap. 6, Sect. 1.2.1), we can put in evidence some important properties of this solution; we thus state:

Theorem 11.1.2 (*on the continuous dependence of the solution on a parameter*). *If the functions* $V_k(X_l; t, \mu)$, $Q_k(X_l, V_l; t, \mu)$ *are continuous with respect to the parameter* $\mu \in [\mu_1, \mu_2]$ *and satisfy the conditions of the theorem of existence and uniqueness, and if the constant* $\mathscr{T}$ *of Lipschitz does not depend on* μ, *then the solution* $X_k(t, \mu)$, $V_k(t, \mu)$, $k = 1,2,...,3n$, *of the system* (11.1.11') *which satisfies the conditions* (11.1.11") *depends continuously on* μ.

Theorem 11.1.3 (*on the analytical dependence of the solution on a parameter; Poincaré*). *The solution* $X_k(t, \mu)$, $V_k(t, \mu)$, $k = 1,2,...,3n$, *of the system* (11.1.11') *which satisfies the conditions* (11.1.11"), *depends analytically on the parameter* $\mu \in [\mu_1, \mu_2]$ *in the neighbourhood of the value* $\mu = \mu_0$ *if, in the interval* $\mathscr{D} \times [\mu_1, \mu_2]$, *the functions* V_k *and* Q_k *are continuous with respect to t and analytic with respect to* X_k, V_k, $k = 1,2,...,3n$, *and* μ.

Theorem 11.1.4 (*on the differentiability of the solutions*). *If in the vicinity of a point* $\mathscr{P}(X_l^0, V_l^0; t)$ *the functions* $V_k(X_l; t)$ *and* $Q_k(X_l, V_l; t)$, $k = 1,2,...,3n$, *are of class* C^m, *then the solutions* $X_k(t)$ *and* $V_k(t)$ *of the system* (11.1.11'), *which satisfy the initial conditions* (11.1.11''), *are of class* C^{m+1} *in that vicinity*.

One can state theorems (analogue to the Theorem 11.1.2) concerning the continuous dependence of the solution on the initial conditions or on several parameters.

The points $\mathscr{P}$ in the vicinity of which the boundary value problem (11.1.11'), (11.1.11'') has not solution or, even if the solution exists, that one is not unique, are called *singular points*; the integral curves (in the space E_{3n}) formed only of singular points are called *singular curves*, the corresponding solution being a *singular solution*. For the singular points there are necessary supplementary conditions, which can lead to the choice of one of *the branches of the multiple solution*.

11.1.1.5 First Integrals. General Integral. Constants of Integration

An *integrable combination* of the system (11.1.11') may be, e.g.,

$$\mathrm{d}f(X_1, X_2, ..., X_{3n}, V_1, V_2, ..., V_{3n}; t) = 0, \tag{11.1.12}$$

obtaining thus a finite relation of the form

$$f(X_1, X_2, ..., X_{3n}, V_1, V_2, ..., V_{3n}; t) = C, \quad C = \text{const}, \tag{11.1.12'}$$

hence a link between the co-ordinates X_k and the components V_k of the velocity at the time t; the function f, which is reduced to a constant along the integral curves, is called *first integral* of the system (11.1.11').

If we determine $h \leq 6n$ first integrals for which

$$f_j(X_k, V_k; t) = C_j, \quad C_j = \text{const}, \quad j = 1,2,...,h, \tag{11.1.13}$$

the matrix

$$\mathbf{M} \equiv \left[\frac{\partial(f_1, f_2, ..., f_h)}{\partial(X_1, X_2, ..., X_{3n}, V_1, V_2, ..., V_{3n})} \right] \tag{11.1.13'}$$

being of rank h, then all these first integrals are *independent* (for the sake of simplicity we say not “functional” independent) and we may express h unknown functions of the system (11.1.13) in terms of the other ones; replacing in (11.1.11'), the problem is reduced to the integration of a system of equations with only $6n - h$ unknowns. If $h = 6n$, then all the first integrals are independent and the system (11.1.13) of first integrals determines all the unknown functions. For $h > 6n$ the first integrals (11.1.13) are no more independent, so that we cannot set up more than $6n$ independent first integrals.

Assuming that $h = 6n$ and solving the system (11.1.13), we get (the matrix **M** is a square matrix of order $6n$ for which $\det \mathbf{M} \neq 0$)

$$X_k = X_k(t; C_1, C_2, ..., C_{6n}), \quad V_k = V_k(t; C_1, C_2, ..., C_{6n}), \quad k = 1,2,...,3n, \tag{11.1.14}$$

where $C_l = \mathrm{const}$, $l = 1,2,...,6n$, hence *the general integral* of the system of equations (11.1.11'). The general integral of the system of vector equations (11.1.8) is, analogously,

$$\mathbf{r}_i = \mathbf{r}_i(t; C_1, C_2, ..., C_{6n}), \quad i = 1,2,...,n; \tag{11.1.15}$$

eventually, we have

$$\mathbf{r}_i = \mathbf{r}_i(t; \mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_{2n}), \quad i = 1,2,...,n, \tag{11.1.15'}$$

where $\mathbf{K}_l = \overrightarrow{\mathrm{const}}$, $l = 1,2,...,2n$. Thus, $6n$ scalar constants or $2n$ vector constants of integration are put into evidence. Because the vector functions (11.1.15) or (11.1.15') verify the equations (11.1.8) for any constants of integration, we can state that the same mechanical system $\mathcal{S}$, acted upon by the same system of forces, may have different motions. Imposing the initial conditions (11.1.9), written in the form (11.1.11''), we obtain

$$X_k(t_0; C_1, C_2, ..., C_{6n}) = X_k^0, \quad V_k(t_0; C_1, C_2, ..., C_{6n}) = V_k^0, \quad k = 1,2,...,3n;$$

the conditions (11.1.11'') being independent, we may write

$$\det\left[\frac{\partial(X_1^0, X_2^0, ..., X_{3n}^0, V_1^0, V_2^0, ..., V_{3n}^0)}{\partial(C_1, C_2, ..., C_{6n})}\right] \neq 0,$$

and, according to the theorem of implicit functions, we deduce

$$C_j = C_j\left(t_0; X_1^0, X_2^0, ..., X_{3n}^0, V_1^0, V_2^0, ..., V_{3n}^0\right), \quad j = 1,2,...,6n.$$

Thus, we obtain, finally,

$$X_k = X_k\left(t; t_0, X_l^0, V_l^0\right), \quad V_k = V_k\left(t; t_0, X_l^0, V_l^0\right), \quad k = 1,2,...,3n, \tag{11.1.16}$$

or

$$\mathbf{r}_i = \mathbf{r}_i\left(t; t_0, \mathbf{r}_j^0, \mathbf{v}_j^0\right), \quad \mathbf{v}_i = \mathbf{v}_i\left(t; t_0, \mathbf{r}_j^0, \mathbf{v}_j^0\right), \quad i = 1,2,...,n. \tag{11.1.16'}$$

In the frame of the conditions of the theorem of existence and uniqueness, the principle of the action of forces and the principle of the initial conditions determine, univocally, the motion of the mechanical system $\mathcal{S}$ in a finite interval of time; by prolongation, this statement may become valid for any t. *The deterministic aspect of Newtonian mechanics* is thus put in evidence.

Sometimes, a calculation in two steps can be made, as in the case of only one particle (see Chap. 6, Sect. 1.2.2). Starting from the system of differential equations of second order (11.1.11), we can find integrable combinations, leading to $3n$ first integrals of the form

$$\varphi_k\left(X_1, X_2, ..., X_{3n}, \dot{X}_1, \dot{X}_2, ..., \dot{X}_{3n}; t\right) = C_k, \quad k = 1,2,...,3n; \tag{11.1.17}$$

if, starting from these relations, in a second stage, we build up other $3n$ integrable combinations, leading to the first integrals

$$\psi_k\left(X_1, X_2, ..., X_{3n}; t; C_1, C_2, ..., C_{3n}\right) = C_{3n+k}, \quad k = 1,2,...,3n, \tag{11.1.17'}$$

then the problem is solved. Indeed, we notice that

$$\det\left[\frac{\partial(\psi_1, \psi_2, ..., \psi_{3n})}{\partial(X_1, X_2, ..., X_{3n})}\right] \neq 0,$$

finding thus the first group of relations (11.1.14).

11.1.2 General Theorems. Conservation Theorems

We present in what follows the general theorems of mechanics (the theorem of momentum, the theorem of moment of momentum and the theorem of kinetic energy) with respect to an *inertial frame of reference* (for the sake of simplicity, we do not mention it in the following) in case of a free discrete mechanical system and in case of a discrete mechanical system subjected to constraints, as well as the corresponding conservation theorems. Among the applications of these results, we mention the problem of n particles. We notice that, in fact, the general theorems are differential consequences of the system of equations of motion (11.1.8); we may also say that these theorems represent necessary conditions which must be verified in the motion of the mechanical system $\mathscr{S}$ with respect to an inertial frame of reference.

11.1.2.1 Theorem of Momentum. Theorem of Motion of the Centre of Mass

Summing the equations of motion with respect to an inertial frame of reference (11.1.8) for all the particles of the mechanical system $\mathscr{S}$ and taking into account the relation (2.2.50) verified by the internal forces, we obtain

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \mathbf{F}_i,$$

where the internal forces disappear. Introducing the momentum (11.1.1'), it results, finally,

$$\dot{\mathbf{H}} = \frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{R}\,, \quad \dot{H}_j = \sum_{i=1}^{n} F_j^{(i)} = R_j\,, \quad j = 1,2,3\,, \tag{11.1.18}$$

and we may state

Theorem 11.1.5 (*theorem of momentum*). *The derivative with respect to time of the momentum of a free discrete mechanical system is equal to the resultant of the given external forces which act upon this system.*

We mention that, in this theorem, both the derivative $\dot{\mathbf{H}}$ and the resultant of the given external forces are free vectors. As well, the first relation (11.1.18) maintains its validity if it is projected on a plane or on an axis (the second group of relations (11.1.18)).

Starting from the relation (3.1.2) which gives the position vector of the centre of mass C of the discrete mechanical system $\mathscr{S}$, differentiating with respect to time in the fixed (inertial frame) and taking into consideration (11.1.1'), we obtain

$$\mathbf{H} = M\dot{\boldsymbol{\rho}} \tag{11.1.19}$$

and we can state

Theorem 11.1.6 *The momentum of a mechanical system is equal to the momentum of the centre of mass of this system, at which we consider that the whole mass of it is concentrated.*

We notice that the relation (11.1.19) is of the type of relation (3.1.9) for the polar static moments; we put thus in evidence the importance of the centre of mass C, which can replace – in certain situations – the whole mechanical system $\mathscr{S}$. The Theorem 11.1.6 is valid for any mechanical system, either if it is free or it is subjected to constraints.

Differentiating the relation (11.1.19) with respect to time, in the considered fixed frame, and taking into account (11.1.18), we may write

$$M\ddot{\boldsymbol{\rho}} = \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{R}\,, \tag{11.1.19'}$$

stating thus

Theorem 11.1.7 (*theorem of motion of the centre of mass; Newton*). *The centre of mass of a free discrete mechanical system is moving as a free particle at which would be concentrated the whole mass of the system and which would be acted upon by the resultant of the given external forces.*

This theorem allows an independent study of the motion of the centre of mass C (even if this one does not belong to the mechanical system $\mathscr{S}$), in a first stage of study of the system in its totality; in a second stage, one can consider the motion of the system with respect to the centre of mass C. As well, the Theorem 11.1.7 represents a justification for the modelling as a particle in the study of mechanical systems.

Starting from the relation (11.1.18), we may write

$$\Delta\mathbf{H} = \mathbf{H}(t_2) - \mathbf{H}(t_1) = \sum_{i=1}^{n}\int_{t_1}^{t_2}\mathbf{F}_i \mathrm{d}t = \int_{t_1}^{t_2}\mathbf{R}\mathrm{d}t\,; \tag{11.1.18'}$$

the variation of the momentum of a free discrete mechanical system in a finite interval of time is thus put into evidence. The quantity $\int_{t_1}^{t_2}\mathbf{R}\mathrm{d}t$ represents *the impulse of the resultant of the given external forces*, corresponding to the interval of time $[t_1, t_2]$.

The notion of hodograph, introduced in Chap. 5, Sect. 1.2.1, leads to a kinematical interpretation of the Theorem 11.1.5; we state thus

Theorem 11.1.5' *The velocity of a point which describes the hodograph of the momentum of a free discrete mechanical system with respect to a fixed pole is equipollent to the resultant of the given external forces which act upon this system.*

11.1.2.2 Theorem of Moment of Momentum

We perform a vector product at the left of each equation of motion (11.1.8) by $\mathbf{r}_i$ and sum for all particles of the mechanical system $\mathscr{S}$; taking into account the relation (2.2.50) which is verified by the internal forces, we obtain

$$\sum_{i=1}^{n} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_{i=1}^{n}\mathbf{r}_i \times \mathbf{F}_i\,,$$

where the internal forces disappear. Introducing the momentum $\mathbf{H}_i$ and the moment of momentum $\mathbf{K}_{Oi}$, we notice that

$$m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = m_i(\mathbf{r}_i \times \ddot{\mathbf{r}}_i + \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i) = m_i\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times \dot{\mathbf{r}}_i) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times \mathbf{H}_i) = \dot{\mathbf{K}}_{Oi},$$

so that we may write

$$\dot{\mathbf{K}}_O = \frac{\mathrm{d}\mathbf{K}_O}{\mathrm{d}t} = \sum_{i=1}^{n}\mathbf{r}_i \times \mathbf{F}_i = \mathbf{M}_O\,, \quad \dot{K}_{Oj} = \in_{jkl}\sum_{i=1}^{n} x_k^{(i)}\mathbf{F}_l^{(i)} = M_{Oj}\,, \quad j = 1,2,3\,. \tag{11.1.20}$$

Thus, we state

Theorem 11.1.8 (*theorem of moment of momentum*). *The derivative with respect to time of the moment of momentum of a free discrete mechanical system, with respect to a fixed pole, is equal to the resultant moment of the given external forces which act upon this system, with respect to the same pole.*

In this theorem, the derivative $\dot{\mathbf{K}}_O$ as well as the resultant moment of the given external forces are bound vectors, applied at the pole O. As well, the first relation (11.1.20) remains valid if it is projected on a plane or on an axis (the second group of relations (11.1.20)). We may also write

$$2\sum_{i=1}^{n} m_i\dot{\mathbf{\Omega}}_i = \mathbf{M}_O\,, \quad 2\sum_{i=1}^{n} m_i\dot{\Omega}_{Oj}^{(i)} = \in_{jkl}\sum_{i=1}^{n} m_i x_k^{(i)}\ddot{x}_l^{(i)} = M_{Oj}\,, \quad j = 1,2,3\,, \tag{11.1.20'}$$

putting thus into evidence the areal accelerations of the component particles of the mechanical system $\mathscr{S}$; we can state

Theorem 11.1.8' (*theorem of areal accelerations*). *The sum of the products of the double masses of the particles of a free discrete mechanical system by their areal accelerations, with respect to a given fixed pole, is equal to the resultant moment of the given external forces which act upon this system with respect to the same pole.*

Another form of the relation (11.1.20) is given by

$$\Delta \mathbf{K}_O = \mathbf{K}_O(t_2) - \mathbf{K}_O(t_1) = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{r}_i \times \mathbf{F}_i \mathrm{d}t = \int_{t_1}^{t_2} \mathbf{M}_O \mathrm{d}t \,; \qquad (11.1.20'')$$

thus, the variation of the moment of momentum of a free discrete mechanical system, in a finite interval of time, is put into evidence. The quantity $\int_{t_1}^{t_2} \mathbf{M}_O \mathrm{d}t$ represents *the impulse of the resultant moment of the given forces* with respect to the pole *O*, corresponding to the interval of time $[t_1, t_2]$.

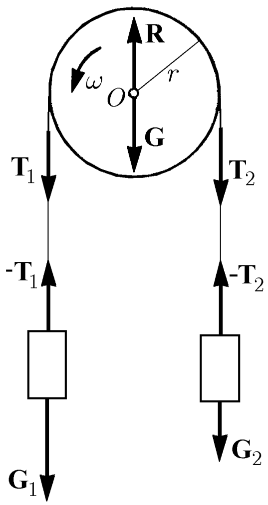

Fig. 11.1 The Atwood engine

With the aid of the hodograph too, the kinematic form of the theorem of moment of momentum is expressed by (for the sake of simplicity, we choose the pole of the moment of momentum as pole of the hodograph)

Theorem 11.1.8'' *The velocity of the point which describes the hodograph of the moment of momentum of a free discrete mechanical system with respect to a given fixed pole is equipollent to the resultant moment of the given external forces which act upon this system, with respect to the same pole.*

We will use these results in the study of *the Atwood engine*, which allows to verify the law of falling of the bodies. This engine is formed by a wheel of radius r and weight $\mathbf{G}$, which is rotating frictionless, with the angular velocity ω, about a horizontal axle, which passes through the pole O; over the wheel passes frictionless an inextensible and perfect flexible and torsionable thread, at the ends of which are suspended the weights $\mathbf{G}_1 = m_1\mathbf{g}$, $\mathbf{G}_2 = m_2\mathbf{g}$, $m_1 > m_2$ (Fig. 11.1); we assume that the mechanical system is moving with the velocity $v = v_1 = v_2 = r\omega$. The theorem of moment of momentum is written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_O\omega + m_1 v_1 r + m_2 v_2 r) = m_1 g r - m_2 g r ,$$

where I_O is the polar moment of inertia of the wheel with respect to the pole O; in this case, the acceleration $a = \dot{v}$ is given by

$$a = \frac{m_1 - m_2}{m'} g , \quad m' = \frac{I_O}{r^2} + m_1 + m_2 , \tag{11.1.21}$$

where m' is the reduced mass of the mechanical system. We notice that $a < g$; if the difference $m_1 - m_2$ is small, then we have $a \ll g$. Thus, we obtain easy the acceleration a, in the frame of an experiment; the acceleration g is then given by the formula (11.1.21) (m_1, m_2, I_O and r are known quantities). The tensions T_1 and T_2 in the thread and the constraint force R are then easily obtained

$$\begin{aligned} T_1 &= m_1(g - a) = \frac{m_1}{m'}(m' + m_2 - m_1)g , \\ T_2 &= m_2(g + a) = \frac{m_2}{m'}(m' + m_1 - m_2)g , \\ R = (m_1 + m_2)g - (m_1 - m_2)a + G &= \frac{m_1 - m_2}{m'}(m' + m_2 - m_1)g + G . \end{aligned} \tag{11.1.21'}$$

The theorem of moment of momentum with respect to the pole O, written only for the subsystem formed by the wheel ($I_O\dot{\omega} = (T_1 - T_2)r$), may be used to verify the results obtained above.

11.1.2.3 Theorem of Torsor

Starting from (11.1.4), we notice that $\dot{\tau}_O\{\mathbf{H}_i\} = \{\dot{\mathbf{H}}, \dot{\mathbf{K}}_O\}$; but

$$\dot{\mathbf{H}} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\mathbf{H}_i = \sum_{i=1}^{n}\dot{\mathbf{H}}_i ,$$

$$\dot{\mathbf{K}}_O = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\mathbf{r}_i \times \mathbf{H}_i = \sum_{i=1}^{n}(\dot{\mathbf{r}}_i \times \mathbf{H}_i + \mathbf{r}_i \times \dot{\mathbf{H}}_i) = \sum_{i=1}^{n}\mathbf{r}_i \times \dot{\mathbf{H}}_i ,$$

so that $\dot{\tau}_O\{\mathbf{H}_i\} = \tau_O\{\dot{\mathbf{H}}_i\}$. Because $\tau_O\{\mathbf{F}_i\} = \{\mathbf{R}, \mathbf{M}_O\}$, starting from (11.1.18) and (11.1.20), we get

$$\dot{\tau}_O\{\mathbf{H}_i\} = \frac{\mathrm{d}}{\mathrm{d}t}\tau_O\{\mathbf{H}_i\} = \tau_O\{\mathbf{F}_i\}, \tag{11.1.22}$$

stating thus

Theorem 11.1.9 (*theorem of torsor*). *The derivative with respect to time of the torsor of momenta of a free discrete mechanical system with respect to a given fixed pole is equal to the torsor of the given external forces which act upon this system, with respect to the same pole.*

Thus, by introducing the notion of torsor, we get a synthesis of the theorems of linear and angular momenta, which has the advantage to contain only the given external forces (the internal forces are eliminated). The theorem of torsor implies two vector relations or six scalar relations.

The variation of the torsor of momenta of a free discrete mechanical system with respect to a given fixed pole, in an interval of time $[t_1, t_2]$, is obtained in the form

$$\Delta\tau_O\{\mathbf{H}_i\} = \tau_O\{\mathbf{H}_i\}_{t=t_2} - \tau_O\{\mathbf{H}_i\}_{t=t_1} = \tau_O\left\{\int_{t_1}^{t_2}\mathbf{F}_i\mathrm{d}t\right\}. \tag{11.1.22'}$$

This relation plays an important rôle in case of a small interval of time and, especially, in case of discontinuous phenomena.

The general theorems stated above take place with respect to an inertial frame of reference $\mathcal{R}$, considered as fixed; the theorem of moment of momentum and the theorem of torsor, which depend on the pole O, maintain their form with respect to another pole Q, rigidly connected to the frame $\mathcal{R}$ (fixed with respect to this frame). If the pole Q is movable, calculating further with respect to the frame $\mathcal{R}$, the momentum remains invariant, but (as in Chap. 6, Sect. 1.2.4) the moment of momentum and the resultant moment of the given external forces become

$$\mathbf{K}_O = \mathbf{K}_Q + \mathbf{r}_Q \times \mathbf{H}, \quad \mathbf{M}_O = \mathbf{M}_Q + \mathbf{r}_Q \times \mathbf{R}, \quad \mathbf{r}_Q = \overrightarrow{OQ};$$

replacing in (11.1.20) and taking into account (11.1.18), we may write ($\overline{\mathbf{r}}_i = \overrightarrow{QP_i}$)

$$\dot{\mathbf{K}}_Q = \frac{\mathrm{d}\mathbf{K}_Q}{\mathrm{d}t} = \sum_{i=1}^{n}\overline{\mathbf{r}}_i \times \mathbf{F}_i - \mathbf{v}_Q \times \mathbf{H} = \mathbf{M}_Q - \mathbf{v}_Q \times \mathbf{H}, \tag{11.1.23}$$

obtaining thus a generalized form of the theorem of moment of momentum. The formula (11.1.22) is generalized in the form

$$\dot{\tau}_Q\{\mathbf{H}_i\} = \frac{\mathrm{d}}{\mathrm{d}t}\tau_Q\{\mathbf{H}_i\} = \tau_Q\{\mathbf{F}_i\} - \{\mathbf{0}, \mathbf{v}_Q \times \mathbf{H}\}. \tag{11.1.23'}$$

11.1.2.4 Theorem of Kinetic Energy

We perform a scalar product of each equation of motion (11.1.8) by $\mathbf{r}_i$ and sum for all the particles of the mechanical system $\mathcal{S}$, and obtain

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \mathrm{d}\mathbf{r}_i = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathrm{d}\mathbf{r}_i + \sum_{i=1}^{n} \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \cdot \mathrm{d}\mathbf{r}_i \;;$$

observing that

$$m_i \ddot{\mathbf{r}}_i \cdot \mathrm{d}\mathbf{r}_i = m_i \frac{\mathrm{d}\dot{\mathbf{r}}_i}{\mathrm{d}t} \cdot \mathrm{d}\mathbf{r}_i = m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} \cdot \mathrm{d}\dot{\mathbf{r}}_i = m_i \dot{\mathbf{r}}_i \cdot \mathrm{d}\dot{\mathbf{r}}_i$$
$$= \frac{1}{2} \mathrm{d}(m_i \dot{\mathbf{r}}_i^2) = \frac{1}{2} \mathrm{d}(m_i v_i^2) = \mathrm{d}T_i$$

and taking into account (3.2.3), (3.2.4) and (11.1.6'), we may write

$$\mathrm{d}T = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}}, \tag{11.1.24}$$

stating thus

Theorem 11.1.10 (*theorem of kinetic energy; Daniel Bernoulli*). *The differential of the kinetic energy of a free discrete mechanical system is equal to the elementary work of the given external and internal forces which act upon this system.*

Unlike the theorem of torsor, in the theorem of kinetic energy intervene also the internal forces in calculation. Dividing the relation (11.1.24) by $\mathrm{d}t$ and taking into account (11.1.7), (11.1.7'), we can write this theorem in the form (closer to the previous ones)

$$\dot{T} = \frac{\mathrm{d}T}{\mathrm{d}t} = P + P_{\mathrm{int}}, \tag{11.1.24'}$$

obtaining thus

Theorem 11.1.10' (*theorem of kinetic energy; second form*). *The derivative with respect to time of the kinetic energy of a free discrete mechanical system is equal to the power of the given external and internal forces which act upon this system.*

In general, the elementary work is not an exact differential (it is a Pfaff form); the theorem of kinetic energy may be written in the form (for $t \in [t_1, t_2]$, between the positions P_1 and P_2 of the representative point P in the space E_{3n})

$$\Delta T = T(t_2) - T(t_1) = T_2 - T_1 = W_{\overparen{P_1 P_2}} + W_{\mathrm{int}\,\overparen{P_1 P_2}}$$
$$= \sum_{i=1}^{n} \int_{\overparen{P_1 P_2}} \mathbf{F}_i \cdot \mathrm{d}\mathbf{r}_i + \sum_{i=1}^{n} \sum_{k=1}^{n}{}' \int_{\overparen{P_1 P_2}} \mathbf{F}_{ik} \cdot \mathrm{d}\mathbf{r}_i$$
$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{F}_i \cdot \mathbf{v}_i \mathrm{d}t + \sum_{i=1}^{n} \sum_{k=1}^{n}{}' \int_{t_1}^{t_2} \mathbf{F}_{ik} \cdot \mathbf{v}_i \mathrm{d}t = \int_{t_1}^{t_2} P \mathrm{d}t + \int_{t_1}^{t_2} P_{\mathrm{int}} \mathrm{d}t \tag{11.1.24''}$$

and we can state

Theorem 11.1.10'' (*theorem of kinetic energy; finite form*). *The variation of the kinetic energy of a free discrete mechanical system in a finite interval of time is equal to the work effected by the external and internal forces which act upon this system in the considered interval of time.*

If we write the relation (11.1.18') for only one particle P_i (we must introduce also the influence of the internal forces) and if we perform a scalar product by the velocity $\mathbf{v}_i^{(2)}$ at the finite moment (the velocity $\mathbf{v}_i^{(1)}$ takes place at the initial moment), and if we sum for all the particles of the discrete mechanical system $\mathscr{S}$, then we obtain

$$\sum_{i=1}^{n} m_i \left(v_i^{(2)} \right)^2 - \sum_{i=1}^{n} m_i \mathbf{v}_i^{(1)} \cdot \mathbf{v}_i^{(2)} = \sum_{i=1}^{n} \mathbf{v}_i^{(2)} \cdot \int_{t_1}^{t_2} \left(\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \right) \mathrm{d}t \,;$$

introducing the notations

$$T_1 = \frac{1}{2} \sum_{i=1}^{n} m_i \left(v_i^{(1)} \right)^2, \quad T_2 = \frac{1}{2} \sum_{i=1}^{n} m_i \left(v_i^{(2)} \right)^2, \quad T_0 = \frac{1}{2} \sum_{i=1}^{n} m_i \left(v_i^{(0)} \right)^2, \tag{11.1.25}$$

where T_0 is *the kinetic energy of the lost velocities*, while $\mathbf{v}_i^{(0)} = \mathbf{v}_i^{(2)} - \mathbf{v}_i^{(1)}$, it results

$$\Delta T + T_0 = \sum_{i=1}^{n} \mathbf{v}_i^{(2)} \cdot \int_{t_1}^{t_2} \left(\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \right) \mathrm{d}t \tag{11.1.26}$$

and we can state

Theorem 11.1.11 *The sum of the variation of the kinetic energy of a free discrete mechanical system in a finite interval of time and the kinetic energy of the lost velocities in the same interval of time is equal to the sum of the scalar products of the impulses of the given external and internal forces which act upon this system, corresponding to the considered interval of time, by the velocities of the particles at the final moment.*

Analogously, we find (we perform a scalar product by the velocities $\mathbf{v}_i^{(1)}$ and sum for all the particles of the mechanical system $\mathscr{S}$)

$$\Delta T - T_0 = \sum_{i=1}^{n} \mathbf{v}_i^{(1)} \cdot \int_{t_1}^{t_2} \left(\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \right) \mathrm{d}t \tag{11.1.26'}$$

and state

Theorem 11.1.11' *The difference between the variation of the kinetic energy of a free discrete mechanical system in a finite interval of time and the kinetic energy of the lost velocities in the same interval of time is equal to the sum of the scalar products of the impulses of the given external and internal forces which act upon this system, corresponding to the considered interval of time, by the velocities of the particles at the initial moment.*

Summing the relations (11.1.26) and (11.1.26') and taking into account the relation (11.1.24"), we get

$$\Delta T = W_{\widehat{P_1 P_2}} + W_{\mathrm{int}\,\widehat{P_1 P_2}} = \frac{1}{2} \sum_{i=1}^{n} \left(\mathbf{v}_i^{(1)} + \mathbf{v}_i^{(2)} \right) \cdot \int_{t_1}^{t_2} \left(\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \right) \mathrm{d}t \,; \tag{11.1.27}$$

thus, we state

Theorem 11.1.12 (*Kelvin*). *The work effected by the given external and internal forces which act upon a free discrete mechanical system in a finite interval of time* (*the variation of the kinetic energy of the respective mechanical system*) *is equal to the sum of the scalar products of the impulses of the given external and internal forces which act upon this system, corresponding to the considered interval of time, by the semi-sum of the velocities of the particles at the initial and final moments.*

Subtracting the relation (11.1.26') from the relation (11.1.26), we may write

$$T_0 = \frac{1}{2}\sum_{i=1}^{n}\mathbf{v}_i^{(0)} \cdot \int_{t_1}^{t_2}\left(\mathbf{F}_i + \sum_{k=1}^{n}{}'\mathbf{F}_{ik}\right)\mathrm{d}t\,; \tag{11.1.27'}$$

there results

Theorem 11.1.12' (*analogue to Kelvin's theorem*). *The kinetic energy of the lost velocities of a free discrete mechanical system in a finite interval of time is equal to half of the sum of the scalar products of the impulses of the given external and internal forces, which act upon this system, corresponding to the considered interval of time, by the lost velocities, in the same interval of time.*

11.1.2.5 Conservation Theorems of Momentum. Applications

If the given external and internal forces which act upon a free discrete mechanical system $\mathscr{S}$ satisfy certain conditions, then the general theorems stated above lead to conservation theorems (hence, to first integrals of the system of differential equations of motion). Thus, if the resultant $\mathbf{R}$ of the given external forces is parallel to a fixed plane (is normal to a fixed direction of unit vector $\mathbf{u}$, with respect to the frame $\mathscr{R}$, or has a zero component, $\mathbf{R} \cdot \mathbf{u} = 0$), as in the case of a single particle (see Chap. 6, Sect. 1.2.5), then the theorem of momentum allows to write

$$\mathbf{H} \cdot \mathbf{u} = \mathbf{u} \cdot \sum_{i=1}^{n} m_i \mathbf{v}_i = u_j \sum_{i=1}^{n} H_j^{(i)} = u_j \sum_{i=1}^{n} m_i v_j^{(i)} = C\,, \quad C = \text{const}\,; \tag{11.1.28}$$

we obtain thus a scalar first integral. Hence, *if the resultant* $\mathbf{R}$ *of the given external forces is parallel to a fixed plane, then the projection of the momentum of the free discrete mechanical system* $\mathscr{S}$ *on the normal to this plane is conserved* (is constant) *in time*. Because

$$\mathbf{u} \cdot \sum_{i=1}^{n} m_i \mathbf{v}_i = \mathbf{u} \cdot \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{u} \cdot \sum_{i=1}^{n} m_i \mathbf{r}_i\right) = C\,,$$

it results

$$\mathbf{u} \cdot \sum_{i=1}^{n} m_i \mathbf{r}_i = u_j \sum_{i=1}^{n} m_i x_j^{(i)} = Ct + C'\,, \quad C, C' = \text{const}\,, \tag{11.1.28'}$$

obtaining a new scalar first integral, independent of the previous one; thus, the mentioned condition allows to build up two independent scalar first integrals.

Analogously, if *the resultant* $\mathbf{R}$ *has a fixed direction* (is normal to two non-parallel fixed directions with respect to the frame $\mathscr{R}$ or has two zero components), then we obtain four independent scalar first integrals of the form (11.1.28), (11.1.28'), while *the projection of the momentum of the free discrete mechanical system* $\mathscr{S}$ *on a plane normal to the resultant* $\mathbf{R}$ (determined by the two fixed directions) *is conserved in time*.

Finally, if the resultant $\mathbf{R}$ of the given external forces vanishes (is normal to three distinct fixed directions), then we may set up three independent scalar first integrals of the form (11.1.28). As a matter of fact, $\mathbf{R} = \mathbf{0}$ leads to $\dot{\mathbf{H}} = \mathbf{0}$ and to

$$\mathbf{H} = \sum_{i=1}^{n} m_i \mathbf{v}_i = \mathbf{C}\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,, \quad H_i = C_i\,, \quad i = 1,2,3\,, \tag{11.1.28''}$$

hence to a vector first integral, equivalent to three scalar first integrals; we state thus (we take into account (11.1.19) too)

Theorem 11.1.13 (*conservation theorem of momentum*). *The momentum* (*and the velocity of the centre of mass*) *of a free discrete mechanical system is conserved in time if and only if the resultant of the given external forces which act upon it vanishes*.

Starting from (11.1.19), we can write

$$M\boldsymbol{\rho} = \mathbf{C}t + \mathbf{C}'\,, \quad \mathbf{C},\mathbf{C}' = \overrightarrow{\mathrm{const}}\,, \quad M\rho_i = C_i t + C_i'\,, \quad i = 1,2,3\,; \tag{11.1.28'''}$$

thus, we get a new vector first integral, equivalent to three scalar first integrals, and we may state

Theorem 11.1.14 (*theorem of rectilinear and uniform motion of the centre of mass*). *The motion of the centre of mass of a free discrete mechanical system is rectilinear and uniform if and only if the resultant of the given external forces which act upon this system vanishes*.

Observing that the Theorem 11.1.14 is a consequence of the Theorem 11.1.13 (and inversely), it results that the conservation theorem of momentum allows to set up two vector first integrals or six scalar first integrals (the maximal number of independent scalar first integrals which can be obtained).

Returning to the Theorem 11.1.7 of general motion of the centre of mass of a free discrete mechanical system, we can mention many applications. Let thus be a discrete mechanical system $\mathscr{S}$ of free particles launched in vacuum and subjected only to the action of a uniform gravitational field; no matter the internal forces could be, the centre of mass C of the mechanical system $\mathscr{S}$ (considered as a particle acted upon by the resultant of the gravity forces) describes an arc of parabola of vertical axis. For instance, if a projectile launched in the vicinity of the Earth explodes at a certain moment, then the particles thus obtained (various parts of the projectile) are moving so that the instantaneous centre of mass of the system thus formed describes, further, the same parabola; indeed, the forces which are developed by explosion are internal ones and do not intervene in calculation. If, at a certain moment, intervene also other external forces (e.g., the collision of a part of the projectile with the ground or other bodies), then the trajectory of the centre of mass is modified.

We have seen that, in the absence of external forces, the centre of mass has a rectilinear and uniform motion; this result has been verified by many astronomical observations. Indeed, assuming that the solar system is isolated (it is not acted upon by external forces, which can be practically accepted, because the other stars and their planets are at great distances from the solar system, so that their influence may be neglected), its centre of mass (very close to the centre of mass of the Sun) has a rectilinear motion, with a velocity of $19.5\ \mathrm{km/s}$, towards a point called Apex, in the vicinity of the star Vega of the constellation Lyra.

If, in the above case, the initial velocity of the centre of mass is zero with respect to an inertial frame of reference $\mathscr{R}$, then the centre of mass is at rest with respect to this frame at any moment. Assuming that the theorem of motion of the centre of mass can be applied also in case of continuous mechanical systems (as it will be seen in Sect. 12.1.2.1), we will apply the above results to living matter too. The will of living beings puts in action their muscles; these actions are internal forces, which do not intervene in computation, so that a living being can change the rectilinear and uniform motion (or the rest) of its centre of mass only by acting upon some external mechanical systems (bodies). Thus, a man which stays on a perfect smooth horizontal plane cannot advance (cannot "go"); indeed, the external forces which act upon the human body are vertical, so that its centre of mass can move only along the vertical (there exist no horizontal components). *The going* becomes possible only because, in reality, the ground (plane surface) cannot be perfect smooth, a sliding friction taking, practically, place. In this case, the man, immobile at the beginning, raises a foot and advances one step; the other foot, in contact with the ground, tends to make a motion in an opposite direction, not to have a horizontal component of motion of the mass centre. At this moment appears an oblique reaction of the ground, with a horizontal component due to friction, directed towards forward; this reaction transported parallel to itself at the centre of mass, determines a forward motion.

Let us consider the case of a non-deformable mechanical system $\mathscr{S}$ of mass M, which has a motion of translation of velocity $\mathbf{v}$ with respect to a fixed frame of reference $\mathscr{R}$, and a particle P of mass m, which moves with a velocity $\mathbf{u}$ with respect to the mechanical system $\mathscr{S}$, hence with a velocity $\mathbf{v}' = \mathbf{v} + \mathbf{u}$ with respect to the frame $\mathscr{R}$; we assume that the resultant of the external forces which act upon the mechanical system $\mathscr{S}' = \mathscr{S} \cup \{P\}$ vanishes. The conservation theorem of momentum allows to write

$$M\mathbf{v} + m\mathbf{v}' = (M + m)\mathbf{v} + m\mathbf{u} = \mathbf{C}\,, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}\,; \tag{11.1.29}$$

if the mechanical system $\mathscr{S}'$ is at rest with respect to the frame $\mathscr{R}$ at the initial moment t_0 ($\mathbf{v}(t_0) = \mathbf{v}'(t_0) = \mathbf{0}$, hence $\mathbf{u}(t_0) = \mathbf{0}$), then we get $\mathbf{C} = \mathbf{0}$, so that

$$\mathbf{v} = -\frac{m}{M + m}\mathbf{u}\,. \tag{11.1.29'}$$

Hence, if the particle P begins to move with the velocity $\mathbf{u}$ with respect to the non-deformable system $\mathscr{S}$, then a velocity $\mathbf{v}$ of opposite direction is conveyed to the latter one; supposing that $m \ll M$ (hence $m/(M + m) \cong m/M$), it results $|\mathbf{v}| \ll |\mathbf{u}|$.

The motion being rectilinear, in this case, we can consider that it takes place along the Ox-axis, the mass centre of the mechanical system $\mathscr{S}$ (materialized by a rigid straight bar, Fig. 11.2) having the abscissa $x(t)$ and the velocity $v(t)$, while the particle P is of abscissa $x'(t)$ and velocity $v'(t) = v(t) + u(t)$ ($u(t)$ is the velocity with respect to the mechanical system $\mathscr{S}$); we may thus write

$$Mv + mv' = 0\,, \quad v = -\frac{m}{M+m}u\,. \tag{11.1.30}$$

Integrating with respect to time (which corresponds to the Theorem 11.1.14), it results

$$Mx + mx' = Mx_0 + mx_0' = (M+m)\xi = \text{const}\,, \tag{11.1.30'}$$

where $x_0 = x(t_0)$, $x_0' = x'(t_0)$, while $\xi(t) = \xi(t_0) = \text{const}$ is the abscissa of the mass centre C of the mechanical system $\mathscr{S}'$ (Fig. 11.2), which remains at rest with respect to the frame $\mathscr{R}$. Denoting by $\delta = x - x_0$, $\delta' = x' - x_0'$ the displacements of the mechanical system $\mathscr{S}$ and of the particle P, respectively, positive in the positive sense of the Ox-axis, we obtain the displacement of the mechanical system $\mathscr{S}$ as a function of the displacement of the particle P (of an opposite direction to the latter one)

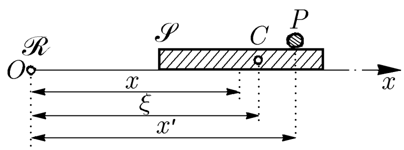

Fig. 11.2 Motion of translation of a mechanical system

$$\delta = -\frac{m}{M}\delta'\,; \tag{11.1.30''}$$

because $m \ll M$, it results $|\delta| \ll |\delta'|$ too. These results explain why the centre of mass of the system formed by a boat (on a non-running water) and by a man is not displacing, no matter the action of the man upon the boat (if the man moves in a direction, then the boat moves in the opposite one); if the boat is near to the border and the man advances in it towards the border, trying to go down, then the boat moves away from the border. Analogously, the repulsion of a gun at the moment of the discharge is thus explained; the velocity of this repulsion is given by the second formula (11.1.30) as a function of the initial velocity of the projectile (bullet), while the formula (11.1.30") gives the displacement back of the gun (of its mass centre) as a function of the distance travelled through by the projectile along its barrel (during the time in which the gun $\mathscr{S}$ and the projectile P form a mechanical system $\mathscr{S}'$).

11.1.2.6 Conservation Theorems of Moment of Momentum. Theorem of Areas. Applications

If the moment $\mathbf{M}_O$ *of the given external forces is contained in a fixed plane* (*is normal to a fixed axis* Δ, $O \in \Delta$, *of unit vector* $\mathbf{u}$, with respect to the frame of reference $\mathcal{R}$ or has a zero component, $\mathbf{M}_O \cdot \mathbf{u} = 0$), as in the case of a single particle (see Chap. 6, Sect. 1.2.5), *then* the theorem of moment of momentum leads to

$$\mathbf{K}_O \cdot \mathbf{u} = \sum_{i=1}^{n} (\mathbf{r}_i, m_i \mathbf{v}_i, \mathbf{u}) = K_{Oj} u_j = \in_{jkl} \sum_{i=1}^{n} m_i x_j^{(i)} v_k^{(i)} u_l = C, \quad C = \text{const}, \tag{11.1.31}$$

obtaining thus a scalar first integral of the equations of motion; in this case, *the projection of the moment of momentum* $\mathbf{K}_O$ *on the axis* Δ *is conserved in time*. If, in particular, the axis Δ coincides with the Ox-axis, then

$$\sum_{i=1}^{n} m_i \left(x_1^{(i)} v_2^{(i)} - x_2^{(i)} v_1^{(i)} \right) = \sum_{i=1}^{n} m_i \left(x_1^{(i)} \dot{x}_2^{(i)} - x_2^{(i)} \dot{x}_1^{(i)} \right) = C, \quad C = \text{const}. \tag{11.1.31'}$$

As well, *if the moment* $\mathbf{M}_O$ *has a fixed support* (is normal to two non-parallel fixed axes Δ_1 and Δ_2 with respect to the frame $\mathcal{R}$ or has two zero components), *then* we get two independent scalar first integrals of the form (11.1.31), while *the projection of the moment of momentum of the discrete mechanical system* $\mathcal{S}$ *on a plane normal to the moment* $\mathbf{M}_O$ (determined by the two fixed directions) *is conserved in time*.

If $\mathbf{M}_O = \mathbf{0}$ (the moment $\mathbf{M}_O$ is normal to three distinct directions), then we can set up three independent scalar first integrals of the form (11.1.31). Besides, the relation $\mathbf{M}_O = \mathbf{0}$ leads to $\dot{\mathbf{K}}_O = \mathbf{0}$ and to

$$\begin{gathered} \mathbf{K}_O = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_i) = \mathbf{C}, \quad \mathbf{C} = \overrightarrow{\text{const}}, \\ K_{Oj} = \in_{jkl} \sum_{i=1}^{n} m_i x_k^{(i)} v_l^{(i)} = C_j, \quad j = 1,2,3, \end{gathered} \tag{11.1.31''}$$

hence to a vector first integral, equivalent to three scalar first integrals; we may thus state

Theorem 11.1.15 (*conservation theorem of moment of momentum*). *The moment of momentum of a free discrete mechanical system, with respect to a fixed pole, is conserved in time if and only if the resultant moment of the given external forces which act upon this system, with respect to the same pole, vanishes.*

The two conservation theorems (of momentum and of moment of momentum) are independent each other and give nine independent scalar first integrals. They can be grouped together in the form (we assume that $\tau_O \{\mathbf{F}_i\} = \mathbf{0}$)

$$\tau_O \{ \mathbf{H}_i \} = \overrightarrow{\text{const}} \tag{11.1.31'''}$$

and we can state (if the torsor of a system of bound vectors vanishes with respect to a pole, then it vanishes with respect to any other pole)

Theorem 11.1.16 (*conservation theorem of torsor*). *The torsor of the momenta of a free discrete mechanical system, with respect to a fixed pole, is conserved in time if and only if the torsor of the given external forces which act upon this system, with respect to the same pole, vanishes.*

Projecting the particles P_i, $i = 1,2,...,n$, of the free discrete mechanical system $\mathscr{S}$ on a fixed plane (e.g., the plane Ox_1x_2) and supplying the projections P_i' with the same masses m_i (or assuming that we have to do with a plane mechanical system), we can write, in polar co-ordinates ($x_1^{(i)} = r_i \cos\theta_i$, $x_2^{(i)} = r_i \sin\theta_i$),

$$K_{O3} = \sum_{i=1}^{n} m_i \left(x_1^{(i)} \dot{x}_2^{(i)} - x_2^{(i)} \dot{x}_1^{(i)} \right) = \sum_{i=1}^{n} m_i r_i^2 \dot{\theta}_i = 2 \sum_{i=1}^{n} m_i \Omega_{O3}^{(i)} .$$

We thus find again the second formula (11.1.3) for $j = 3$. If $M_{O3} = 0$, then the formula (11.1.31') leads to

$$2 \sum_{i=1}^{n} m_i \Omega_{O3}^{(i)} = \sum_{i=1}^{n} m_i r_i^2 \dot{\theta}_i = C, \quad C = \text{const}, \tag{11.1.32}$$

and we may state

Theorem 11.1.17 (*theorem of areal velocities; plane case*). *The sum of the products of the double masses of the particles of a free discrete mechanical system by the areal velocities of their projections on a fixed plane, with respect to a fixed pole in this plane, is conserved in time if and only if the resultant moment of the given external forces which act upon this system, with respect to an axis normal to the considered plane at the same pole, vanishes.*

This theorem can be applied, e.g., if the supports of all the given external forces pierce a fixed straight line. Observing that $\Omega_{O3}^{(i)} = \mathrm{d}\mathscr{A}_{O3}^{(i)} / \mathrm{d}t$, where we supplied the area by the sign +, corresponding to a positive rotation in the considered fixed plane (see also Fig. 6.5), we get (integrating from the initial moment t_0)

$$2 \sum_{i=1}^{n} m_i \mathscr{A}_{O3}^{(i)} = C(t - t_0), \quad C = \text{const}, \tag{11.1.32'}$$

and are led to

Theorem 11.1.18 (*theorem of areas; plane case; L. Euler, D. Bernoulli, d'Arcy*). *The sum of the products of the double masses of the particles of a free discrete mechanical system by the areas described by the radii vectores of their projections on a fixed plane, with respect to a fixed pole in this plane, starting from their initial positions, is in direct proportion to the interval of time travelled through if and only if the resultant moment of the given external forces which act upon this system, with respect to an axis normal to the considered plane at the same pole, vanishes.*

We notice that the Theorems 11.1.17 and 11.1.18 are equivalent.

If the angular velocity is the same for all particles ($\dot{\theta}_i = \omega$, $i = 1,2,\ldots,n$), then the relation (11.1.32) reads

$$I_{33}\omega = C\,, \tag{11.1.32''}$$

where $I_{33} = I_{x_3}$ is the moment of inertia of the mechanical system $\mathscr{S}$ with respect to the Ox_3-axis, defined by the relation (3.1.21'). This relation takes place, e.g., in case of a non-deformable mechanical system $\mathscr{S}$ in motion of rotation with respect to a fixed axis. The constant C is called the *constant of areas*.

In particular, if at a moment $t = t_0$ the discrete mechanical system $\mathscr{S}$ is at rest with respect to a fixed (inertial) frame of reference $\mathscr{R}$ ($\dot{\theta}_i(t_0) = 0$ and $\Omega_{O3}^{(i)}(t_0) = 0$), hence if the mechanical system $\mathscr{S}$ begins to move starting from a position of rest, then the constant of areas vanishes ($C = 0$); thus, it results

$$2\sum_{i=1}^{n} m_i \Omega_{O3}^{(i)} = \sum_{i=1}^{n} m_i r_i^2 \dot{\theta}_i = 0\,, \quad \sum_{i=1}^{n} m_i \mathscr{A}_{O3}^{(i)} = 0\,, \quad I_{33}\omega = 0\,. \tag{11.1.32'''}$$

Hence, if some particles have an angular velocity in one sense (e.g., $\dot{\theta}_i > 0$), then for the other particles the velocity must be of opposite sense ($\dot{\theta}_i < 0$); one can make this observation for the areas $\mathscr{A}_{O3}^{(i)}$ too. From the third relation (11.1.32''') one can see that $\omega = 0$; hence, the mechanical system $\mathscr{S}$ cannot rotate around the fixed axis if it is at rest with respect to the latter one at the initial moment. For instance, a man (in general, a living being) in a vertical position on a perfect smooth ground cannot rotate about a vertical axis passing through its centre of mass (he is acted upon only by his own weight and by the normal constraint force of the ground, the resultant moment of the external forces with respect to the centre of mass vanishing). The rotation takes place only if a pivoting friction between man and ground intervene too (as in case of walking) or by various complicated motions made by the man, as we will see later.

The above results have many applications. Thus, we will consider a non-deformable mechanical system $\mathscr{S}$, which has a moment of inertia $I_{\mathscr{S}}$ with respect to a fixed axis in the inertial frame of reference $\mathscr{R}$ and rotates with an angular velocity $\boldsymbol{\omega}_{\mathscr{S}}$ about this axis, and a particle P (or another non-deformable mechanical system $\bar{\mathscr{S}}$), which has a moment of inertia I with respect to the same fixed axis and which rotates with an angular velocity $\boldsymbol{\omega}$ with respect to the system $\mathscr{S}$, hence with an angular velocity $\boldsymbol{\omega}' = \boldsymbol{\omega}_{\mathscr{S}} + \boldsymbol{\omega}$ with respect to the frame $\mathscr{R}$; we assume that the resultant moment of the external forces which act upon the mechanical system $\mathscr{S}' = \mathscr{S} \cup \{P\}$ (or $\mathscr{S}' = \mathscr{S} \cup \bar{\mathscr{S}}$) vanishes. The conservation theorem of moment of momentum allows to write

$$I_{\mathscr{S}}\boldsymbol{\omega}_{\mathscr{S}} + I\boldsymbol{\omega}' = (I_{\mathscr{S}} + I)\boldsymbol{\omega}_{\mathscr{S}} + I\boldsymbol{\omega} = \mathbf{C}\,, \quad \mathbf{C} = \overrightarrow{\text{const}}\,; \tag{11.1.33}$$

if the mechanical system $\mathscr{S}'$ is at rest with respect to the frame $\mathscr{R}$ at the initial moment t_0 ($\boldsymbol{\omega}_{\mathscr{S}}(t_0) = \boldsymbol{\omega}'(t_0) = \mathbf{0}$, hence $\boldsymbol{\omega}(t_0) = \mathbf{0}$), then we get $\mathbf{C} = \mathbf{0}$, so that

$$\boldsymbol{\omega}_{\mathscr{S}} = -\frac{I}{I_{\mathscr{S}} + I}\boldsymbol{\omega}. \tag{11.1.33'}$$

Hence, if the particle P begins to rotate with the angular velocity $\boldsymbol{\omega}$ with respect to the non-deformable system $\mathscr{S}$, the latter one also moves with an angular velocity $\boldsymbol{\omega}_{\mathscr{S}}$ of opposite sense (Fig. 11.3); supposing that $I \ll I_{\mathscr{S}}$ (hence $I/(I_{\mathscr{S}} + I) \cong I/I_{\mathscr{S}}$), it results $|\boldsymbol{\omega}_{\mathscr{S}}| \ll |\boldsymbol{\omega}|$. We may assume that the rotations take place about the same Ox_3 - axis, which passes through the pole O, the angular velocities being specified by the scalar magnitudes $\omega_{\mathscr{S}}(t)$ and $\omega(t)$, respectively (hence, $\omega'(t) = \omega_{\mathscr{S}}(t) + \omega(t)$ too); it results

$$I_{\mathscr{S}}\omega_{\mathscr{S}} + I\omega' = 0\,, \quad \omega_{\mathscr{S}} = -\frac{I}{I_{\mathscr{S}} + I}\omega\,. \tag{11.1.34}$$

Integrating with respect to time, we obtain

$$I_{\mathscr{S}}\theta_{\mathscr{S}} + I\theta' = I_{\mathscr{S}}\theta_{\mathscr{S}}^0 + I\theta_0' = \text{const}\,, \tag{11.1.34'}$$

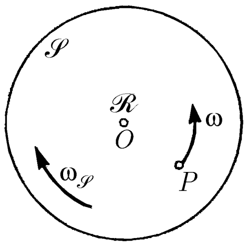

Fig. 11.3 Motion of rotation of a mechanical system

where $\theta_{\mathscr{S}}^0 = \theta_{\mathscr{S}}(t_0)$, $\theta_0' = \theta'(t_0)$. Denoting by $\vartheta = \theta_{\mathscr{S}} - \theta_{\mathscr{S}}^0$, $\vartheta' = \theta' - \theta_0'$ the rotation of the mechanical system $\mathscr{S}$ and the rotation of the particle P, respectively, considered to be positive for the positive sense of rotation, we get the angle of rotation of the mechanical system $\mathscr{S}$ as a function of the angle of rotation of the particle P (of an opposite sense with respect to the latter one)

$$\vartheta_{\mathscr{S}} = -\frac{I}{I_{\mathscr{S}}}\vartheta'; \tag{11.1.34''}$$

if $I \ll I_{\mathscr{S}}$, then it results $|\vartheta_{\mathscr{S}}| \ll |\vartheta'|$ too. These results are analogous to those obtained at the precedent subsection for a motion of translation, in case of a similar problem.

Returning to the rotation of a man about a vertical axis on a perfect smooth ground, case considered above, we notice that to any tendency of rotation of the upper part of his body (determined by internal forces) corresponds a tendency of rotation of the lower part of his body in the opposite sense. If the man would hold up his hands, with the fists in symmetrical positions with respect to the axis, and would effect by each fist a motion of rotation in the same sense, in a horizontal plane, the symmetry with respect to the axis being preserved, then his body would rotate in an opposite sense. As well, as it was shown by Saint-Germain, if the man would rotate several loads hanged simultaneously on a belly-band (by the action of internal forces) in the same sense, then his body would rotate in an opposite sense. Let us consider also the case of a circular disc, in a horizontal plane, which can rotate without friction about a vertical axle which passes through its centre (*Prandtl's disc*); we have seen that a particle which describes a circle with the centre on the vertical axis induces a rotation of the disc in the opposite sense (Fig. 11.3). If a man is in a vertical position on the disc, his centre of mass being on its axis, and keeps in his hands a wheel, the axle of which is along the same vertical axis, then any rotation of the wheel (provoked by an external cause) leads to a rotation of the system (formed by the man and the disc) in an opposite sense.

If the constant of areas is non-zero ($\mathbf{C} \neq \mathbf{0}$), corresponding to the formula (11.1.33), then one can obtain an angular velocity non-parallel to $\mathbf{C}$ only if the angular velocity $\boldsymbol{\omega}$ has a direction different from that of $\mathbf{C}$; returning to Prandtl's disc, the axle of the wheel must be inclined with respect to the vertical. As well, the formula (11.1.32") shows that the angular velocity ω is in inverse proportion to the axial moment of inertia I_{33}; for instance, a ballerina (or a skater) rotates with a greater angular velocity if she (he) has the hands pressed to the body (the axial moment of inertia is smaller) or with a smaller angular velocity if she (he) stretches the hands from the body (the axial moment of inertia is greater).

Analogously, a cosmic vehicle may change its direction of motion only if it is in a gravitational field. The manoeuvres effected in the vehicle are, in fact, actions of internal forces; an intervention of external forces, to give the vehicle the possibility of rotation, is necessary.

If the Theorem 11.1.15 takes place, then we can write conservation relations of the form (11.1.32) for the three planes of co-ordinates; we obtain thus

$$2\sum_{i=1}^{n} m_i \boldsymbol{\Omega}_O^{(i)} = \mathbf{C}, \quad \mathbf{C} = \overrightarrow{\mathrm{const}}, \tag{11.1.35}$$

and may state

Theorem 11.1.17' (*theorem of areal velocities; space case*). *The sum of the products of the double masses of the particles of a free discrete mechanical system by their areal velocities, with respect to a fixed pole, is conserved in time if and only if the resultant moment of the given external forces which act upon this system, with respect to the same pole, vanishes.*

In this case, the constant moment of momentum $\mathbf{K}_O$ is equal to the constant of areas $\mathbf{C}$ ($\mathbf{K}_O = \mathbf{C}$). We can write a theorem of areal velocities in the plane case for any plane passing through O and of normal of unit vector $\mathbf{k}$, the corresponding constant of areas being $C_k = \mathbf{C} \cdot \mathbf{k}$. In case of a plane Π normal to $\mathbf{K}_O$, the constant of areas will take its maximal value (equal in modulus to $|\mathbf{C}|$); the respective plane is called *the plane of the maximum of areas*. If $\mathbf{k} \perp \mathbf{C}$, hence if the moment of momentum $\mathbf{K}_O$ is contained in the plane Π, then the constant of areas vanishes. By integration in the plane Π, starting from the initial moment t_0, we can write

$$2\sum_{i=1}^{n} m_i \mathscr{A}_{Ok}^{(i)} = C_k \left(t - t_0 \right), \tag{11.1.35'}$$

where C_k is the corresponding scalar constant of areas; we can state

Theorem 11.1.18' (*theorem of areas; space case*). *The sum of the products of the double masses of the particles of a free discrete mechanical system by the areas described by the radii vectores of their projections on any fixed plane passing through a fixed pole, with respect to that pole, starting from their initial positions is in direct proportion to the interval of time travelled through if and only if the resultant moment of the given external forces which act upon this system, with respect to the same pole, vanishes.*

11.1.2.7 Conservation Theorems of Mechanical Energy. Applications

In the theorem of kinetic energy, expressed by the formula (11.1.24), $\mathrm{d}T$ is a differential, but $\mathrm{d}W$ and $\mathrm{d}W_{\mathrm{int}}$ are not, in general, exact differentials; if the sum of the elementary works is an exact differential

$$\mathrm{d}W + \mathrm{d}W_{\mathrm{int}} = \mathrm{d}\Phi, \quad \Phi = \Phi(\mathbf{r}_i, \dot{\mathbf{r}}_i; t), \tag{11.1.36}$$

then, by integration, we can obtain a first integral of energy in the form

$$T = \Phi + h, \quad h = \mathrm{const}. \tag{11.1.36'}$$

A first important case to be considered is that in which the given internal forces are conservative ones, deriving from a simple potential ($\mathrm{d}W_{\mathrm{int}} = \mathrm{d}U$) or from a generalized potential ($\mathrm{d}W_{\mathrm{int}} = \mathrm{d}U_0$); we say also that the mechanical system $\mathscr{S}$ is, in this case, a natural mechanical system. Going from the expression (3.2.4) of the internal elementary work, we notice that, in the particular case in which the internal forces are of the form $F_{ik} = F_{ik}(r_{ik})$, we obtain a simple potential (the prime to the sign Σ indicates a summation for $k \neq i$)

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}{}' \int F_{ik} \mathrm{d}r_{ik} + \mathrm{const}, \tag{11.1.37}$$

where r_{ik} is the distance between two particles P_i and P_k of the discrete mechanical system $\mathscr{S}$. For instance, in case of internal forces of attraction, in direct proportion to the distance, we read

$$\mathbf{F}_{ik} = -\mathbf{F}_{ki} = k_{ik}\mathbf{r}_{ik} = k_{ik}(\mathbf{r}_k - \mathbf{r}_i), \quad F_{ik} = -k_{ik}r_{ik},$$
$$U = -\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}{}' k_{ik}r_{ik}^2 > 0, \quad k_{ik} > 0, \quad i,k = 1,2,\ldots,n. \tag{11.1.37'}$$

Passing to the space E_{3n} of co-ordinates X_k, specified by the relation (11.1.10), we may write

$$\mathrm{d}W_{\mathrm{int}} = \sum_{i=1}^{n}\sum_{k=1}^{n}{}'\mathbf{F}_{ik}\cdot\mathrm{d}\mathbf{r}_i = \sum_{k=1}^{3n}Q_k\mathrm{d}X_k, \quad Q_k = Q_k(X_l, \dot{X}_l), \tag{11.1.38}$$

so that, in case of a simple potential U, it results

$$Q_k = \frac{\partial U}{\partial X_k}, \quad U = U(X_l), \quad k = 1,2,\ldots,3n, \tag{11.1.38'}$$

while, in case of a generalized potential, we get

$$Q_k = [U]_k = \frac{\partial U}{\partial X_k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{X}_k}\right), \quad U = U(X_l, \dot{X}_l), \quad k = 1,2,\ldots,3n. \tag{11.1.38''}$$

As we have seen in Chap. 1, Sect. 1.1.12 and in Chap. 6, Sect. 1.1.2, the generalized potential can be only of the form

$$U = \sum_{k=1}^{3n}U_k\dot{X}_k + U_0, \quad U_k = U_k(X_l), \quad U_0 = U_0(X_l), \tag{11.1.39}$$

so that it results

$$Q_k = \sum_{l=1}^{3n}\left(\frac{\partial U_l}{\partial X_k} - \frac{\partial U_k}{\partial X_l}\right)\dot{X}_l + \frac{\partial U_0}{\partial X_k} \tag{11.1.39'}$$

and then

$$\mathrm{d}W_{\mathrm{int}} = \sum_{k=1}^{3n}Q_k\mathrm{d}X_k = \mathrm{d}U_0. \tag{11.1.39''}$$

Hence, in case of conservative internal forces we may write the relation

$$\mathrm{d}T = \mathrm{d}U + \mathrm{d}W \tag{11.1.40}$$

or the relation

$$\mathrm{d}T = \mathrm{d}U_0 + \mathrm{d}W; \tag{11.1.40'}$$

introducing *the potential energy* $V = -U$, in case of a *simple potential*, or the potential energy $V = -U_0$, in case of a *generalized potential*, we can write

$$W_{\text{int}\,\widehat{P_1P_2}} = V_1 - V_2 = -\Delta V\,, \quad V = -W_{\text{int}\,\widehat{P_0P}} + V_0\,. \tag{11.1.41}$$

It results that the potential energy of a free discrete mechanical system acted upon by conservative internal forces (*natural mechanical system*) is equal to the internal work with changed sign, effected by the internal forces, starting from the initial moment (excepting an arbitrary constant V_0, which represents the potential energy at the initial moment; frequently, one chooses V_0 so as a minimum of the potential energy be equal to zero).

Denoting *the mechanical energy* by $E = T + V$, we get

$$\mathrm{d}E = \mathrm{d}(T + V) = \mathrm{d}W, \tag{11.1.42}$$

stating thus

Theorem 11.1.19 (*theorem of mechanical energy; Helmholtz*). *The differential of the mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces* (*natural mechanical system*), *is equal to the elementary work of the given external forces which act upon this system.*

Dividing the relation (11.1.42) by $\mathrm{d}t$ and taking into account (11.1.7), we have

$$\dot{E} = \frac{\mathrm{d}E}{\mathrm{d}t} = P \tag{11.1.42'}$$

and may state

Theorem 11.1.19' (*theorem of mechanical energy; second form*). *The derivative with respect to time of the mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces* (*natural mechanical system*), *is equal to the power of the given external forces which act upon this system.*

In general, the elementary work of the given external forces is not an exact differential, so that we can write

$$\Delta E = E(t_2) - E(t_1) = E_2 - E_1 = W_{\widehat{P_1P_2}} = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{F}_i \cdot \mathbf{v}_i \mathrm{d}t = \int_{t_1}^{t_2} P\mathrm{d}t\,, \tag{11.1.42''}$$

stating thus

Theorem 11.1.19'' (*theorem of mechanical energy; finite form*). *The variation of the mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces* (*natural mechanical system*), *in a finite interval of time, is equal to the work effected by the given external forces which act upon this system in the considered interval of time.*

Applying the principle of action and reaction to the external forces $\mathbf{F}_i$ correspond the forces $\mathbf{f}_i = -\mathbf{F}_i$, which represent the actions of the considered discrete mechanical system upon some external mechanical systems. The relation (11.1.42'') allows to write

$$E_1 - E_2 = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{f}_i \cdot \mathbf{v}_i \mathrm{d}t\,, \tag{11.1.42'''}$$

so that we can state

Theorem 11.1.19''' (*theorem of mechanical energy; second finite form*). *The lost mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces* (*natural mechanical system*), *in a finite interval of time, is equal to the work effected by this system on the external mechanical systems, in the same interval of time.*

We can say that this represents an *exit* for the considered mechanical system.

We may write the relation (11.1.42") also in the form

$$E = W_{\widehat{P_0 P}} + E_0 \,; \tag{11.1.42$^{\text{iv}}$}$$

hence, the mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces (natural mechanical system), is equal to the external work effected by the given external forces, starting with the initial moment (excepting an arbitrary constant E_0, which represents the mechanical energy at the initial moment, as in the case of the potential energy; as well, we can choose E_0 so that a minimum of the potential energy be equal to zero).

If $\mathrm{d}W = 0$ for a certain interval of time, then the relation (11.1.42) leads to

$$E = T + V = h \,, \quad h = T_0 + V_0 = \text{const} \,, \tag{11.1.43}$$

and we can state

Theorem 11.1.20 (*conservation theorem of mechanical energy*). *The mechanical energy of a free discrete mechanical system, acted upon by conservative internal forces* (*natural mechanical system*), *is conserved in a certain interval of time if and only if the elementary work of the given external forces which act upon this system vanishes in the same interval of time.*

In particular, this theorem can be applied to a closed (isolated) mechanical system, for which intervene only internal forces. By applying the conservation theorem of mechanical energy, a part of the kinetic energy is transformed in potential energy, if T decreases, or a part of the potential energy is transformed in kinetic energy, if T increases; hence, the mechanical energy is conserved as a whole, but it is in continuous transformation. For instance, for a heavy particle P of mass m, which falls on the Earth surface from the height h, without initial velocity, along the Ox-axis (situated along the ascendent vertical), we can write

$$\frac{1}{2} m v^2 + m g x = m g h \,; \tag{11.1.44}$$

if x varies from h to 0, then v increases from 0 to $v_{\max}$ (the kinetic energy increases from 0 to $T_{\max} = mgh$), while the potential energy decreases from $V_{\max} = mgh$ to 0 (excepting an arbitrary constant). The influence of the motion of the Earth has been neglected in this formula. The above results remain valid also for the continuous mechanical systems, as we will see in the next chapter.

Let thus be an elastic blade AB, built-in at A, at rest (of zero potential and zero kinetic energy, hence zero mechanical energy too) in a vertical position (Fig. 11.4); we neglect the weight of the blade, the resistance of the air as well as other external forces, and we assume that the internal forces are conservative. We apply an external force at the free end B, which becomes a new position B'; one obtains thus a mechanical energy equal to the external work W (one uses the theorem of mechanical energy). At this position, the kinetic energy vanishes, while the potential energy is equal to W. If the action of the external force ceases, then the elastic blade tends to come back at the stable position of equilibrium AB; the potential energy decreases till it vanishes, while the kinetic energy increases till its maximal value at the position AB (the conservation theorem of mechanical energy is applied). Due to the velocities of various points of the blade, that one passes beyond the position of equilibrium, in a new position AB'', symmetric to the position AB' with respect to AB; the potential energy has of new a maximal value, while the kinetic energy vanishes. Thus, by transformations of energy, the blade oscillates about the stable position of equilibrium AB.

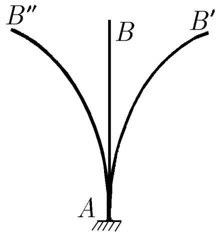

Fig. 11.4 Application of the conservation theorem of mechanical energy

If, in general, we assume that the external forces are also conservative (deriving from a simple or a generalized potential), we can write

$$\mathrm{d}W = \mathrm{d}U_{\mathrm{ext}}; \tag{11.1.45}$$

introducing an *external potential energy* $\overline{V} = -U_{\mathrm{ext}}$ (unlike V, which is called *internal potential energy* too), the relation (11.1.42) leads to a conservation theorem of energy of the form

$$T + V + \overline{V} = h, \quad h = \mathrm{const}. \tag{11.1.46}$$

If we transmit to a mechanical system a certain mechanical energy (as in the case of the elastic blade) and then we leave it free, the motion continues without loss of mechanical energy; only transformations of kinetic energy in potential energy and vice versa take place. We obtain thus a *mechanical perpetuum mobile*. But the experiment shows that one cannot realize apparatuses corresponding to such a phenomenon (e.g., the elastic blade considered above cannot reach practically a position AB'', symmetric to AB', but reaches a position closer to that of stable equilibrium AB; after a certain number of oscillations, the blade remains at rest). The consumption of energy due to which it is impossible to increase the sum $T + V$ (leading to a useful work) or, at least, to let it remain constant (realizing a *perpetuum mobile of the first species*) can be explained by the apparition of certain internal forces of friction and of certain external resistant forces (e.g., the resistance of the air). Hence, not all internal forces are conservative; *non-conservative internal forces* (e.g., forces of friction, which have a sense opposite to that of the displacements, leading to a negative internal work) can be put into evidence too. If the discrete mechanical system $\mathscr{S}$ is *adiabatically isolated* (has changes of energy with the exterior only if upon it is effected an external work), we are led to the notion of internal energy, E_{int}, which results just from the work of non-conservative internal forces and can depend on temperature, mass, volume etc. If upon an adiabatically isolated discrete mechanical system no external work is exerted, then we may write

$$\mathrm{d}(T + V) + \mathrm{d}E_{\text{int}} = 0\,; \tag{11.1.47}$$

but if an external work is exerted too, then we can write, in general,

$$\mathrm{d}(T + V) + \mathrm{d}E_{\text{int}} = \mathrm{d}W\,. \tag{11.1.47'}$$

An *adiabatically non-isolated* discrete mechanical system may receive energy from the exterior, even if upon it no external work is exerted; but a flux of heat $\mathrm{d}Q$ can intervene. We may thus write

$$\mathrm{d}(T + V) + \mathrm{d}E_{\text{int}} = \mathrm{d}W + \mathrm{d}Q\,. \tag{11.1.47''}$$

Introducing also an elementary work of non-mechanical and non-caloric nature $\mathrm{d}\overline{W}$ (e.g., of electromagnetic nature), we can write (after R.J. Mayer and J. Joule)

$$\mathrm{d}(T + V) + \mathrm{d}E_{\text{int}} = \mathrm{d}W + \mathrm{d}Q + \mathrm{d}\overline{W}\,; \tag{11.1.47'''}$$

we obtain thus *the first principle of thermodynamics*. If $\mathrm{d}E_{\text{int}} = 0$ and the sum $\mathrm{d}W + \mathrm{d}Q + \mathrm{d}\overline{W}$ is an exact differential, we may state, after H. von Helmholtz, a conservation theorem of energy of the form (11.1.46). One could thus say that it is possible to realize a perpetuum mobile, where intervene transformations of kinetic, potential and caloric energy or of other energies; but the experiment shows that the caloric energy generated by frictions can be transformed in mechanical energy only by loss of energy, the process being irreversible (one passes from mechanical energy to a caloric one only by degradation).

11.1.2.8 Problem of n Particles

Let be an isolated (closed) free discrete mechanical system $\mathscr{S}$, formed by n particles P_i of masses m_i and position vectors $\mathbf{r}_i$, subjected only to the action of some internal forces $\mathbf{F}_{ik} = -\mathbf{F}_{ki}$, $i \neq k$, $i,k = 1,2,...,n$; if these forces are of Newtonian attraction (or of Coulombian nature, of attraction or repulsion), then the corresponding problem is known as *the problem of n particles*. At a cosmic level, the celestial bodies can be modelled, in a first approximation, as particles (e.g., the solar system, formed by the Sun, the planets and their satellites, acted upon by other celestial bodies; their actions are negligible external forces, due to the great distances at which these bodies are, as well as due to their quasi-spherical distribution, their effects annulling each other). The respective problem is called also *the problem of n bodies* and is the basic problem in *celestial mechanics*. At the atomic level, one has the same problem for the system formed by nucleus and electrons.

The equations of motion read

$$m_i \ddot{\mathbf{r}}_i = \sum_{k=1}^{n}{}' \mathbf{F}_{ik}, \quad \mathbf{F}_{ik} = f \frac{m_i m_k}{r_{ik}^3} \mathbf{r}_{ik}, \quad \mathbf{r}_{ik} = \mathbf{r}_k - \mathbf{r}_i, \quad i,k = 1,2,...,n, \tag{11.1.48}$$

and lead to the vector first integrals

$$\mathbf{H} = \sum_{i=1}^{n} m_i \mathbf{v}_i = \mathbf{C}, \quad M\boldsymbol{\rho} = \sum_{i=1}^{n} m_i \mathbf{r}_i = \mathbf{C}t + \mathbf{C}', \quad \mathbf{C}, \mathbf{C}' = \overrightarrow{\mathrm{const}}, \tag{11.1.49}$$

which correspond to the conservation theorems of momentum and of rectilinear and uniform motion of the centre of mass, respectively, to the vector first integral

$$\mathbf{K}_O = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = 2\sum_{i=1}^{n} m_i \boldsymbol{\Omega}_i = \bar{\mathbf{C}}, \quad \bar{\mathbf{C}} = \overrightarrow{\mathrm{const}}, \tag{11.1.49'}$$

which corresponds to the conservation theorem of moment of momentum, and to the scalar first integral

$$T - U = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 - f\sum_{i=1}^{n}\sum_{k=1}^{n}{}' \frac{m_i m_k}{r_{ik}} = h, \quad h = \mathrm{const}, \tag{11.1.49''}$$

which corresponds to the conservation theorem of mechanical energy, where $\mathbf{r}_{ik} = \overrightarrow{P_i P_k}$, while f is the universal constant of attraction (we took into account (1.1.84) and (3.2.6') for the forces of Newtonian attraction). We obtain thus ten scalar first integrals for the $3n$ scalar differential equations of motion of second order. In 1887, H. Burns showed that these first integrals are the only algebraic first integrals which can be obtained; as well, in 1889, H. Poincaré stated that, besides these first integrals, one cannot obtain other uniform and analytic first integrals, while Painlevé showed that there does not exist other first integrals algebraic only with respect to the components of the velocity vectors (neither for larger conditions). But, taking into account that time does not appear explicitly in those first integrals, they are equivalent to 11 scalar first integrals; moreover, observing that the internal forces depend only on the reciprocal distances, one can obtain one more first integral. But there are necessary $6n$ scalar first integrals to solve completely the problem, so that these 12 scalar first integrals are sufficient only for $n = 2$.

We may write the n vector equations or the $3n$ scalar equations of motion, respectively, in the form

$$m_i \ddot{\mathbf{r}}_i = \nabla_i U \,, \quad \nabla_i = \mathbf{i}_j \frac{\partial}{\partial x_j^{(i)}} \,, \quad \ddot{x}_j^{(i)} = \frac{\partial U}{\partial x_j^{(i)}} \,, \quad i = 1,2,\ldots,n \,, \quad j = 1,2,3 \,. \tag{11.1.48'}$$

The centre of mass C of an isolated discrete mechanical system $\mathscr{S}$ (e.g., the solar system) has a uniform and rectilinear motion, so that remains to be studied its motion of rotation about that centre.

Starting from the equations (11.1.48'), we get

$$\sum_{i=1}^{n} m_i x_j^{(i)} \ddot{x}_j^{(i)} = \sum_{i=1}^{n} x_j^{(i)} \frac{\partial U}{\partial x_j^{(i)}} = -U \,,$$

where we used Euler’s formula for a function U, homogeneous of degree -1. Taking into account the first integral (11.1.49") too, we get

$$\sum_{i=1}^{n} m_i \left(x_j^{(i)} \ddot{x}_j^{(i)} + \dot{x}_j^{(i)} \dot{x}_j^{(i)} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} m_i x_j^{(i)} \dot{x}_j^{(i)} = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{i=1}^{n} m_i x_j^{(i)} x_j^{(i)} = U + 2h \,;$$

one obtains thus *the Lagrange-Jacobi formula* (obtained by Lagrange in 1772 for $n = 2$ and generalized by Jacobi in 1842 for any n)

$$\ddot{I}_O = 2(U + 2h) \,, \tag{11.1.50}$$

where I_O is the polar moment of inertia of the discrete mechanical system $\mathscr{S}$ with respect to the origin O.

Starting from the algebraic identity

$$2 \sum_{k=1}^{n} m_k \sum_{i=1}^{n} m_i \left(x_j^{(i)} \right)^2 = 2 \left(\sum_{i=1}^{n} m_i x_j^{(i)} \right)^2 + \sum_{k=1}^{n} \sum_{i=1}^{n}{}' m_k m_i \left(x_j^{(k)} - x_j^{(i)} \right)^2 \,, \quad j = 1,2,3 \,,$$

which can be easily verified, summing for j from 1 to 3, introducing the mass M and the moment of inertia I_O of the mechanical system $\mathscr{S}$ and taking into account the first integral (11.1.28'''), it results

$$MI_O = (\mathbf{C}t + \mathbf{C}')^2 + R^2 \,, \quad 2R^2 = \sum_{k=1}^{n} \sum_{i=1}^{n}{}' m_k m_i r_{ik}^2 \,; \tag{11.1.50'}$$

introducing in (11.1.50), we can write also the formula

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} R^2 = 2M\left(U + 2\bar{h}\right), \quad \bar{h} = h - \frac{C^2}{2M}, \tag{11.1.50''}$$

equivalent to the formula (11.1.50) and called the Lagrange-Jacobi formula too. If, in particular, $O \equiv C$, then we have $\mathbf{C} = \mathbf{0}$, so that $\bar{h} = h$. Integrating twice with respect to time, we get

$$R^2 = R_0^2 + QT + 2M\int_0^t \mathrm{d}\tau \int_0^\tau \left[U(\bar{\tau}) + 2\bar{h}\right]\mathrm{d}\bar{\tau}, \quad R_0, Q = \text{const}. \tag{11.1.50'''}$$

This last result allows to study the stability of the discrete mechanical system $\mathscr{S}$; after Jacobi, we say that this system is stable if the distances between the particles remain finite, hence if R is finite for $t \to \infty$. If there exists $\bar{t}$ so that $\forall t > \bar{t}$ to have $U + 2\bar{h} > \varepsilon$, $\varepsilon > 0$, then the discrete mechanical system $\mathscr{S}$ is labile; this takes place, e.g., for $\bar{h} > 0$, because $U > 0$. We notice that one cannot have $U + 2\bar{h} < \varepsilon$, because the second member of the relation (11.1.50''') would become negative for t sufficiently great. It results that, to ensure the stability of the discrete mechanical system $\mathscr{S}$ it is necessary that $\bar{h} < 0$ and $\left|U + 2\bar{h}\right| < \varepsilon$ for $t > \bar{t}$.

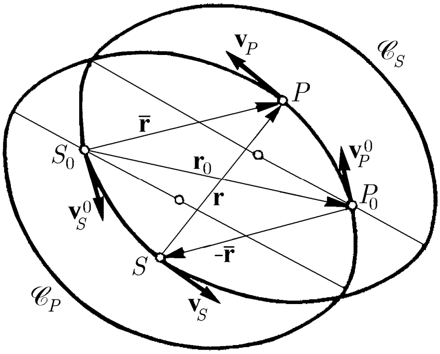

Fig. 11.5 Problem of two particles (the Sun and a planet)

The case $n = 2$ (*the problem of two particles*) has been considered in Chap. 8, Sect. 1.2.1, being reduced to the case of central forces (of attraction or of repulsion). Assuming, e.g., that one of the particles is the Sun, of mass M, the other particle being a planet P, of mass m, the equation of motion of a particle with respect to the other one (for instance, the motion of the planet with respect to the Sun) reads (see the formula (8.1.14))

$$\ddot{\mathbf{r}} = -f\frac{M + m}{r^2}\mathbf{u}, \tag{11.1.51}$$

where $\mathbf{u} = \operatorname{vers}\mathbf{r} = \operatorname{vers}\overrightarrow{SP}$ (Fig. 11.5); hence, the motion of the particle P with respect to the particle S is identical (for analogous initial conditions) to the motion of a particle P, of mass m, attracted with a Newtonian force by the fixed centre S, of mass $M + m$, and one can use the results obtained in Chap. 9, Sects. 2.1.2–2.1.4. The Theorems 9.2.1 and 9.2.2 (the first two laws of Kepler) remain, further, valid. It results that the particle P moves with respect to the particle S along the ellipse, after the law of areas, the particle S being situated at one of the foci; concerning the motion of the particle S with respect to the particle P, one can make analogous statements. The formula (9.2.17') allows to write

$$T = 2\pi a\sqrt{\frac{a}{f(M+m)}} = 2\pi a\sqrt{\frac{a}{fM}}\left(1+\frac{m}{M}\right)^{-1/2} \cong 2\pi a\sqrt{\frac{a}{fM}}\left(1-\frac{m}{2M}\right), \quad (11.1.51')$$

where a is the semi-major axis of the ellipse described by the particle P, T being the period in which this ellipse is entirely travelled through; hence, it results

$$\frac{T^2}{a^3} = \frac{4\pi^2}{f(M+m)} = \frac{4\pi^2}{fM}\left(1+\frac{m}{M}\right)^{-1} \cong \frac{4\pi^2}{fM}\left(1-\frac{m}{M}\right). \quad (11.1.51'')$$

Observing that, in case of the solar system, the ratio m/M is negligible with respect to unity (one obtains the greatest ratio $m/M = \mathscr{O}(10^{-3})$ for Jupiter, while for the Earth we have $m/M = \mathscr{O}(10^{-6})$), we can admit the third law of Kepler too, with a good approximation (as a matter of fact, the astronomical observations of Tycho Brahe are thus explained). In case of binary stars, the relation (11.1.51') allows to deduce the sum of the two masses $M + m$, taking into account that the quantities a and T can be measured by astronomical observations.

Let $\mathbf{v}_P^0 - \mathbf{v}_S^0$ be the relative initial velocity of the particle P with respect to the particle S at the initial moment t_0, where $\mathbf{v}_P^0$ and $\mathbf{v}_S^0$ are the initial velocities of the particles P and S, respectively. Taking into account the results in Chap. 9, Sect. 2.1.2, it results that the trajectory of the particle P is an ellipse if $r_0(\mathbf{v}_P^0 - \mathbf{v}_S^0)^2 < 2f(M+m)$, a parabola if $r_0(\mathbf{v}_P^0 - \mathbf{v}_S^0)^2 = 2f(M+m)$ or a hyperbola if $r_0(\mathbf{v}_P^0 - \mathbf{v}_S^0)^2 > 2f(M+m)$, where $\mathbf{r}_0 = \overrightarrow{S_0P_0}$ is the position vector at the initial moment; hence, the genus of the conic depends on the initial conditions. Analogously, the motion of the particle S with respect to the particle P takes place as that particle would be attracted by the centre P, the attractive mass being $M + m$; the genus of the conic $\mathscr{C}_S$ described by the particle S is specified by the same conditions, so that this particle describes a conic of the same genus as that of the conic $\mathscr{C}_P$, described by the particle P. The two conics are both in the plane determined by the vectors $\mathbf{r}_0$ and $\mathbf{v}_P^0 - \mathbf{v}_S^0$, and are obtained one from the other by a plane translation;

hence, the axes of the two conics are parallel. If P is a given position on $\mathscr{C}_P$, specified by the vector $\bar{\mathbf{r}} = \overrightarrow{S_0 P}$, we set up the vector $\overrightarrow{P_0 S} = -\bar{\mathbf{r}}$, obtaining thus the corresponding position S on $\mathscr{C}_S$. We notice also that the sense of motion of the particles P and S is the same on the conics $\mathscr{C}_P$ and $\mathscr{C}_S$, respectively. In case of the solar system, both conics are ellipses, as we have assumed in Fig. 11.5.

In the case $n = 3$ (*the problem of three particles*) one can no more obtain final results in a finite form. If one of the particles, S, has a mass M much greater than the masses m_1 and m_2 of the other two particles P_1 and P_2, respectively (e.g., the Sun and two planets), then one can apply some methods of successive approximations. Thus, one can consider, in a first approximation, the motion of the particles S and P_1 (as a problem of two particles). Further, one assumes that the particle P_2 perturbs the Keplerian motion by the attraction exerted upon both particles S and P_1; a perturbing term which modifies the previous results is thus introduced. As a matter of fact, the particle P_1 perturbs the motion of the particle P_2 too, introducing thus supplementary terms. The iterative process of computation is convergent. The inverse problem can be also put: to determine the position and the characteristics of a perturbing particle if (by measurement or observations, eventually) the perturbations in the motion of another particle are known. Thus, Leverrier discovered in 1845 the planet Neptune, studying the perturbations of motion of the neighbouring planet Uranus; in recent years (in 1961), from observations concerning the perturbations of the component A of the binary star 61 of the constellation Swan, one has concluded that around this component, at a distance of 11 light years, moves a planet greater than Jupiter, on an elliptic trajectory. In 2005, a tenth planet of the solar system, at a greater distance from the Sun than Pluto, seems to be discovered.

In general, the problem of n particles is decomposed in several problems corresponding to $n = 2$ or $n = 3$.

11.1.2.9 Discrete Mechanical Systems Subjected to Constraints

The discrete mechanical system $\mathscr{S}$ of n particles P_i, of masses m_i, and position vectors $\mathbf{r}_i$, $i = 1, 2, \ldots, n$, with respect to a fixed (inertial) frame of reference $\mathscr{R}$, previously considered, has $3n$ degrees of freedom, its position being specified by the position of the representative point $P(X_k)$ in the representative space E_{3n}. If m holonomic (rheonomic or scleronomic) ideal scalar constraints intervene, then the equations of motion, the general theorems and, as a consequence, the conservation theorems must be completed, introducing the constraint forces too. In general, we assume that the mechanical system $\mathscr{S}$ is subjected to $m < 3n$ bilateral, holonomic (finite, of geometric nature) constraints of the form (3.2.8); if we would have $3n$ distinct constraints, the position of the representative point P, hence of the mechanical system $\mathscr{S}$, would be specified from a geometric point of view (uniqueness or not). The case of non-holonomic constraints of the form (3.2.13) will be studied later by analytical methods. Returning to the representative point P, the m bilateral holonomic constraints specify that this point must be at the intersection of m hypersurfaces (fixed or movable, as the constraints are scleronomic or rheonomic, respectively) in the representative space E_{3n}. In some cases, we can consider also unilateral constraints (the representative point is at one side of a hypersurface).

Using the axiom of liberation from constraints and introducing the external constraint forces $\mathbf{R}_i$ (the resultant of the external constraint forces applied upon each particle P_i) as well as the internal constraint forces $\mathbf{R}_{ik}$, $i \neq k$, $k = 1,2,...,n$, the equations of motion (11.1.8) read

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i + \sum_{k=1}^{n}{}' (\mathbf{F}_{ik} + \mathbf{R}_{ik}), \quad i = 1,2,...,n; \tag{11.1.52}$$

in components, we can write

$$m_i \ddot{x}_j^{(i)} = F_j^{(i)} + R_j^{(i)} + \sum_{k=1}^{n}{}' \left(F_j^{(ik)} + R_j^{(ik)} \right), \quad i = 1,2,...,n, \quad j = 1,2,3. \tag{11.1.52'}$$

The position of the discrete mechanical system $\mathscr{S}$ subjected to constraints can be determined, at a given moment, by the equations (11.1.52') and by the constraint relations, obtaining the functions $x_j^{(i)} = x_j^{(i)}(t)$, $i = 1,2,...,n$, $j = 1,2,3$; in general, we can state (as for $\mathbf{F}_i$ and $\mathbf{F}_{ik}$, see Sect. 11.1.1.3) that $\mathbf{R}_i = \mathbf{R}_i(\mathbf{r}_l, \dot{\mathbf{r}}_l; t)$, $\mathbf{R}_{ik} = \mathbf{R}_{ik}(\mathbf{r}_l, \dot{\mathbf{r}}_l; t)$, $i \neq k$, $i,k = 1,2,...,n$, the motion of each particle depending on the motion of all other particles. In this case, the mechanical system $\mathscr{S}$ subjected to constraints works as a free discrete mechanical system.

We notice that all the results obtained in the case of free discrete mechanical systems can be transcribed for the discrete mechanical system $\mathscr{S}$ subjected to constraints if we add the constraint (unknown) forces to the given ones. In the first fundamental problem, besides the trajectories (the vector functions $\mathbf{r}_i = \mathbf{r}_i(t)$, $i = 1,2,...,n$), one must determine also the constraint forces $\mathbf{R}_i$ and $\mathbf{R}_{ik}$, $i \neq k$, $i,k = 1,2,...,n$. In the second fundamental problem, one must determine the forces $\mathbf{F}_i$, $\mathbf{F}_{ik}$ and $\mathbf{R}_i$, $\mathbf{R}_{ik}$; the problem has not a unique solution. As well, nor in case of a mixed fundamental problem the solution is unique.

In what concerns the theorems of existence and uniqueness, they remain further valid if the functions (3.2.8) fulfil the conditions asked in Sect. 11.1.1.4 (are functions of class C^1 and fulfil conditions of Lipschitz type).

Corresponding to the results in Sects. 11.1.2.1 and 11.1.2.2, we can write

$$\dot{\mathbf{H}} = \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{R} + \overline{\mathbf{R}}, \quad \dot{H}_j = R_j + \overline{R}_j, \quad j = 1,2,3, \tag{11.1.53}$$

$$M\dot{\boldsymbol{\rho}} = \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{R} + \overline{\mathbf{R}}, \quad M\dot{\rho}_j = R_j + \overline{R}_j, \quad j = 1,2,3, \tag{11.1.53'}$$

$$\dot{\mathbf{K}}_O = \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{M}_O + \bar{\mathbf{M}}_O , \quad \dot{K}_{Oj} = M_{Oj} + \bar{M}_{Oj} , \; j = 1,2,3 , \tag{11.1.53''}$$

stating thus (the theorems take place in case of holonomic constraints, as well as in case of non-holonomic ones; as well, we can assume to have unilateral constraints):

Theorem 11.1.21 (*theorem of momentum*). *The derivative with respect to time of the momentum of a discrete mechanical system subjected to constraints is equal to the resultant of the given and constraint external forces which act upon this system.*

Theorem 11.1.22 (*theorem of motion of the centre of mass*). *The centre of mass of a discrete mechanical system subjected to constraints moves as a free particle at which would be concentrated the whole mass of this system and which would be acted upon by the resultant of the given and constraint external forces.*

Theorem 11.1.23 (*theorem of moment of momentum*). *The derivative with respect to time of the moment of momentum of a discrete mechanical system subjected to constraints, with respect to a fixed pole, is equal to the resultant moment of the given and constraint external forces which act upon this system, with respect to the same pole.*

Introducing the areal accelerations as well as the notion of hodograph, we can state theorems analogous to Theorems 11.1.8', 11.1.5' and 11.1.8'', respectively.

Observing that

$$\dot{\tau}_O \{\mathbf{H}_i\} = \tau_O \{\mathbf{F}_i\} + \tau_O \{\mathbf{R}_i\} , \tag{11.1.53'''}$$

we may state

Theorem 11.1.24 (*theorem of torsor*). *The derivative with respect to time of the torsor of momenta of a discrete mechanical system subjected to constraints, with respect to a fixed pole, is equal to the torsor of the given and constraint external forces which act upon this system, with respect to the same pole.*

Introducing *the impulse of the resultant of the external constraint forces* $\sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{R}_i \mathrm{d}t$ and *the impulse of the resultant moment of the external constraint forces* $\sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{r}_i \times \mathbf{R}_i \mathrm{d}t$, corresponding to the time interval $[t_1, t_2]$, we can write (the internal constraint forces do not intervene in computation)

$$\Delta \mathbf{H} = \sum_{i=1}^{n} \int_{t_1}^{t_2} (\mathbf{F}_i + \mathbf{R}_i) \mathrm{d}t = \int_{t_1}^{t_2} \mathbf{R} \mathrm{d}t + \int_{t_1}^{t_2} \bar{\mathbf{R}} \mathrm{d}t , \tag{11.1.54}$$

$$\Delta \mathbf{K}_O = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{r}_i \times (\mathbf{F}_i + \mathbf{R}_i) \mathrm{d}t = \int_{t_1}^{t_2} \mathbf{M}_O \mathrm{d}t + \int_{t_1}^{t_2} \bar{\mathbf{M}}_O \mathrm{d}t , \tag{11.1.54'}$$

$$\Delta \tau_O \{\mathbf{H}_i\} = \tau_O \left\{ \int_{t_1}^{t_2} \mathbf{F}_i \mathrm{d}t \right\} + \tau_O \left\{ \int_{t_1}^{t_2} \mathbf{R}_i \mathrm{d}t \right\} . \tag{11.1.54''}$$

The relation (11.1.24) is completed in the form

$$\mathrm{d}T = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}} + \mathrm{d}W_R + \mathrm{d}W_{R\mathrm{int}} \tag{11.1.55}$$

and we can state

Theorem 11.1.25 (*theorem of kinetic energy*). *The differential of the kinetic energy of a discrete mechanical system subjected to constraints is equal to the elementary work of the given and constraint external and internal forces which act upon this system.*

As it was shown in Chap. 3, Sect. 2.2.9, in case of scleronomic constraints (or even in a more general case, that is in case of catastatic constraints) we have $\mathrm{d}W_R = \mathrm{d}W_{R\mathrm{int}} = 0$; in this case, the Theorem 11.1.25, corresponding to a discrete mechanical system subjected to constraints, is stated in the same form as the Theorem 11.1.10, corresponding to a free mechanical system. In what concerns the Theorems 11.1.10' and 11.1.10", one can make analogous observations.

As well, if the moment of momentum is written with respect to a pole Q, movable with respect to the fixed origin O, the formula (11.1.23') leads to

$$\dot{\tau}_Q\{\mathbf{H}_i\} = \tau_Q\{\mathbf{F}_i\} + \tau_Q\{\mathbf{R}_i\} - \{\mathbf{0}, \mathbf{v}_Q \times \mathbf{H}\}. \tag{11.1.56}$$

Using the results given in Sects. 1.2.5–1.2.7, we may set up first integrals, in certain conditions, in case of a discrete mechanical system subjected to constraints too. Thus, if the sum $\mathbf{R} + \bar{\mathbf{R}}$ is parallel to a fixed plane (is normal to a fixed direction of unit vector $\mathbf{u}$, $(\mathbf{R} + \bar{\mathbf{R}}) \cdot \mathbf{u} = 0$), then we can write the first integral (11.1.28); analogously, if the sum $\mathbf{M}_O + \bar{\mathbf{M}}_O$ is contained in a fixed plane (is normal to a fixed axis Δ, $O \in \Delta$, of unit vector $\mathbf{u}$, $(\mathbf{M}_O + \bar{\mathbf{M}}_O) \cdot \mathbf{u} = 0$), then one obtains the first integral (11.1.31).

If $\mathbf{R} + \bar{\mathbf{R}} = \mathbf{0}$ (necessary condition of static equilibrium), then we can state a conservation theorem of momentum and a theorem of rectilinear and uniform motion of the mass centre, while if $\mathbf{M}_O + \bar{\mathbf{M}}_O = \mathbf{0}$ (necessary condition of static equilibrium), then we may state a conservation theorem of moment of momentum. If both conditions are fulfilled simultaneously (necessary and sufficient conditions of static equilibrium in case of a non-deformable mechanical system), then one can state a conservation theorem of torsor.

In case of scleronomic (or even catastatic) constraints and of conservative internal forces, we can state a theorem of mechanical energy in the form of Theorems 11.1.19–11.1.19", and if $\mathrm{d}W = 0$ too, for a certain interval of time, we obtain a conservation theorem of mechanical energy, stated as the Theorem 11.1.20.

We mention that the conservation theorem of moment of momentum allows to state in this case a theorem of areal velocities, and a theorem of areas too.

As in case of a free discrete mechanical system, one can obtain only $6n$ independent first integrals; but these ones are sufficient to determine the motion. Even if the constraint forces are not known a priori, the conditions imposed above are sometimes fulfilled, so that one can build up first integrals also in case of discrete mechanical systems subjected to constraints.

In case of constraints with friction one can make considerations analogous to those in case of the motion of a single particle (see Chap. 6, Sect. 2.2.3).

11.1.2.10 Differential Principles of Mechanics

The equations of motion of a discrete mechanical system, free or subjected to constraints, are written in the form (11.1.8) or in the form (11.1.52), respectively; these equations correspond to the second principle of Newton, the first differential principle of mechanics. As in the case of a single particle, this basic principle can be expressed also in other equivalent forms, which are useful in various particular cases; if they are considered as consequences of Newton's principle, these equivalent forms are theorems.

Introducing *the forces of inertia*

$$\overline{\mathbf{F}}_i = -m_i \ddot{\mathbf{r}}_i , \quad i = 1,2,...,n , \tag{11.1.57}$$

the law of motion reads (we consider the more general case of a discrete mechanical system subjected to constraints)

$$\mathbf{F}_i + \overline{\mathbf{F}}_i + \mathbf{R}_i + \sum_{k=1}^{n}{}'(\mathbf{F}_{ik} + \mathbf{R}_{ik}) = \mathbf{0} , \quad i = 1,2,...,n , \tag{11.1.58}$$

and we can state

Theorem 11.1.26 (*d'Alembert*). *The motion of a discrete mechanical system subjected to constraints takes place so that, at any moment, it is in dynamic equilibrium under the action of the given and constraint, external and internal forces which act upon it and of the forces of inertia.*

Obviously, each particle (more general, each subsystem) of the system is in dynamic equilibrium.

We introduce the forces

$$\mathbf{\Phi}_i = \mathbf{F}_i + \overline{\mathbf{F}}_i + \sum_{k=1}^{n}{}'\mathbf{F}_{ik} = \mathbf{F}_i + \sum_{k=1}^{n}{}'\mathbf{F}_{ik} - m_i \ddot{\mathbf{r}}_i , \quad i = 1,2,...,n , \tag{11.1.59}$$

which are called *the lost forces of d'Alembert*; the equations (11.1.58) become, in this case,

$$\mathbf{\Phi}_i + \mathbf{R}_i + \sum_{k=1}^{n}{}'\mathbf{R}_{ik} = \mathbf{0} , \quad i = 1,2,...,n , \tag{11.1.60}$$

and we may state

Theorem 11.1.26' (*d'Alembert*). *The motion of a discrete mechanical system subjected to constraints takes place so that, at any moment, the constraint external and internal forces are in equilibrium with the lost forces of d'Alembert.*

This equilibrium takes place for each particle, hence for any subsystem of the considered mechanical system. We can thus state:

Theorem 11.1.27 (*theorem of dynamic equilibrium of parts*). *If a discrete mechanical system $\mathcal{S}$ subjected to constraints is, at a given moment, in dynamic equilibrium under the action of the lost forces of d'Alembert and of the constraint external and internal forces which act upon it, then any part of the system* (*any subsystem* $S \subset \mathcal{S}$) *will be in* *dynamic equilibrium too, at that moment, under the action of the lost forces of d'Alembert and of the constraint forces which act upon that part.*

Theorem 11.1.27' (*theorem of rigidity*). *Assuming that a given discrete mechanical system subjected to constraints becomes rigid at a certain moment, the conditions of dynamic equilibrium of the new mechanical system represent necessary conditions for the motion of the given mechanical system at that moment.*

We notice that, applying the theorem of rigidity to all parts of the discrete mechanical system $\mathscr{S}$ (to all subsystems $S \subset \mathscr{S}$), we get sufficient conditions to describe the motion. Indeed, taking, e.g., all the subsystems formed by two particles, there result the conditions of vanishing torsor (it is sufficient to mention only the conditions concerning the resultants)

$$\mathbf{\Phi}_i + \mathbf{R}_i + \sum_{k=1}^{n}{}' \mathbf{R}_{ik} + \mathbf{\Phi}_j + \mathbf{R}_j + \sum_{k=1}^{n}{}' \mathbf{R}_{jk} = \mathbf{0}, \quad i, j = 1, 2, ..., n,$$

which, obviously, lead to the conditions (11.1.60).

The relation (11.1.59) may be written also in the form

$$\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} = \mathbf{\Phi}_i + (-\overline{\mathbf{F}}_i) = \mathbf{\Phi}_i + m_i \ddot{\mathbf{r}}_i, \quad i = 1, 2, ..., n;$$

it results that only the components $m_i \ddot{\mathbf{r}}_i = -\overline{\mathbf{F}}_i$ of the forces $\mathbf{F}_i$ have a contribution to the motion of the discrete mechanical system, the components $\mathbf{\Phi}_i$ being lost by equilibrating the constraint forces (the given denomination is thus justified).

We notice that each of the Theorems 11.1.26 and 11.1.26' can stay at the basis of the Newtonian mathematical model of mechanics, representing each of them a *differential principle of mechanics*.

Formally, the Equations (11.1.60), which represent *the necessary and sufficient conditions of dynamic equilibrium* (characterizing entirely the motion of the discrete mechanical system subjected to constraints) do not differ from the relations (4.1.55), which represent *the necessary and sufficient conditions of static equilibrium* of such a system. Hence, all the considerations made for the problems of statics (including the Theorems 4.1.6' and 4.1.7'), starting from the relations (4.1.55), can be transposed for the similar problems with a dynamic character, replacing the given forces $\mathbf{F}_i$ and $\mathbf{F}_{ik}$, $i \neq k$, $i, k = 1, 2, ..., n$, by the lost forces $\mathbf{\Phi}_i$ of d'Alembert; e.g., the condition (4.1.56) leads to the theorem of torsor, characterized by the formula (11.1.53'''). In fact, we can use all the results in Chap. 4, Sects. 1.2.1–1.2.3. Thus, we may write a *necessary condition* for the motion in the form

$$\tau_O \{\mathbf{\Phi}_i\} + \tau_O \{\mathbf{R}_i\} = \mathbf{0} \tag{11.1.61}$$

or in the form

$$\sum_{i=1}^{n} (\mathbf{\Phi}_i + \mathbf{R}_i) = \mathbf{0}, \quad \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{\Phi}_i + \mathbf{R}_i) = \mathbf{0}, \tag{11.1.61'}$$

stating

Theorem 11.1.24' (*theorem of torsor; second form*). *The motion of a discrete mechanical system subjected to constraints takes place so that, at any moment, the sum of the torsor of the lost forces of d'Alembert with respect to a fixed pole and the torsor of the constraint external forces with respect to the same pole vanishes*.

Let be a discrete mechanical system $\mathcal{S}$ subjected to ideal constraints (for which the virtual work of the constraint forces (3.2.36) vanishes). We start from the necessary and sufficient equations of motion (11.1.60), written in the form (by $\mathbf{R}_i$ we mean the resultant of all constraint forces, without distinction between external and internal ones)

$$\boldsymbol{\Phi}_i + \mathbf{R}_i = \mathbf{0}, \quad i = 1,2,...,n; \tag{11.1.62}$$

if we perform a scalar product by the virtual displacements $\delta\mathbf{r}_i$, we sum for all particles of the mechanical system $\mathcal{S}$ and take into account the relation of definition of ideal constraints (3.2.36), then we obtain the relation

$$\sum_{i=1}^{n} \boldsymbol{\Phi}_i \cdot \delta\mathbf{r}_i = 0, \tag{11.1.63}$$

which represents a necessary condition to describe the motion. Assuming that the condition (11.1.63) is fulfilled and that the system is subjected to p holonomic constraints of the form (3.2.21") and to m non-holonomic constraints of the form (3.2.15), we can use the method of Lagrange's multipliers; we may thus write

$$\sum_{i=1}^{n} \left(\boldsymbol{\Phi}_i + \sum_{l=1}^{p} \lambda_l \nabla_i f_l + \sum_{k=1}^{m} \mu_k \boldsymbol{\alpha}_{ki} \right) \cdot \delta\mathbf{r}_i = 0,$$

where λ_l, $l = 1,2,...,p$, μ_k, $k = 1,2,...,m$ are non-determinate scalars (Lagrange's multipliers) and where we noticed that in a finite double sum one can invert the order of summation. By a demonstration analogous to that in Chap. 3, Sect. 2.2.9, we get finally,

$$\boldsymbol{\Phi}_i + \sum_{l=1}^{p} \lambda_l \nabla_i f_l + \sum_{k=1}^{m} \mu_k \boldsymbol{\alpha}_{ki} = \mathbf{0}, \quad i = 1,2,...n. \tag{11.1.64}$$

We find again the relations (3.2.37) which give the constraint forces, the relations (11.1.64) being thus equivalent to the relations (11.1.62). We can state (the relation (11.1.63) becomes a sufficient condition too)

Theorem 11.1.28 (*theorem of virtual work; d'Alembert-Lagrange*). *The motion of a discrete mechanical system subjected to ideal constraints takes place so that the virtual work of the lost forces of d'Alembert, which act upon it, vanishes for any system of virtual displacements of the respective mechanical system*.

Taking into account the equivalence between the relation (11.1.63), which represents the theorem of virtual work, and the relations (11.1.62), which represent the form taken by Newton's equations, it results that the theorem of virtual work can be considered as being a principle (*the principle of virtual work* or *the principle of virtual* *displacements*), because – starting from it – one can solve the fundamental problems of dynamics.

The equations (11.1.64) are known as *Lagrange's equations of the first kind*.

Introducing the virtual velocities (3.2.1'), we may write the condition (11.1.63) in the form

$$\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \mathbf{v}_i^* = 0\,, \tag{11.1.63'}$$

the considered principle being thus called *the principle of virtual velocities* too.

In case of holonomic (of the form (3.2.16$^{\text{iv}}$)) or non-holonomic (of the form (3.2.16)) *ideal unilateral constraints*, the virtual work of the constraint forces verifies the inequality (3.2.36'). The principle of virtual work is expressed, in this case, in the form

$$\sum_{i=1}^{n} \mathbf{\Phi}_i \cdot \delta\mathbf{r}_i \leq 0 \tag{11.1.63''}$$

for any system of virtual displacements, representing the necessary and sufficient condition to describe the motion in case of a discrete mechanical system subjected to ideal unilateral constraints; in this case too, one can make considerations analogous to those above.

11.1.2.11 General Considerations

The general (*universal*) *theorems of mechanics* are expressed in a *torsor form* (containing two vector relations) or in a *scalar form*. The theorem of torsor (including the theorem of momentum and the theorem of moment of momentum) is expressed by vector relations between quantities of kinetic nature and given and constraint external forces (the internal forces do not intervene – this is a great advantage from the point of view of practical computation). The theorem of kinetic energy is expressed by a scalar relation between quantities of kinetic nature and given and constraint, external and internal forces (in this theorem intervene – in general – all types of forces; in case of catastatic constraints, the constraint forces do not intervene). These seven scalar relations allow to obtain, in certain conditions, till ten scalar first integrals (the conservation theorem of momentum allows to write six first integrals, the conservation theorem of moment of momentum leads to three first integrals, while the conservation theorem of mechanical energy represents only one first integral).

We notice, after V. Vâlcovici, that one can group the three general Theorems 11.1.5, 11.1.8 and 11.1.10' in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}_0 \cdot \mathbf{H} + \boldsymbol{\omega} \cdot \mathbf{K}_O + cT\right) = \mathbf{v}_0 \cdot \mathbf{R} + \boldsymbol{\omega} \cdot \mathbf{M}_O + c\left(P + P_{\text{int}}\right), \tag{11.1.65}$$

where $\{\boldsymbol{\omega}, \mathbf{v}_0\}$ represents a finite rototranslation $\left(\boldsymbol{\omega}, \mathbf{v}_0 = \overrightarrow{\text{const}}\right)$, while $c = \text{const}$ is a scalar. One can obtain this result effecting a scalar product of each equation (11.1.8) by the vector $\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_i + c\mathbf{v}_i$, $i = 1,2,...,n$, and summing for all the values of the index i. If, in particular, the constant quantities $\mathbf{v}_0$, $\boldsymbol{\omega}$ and c (equivalent to seven scalar constants) are so that $\mathbf{v}_0 \cdot \mathbf{R} + \boldsymbol{\omega} \times \mathbf{M}_O + c(P + P_{\text{int}}) = 0$ (eventually, $P\mathrm{d}t$ or/and $P_{\text{int}}\mathrm{d}t$ are exact differentials), then we have $\mathbf{v}_0 \cdot \mathbf{H} + \boldsymbol{\omega} \times \mathbf{K}_O + cT = \text{const}$, obtaining a first integral of the system of equations of motion. If we take $c = 0$, it results

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}_0 \cdot \mathbf{H} + \boldsymbol{\omega} \cdot \mathbf{K}_O) = \mathbf{v}_0 \cdot \mathbf{R} + \boldsymbol{\omega} \cdot \mathbf{M}_O \tag{11.1.65'}$$

and we can state

Theorem 11.1.29 (*theorem of power of the kinetic torsor; V. Vâlcovici*). *The derivative with respect to time of the power of the torsor of momenta of a free discrete mechanical system, with respect to a fixed pole, by a constant finite rototranslation, is equal to the power of the torsor of the given external forces which act upon this system, with respect to the same pole.*

These relations represent *necessary conditions* to describe the motion of a deformable (in general) discrete mechanical system; indeed, in this case the lost forces of d'Alembert are modelled by bound vectors. In case of a non-deformable discrete mechanical system, these forces are modelled by sliding vectors, so that these relations become also *sufficient conditions* to describe the motion; in fact (as it will be seen in next chapter), in this case the theorem of torsor entirely characterizes the motion, the theorem of kinetic energy being a linear consequence of it. In the case of a non-deformable discrete mechanical system too, we notice that $\mathrm{d}W_{\text{int}} = \mathrm{d}W_{R\text{int}} = 0$, so that the theorem of kinetic energy takes the simpler form

$$\mathrm{d}T = \mathrm{d}W + \mathrm{d}W_R, \tag{11.1.55'}$$

while, in the case of scleronomic constraints, we get

$$\mathrm{d}T = \mathrm{d}W. \tag{11.1.55''}$$

Unlike the universal theorems, the theorem of virtual work expresses always a necessary and sufficient condition to describe the motion, in the hypothesis of ideal constraints of the discrete mechanical system. This theorem has the advantage to contain only the given external and internal forces (the constraint forces do not intervene in computation); the motion can be thus studied without determining, previously, the constraint forces.

Besides the momentum $\mathbf{H}$, which is called also *kinetic resultant*, one can introduce also *the dynamic resultant*

$$\mathbf{A} = \sum_{i=1}^{n} \mathbf{A}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i; \tag{11.1.66}$$

we notice the obvious relation

$$\mathbf{A} = \dot{\mathbf{H}}. \tag{11.1.66'}$$

Analogously, besides the moment of momentum $\mathbf{K}_O$ (which is a kinetic moment), with respect to the pole O, one can introduce also *the dynamic moment*, with respect to the same pole,

$$\mathbf{D}_O = \sum_{i=1}^{n} \mathbf{D}_{Oi} = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{a}_i) = \sum_{i=1}^{n} \mathbf{r}_i \times (m_i \ddot{\mathbf{r}}_i); \tag{11.1.67}$$

we obtain the obvious relation

$$\mathbf{D}_O = \dot{\mathbf{K}}_O . \tag{11.1.67'}$$

The relations (11.1.53) and (11.1.53'') may be written, in this case, in the form

$$\mathbf{A} = \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{R} + \bar{\mathbf{R}}, \quad A_j = R_j + \bar{R}_j, \ j = 1,2,3, \tag{11.1.66''}$$

$$\mathbf{D}_O = \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{M}_O + \bar{\mathbf{M}}_O, \ D_{Oj} = M_{Oj} + \bar{M}_{Oj}, \ j = 1,2,3, \tag{11.1.67''}$$

stating thus:

Theorem 11.1.30 (*theorem of dynamic resultant*). *The dynamic resultant of a discrete mechanical system subjected to constraints is equal to the resultant of the given and constraint external forces which act upon this system.*

Theorem 11.1.31 (*theorem of dynamic moment*). *The dynamic moment of a discrete mechanical system subjected to constraints, with respect to a fixed pole, is equal to the resultant moment of the given and constraint external forces which act upon this system, with respect to the same pole.*

The torsor of momenta $\tau_O\{\mathbf{H}_i\}$ is called also kinetic torsor; analogously, one can introduce the dynamic torsor $\tau_O\{\mathbf{A}_i\}$. Observing that

$$\tau_O\{\mathbf{A}_i\} = \dot{\tau}_O\{\mathbf{H}_i\}, \tag{11.1.68}$$

it results

$$\tau_O\{\mathbf{A}_i\} = \tau_O\{\mathbf{F}_i\} + \tau_O\{\mathbf{R}_i\} \tag{11.1.68'}$$

and we can state

Theorem 11.1.32 (*theorem of dynamic torsor; Newton-Euler*). *The dynamic torsor of a discrete mechanical system subjected to constraints, with respect to a fixed pole, is equal to the torsor of the given and constraint external forces which act upon this system, with respect to the same pole.*

Taking into account (11.1.65), we may write

$$\mathbf{v}_0 \cdot \mathbf{A} + \boldsymbol{\omega} \cdot \mathbf{D}_O + c\dot{T} = \mathbf{v}_0 \cdot \mathbf{R} + \boldsymbol{\omega} \cdot \mathbf{M}_O + c(P + P_{\text{int}}) \tag{11.1.65''}$$

too; in the particular case $c = 0$, we state

Theorem 11.1.29' (*theorem of power of the dynamic torsor*). *The power of the dynamic torsor of a free discrete mechanical system, with respect to a fixed pole, by a constant finite rototranslation, is equal to the power of the torsor of the given external forces which act upon this system, with respect to the same pole.*

Obviously, the influence of the constraint forces may be also introduced in the Theorems 11.1.29 and 11.1.29'.

These results are equivalent to those previously obtained; besides, they can give sometimes useful information concerning the motion of a discrete mechanical system $\mathcal{S}$.

From the above exposition (as a completion to Chap. 1, Sect. 1.2.3), one can affirm that, by conjugating elements of kinematics with elements of mass geometry, one obtains *the kinetics*, which deals with the motion of mechanical systems supplied with mass, without taking into account the forces which act upon them; if the forces intervene too, then one has to do with *dynamics*.

11.1.2.12 Group Properties

Let be the system of equations of motion (11.1.8). Let us also assume that the given forces are conservative (or quasi-conservative) in their totality, deriving from the simple potential (or quasi-potential)

$$U(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, ..., \dot{\mathbf{r}}_n; t) = \sum_{i=1}^{n} U_i(\mathbf{r}_i, \dot{\mathbf{r}}_i; t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}{}' U_{ik}(\mathbf{r}_i, \mathbf{r}_k, \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_k; t), \tag{11.1.69}$$

in the form

$$\mathbf{F}_i + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} = \nabla_i U, \quad \mathbf{F}_i = \nabla_i U_i, \quad \mathbf{F}_{ik} = \frac{1}{2}\nabla_i (U_{ik} + U_{ki}), \quad i, k = 1, 2, ..., n; \tag{11.1.69'}$$

for the sake of generality we assume that the potential (quasi-potential) depends on velocities too.

In case of a transformation of the form

$$\mathbf{r}_i' = \mathbf{r}_i + \mathbf{r}_0, \quad \mathbf{r}_0 = \overrightarrow{\mathrm{const}}, \quad \dot{\mathbf{r}}_i' = \dot{\mathbf{r}}_i, \quad i = 1, 2, ..., n, \quad t' = t, \tag{11.1.70}$$

which belongs to the group of space translations T, with three parameters (see Chap. 6, Sect. 1.2.3 too), we may write

$$U(\mathbf{r}_1', \mathbf{r}_2', ..., \mathbf{r}_n') - U(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n) = \mathbf{r}_0 \cdot \sum_{i=1}^{n} \nabla_i U,$$

corresponding to Lagrange's theorem. Hence, if

$$U(\mathbf{r}_1', \mathbf{r}_2', ..., \mathbf{r}_n', \dot{\mathbf{r}}_1', \dot{\mathbf{r}}_2', ..., \dot{\mathbf{r}}_n'; t) = U(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, ..., \dot{\mathbf{r}}_n; t), \tag{11.1.71}$$

then we have

$$\sum_{i=1}^{n} \nabla_i U = \mathbf{0} \tag{11.1.71'}$$

too, the sum of the given forces vanishing. We can thus state

Theorem 11.1.33 *If the simple potential (quasi-potential) of the given conservative (quasi-conservative) forces which act upon a free discrete mechanical system is invariant with respect to the group of space translations T, then the momentum of this system is conserved in time.*

Taking into account the relation $|\mathbf{r}_i' - \mathbf{r}_k'| = |\mathbf{r}_i - \mathbf{r}_k|$, we notice that the relation (11.1.71) takes place if

$$U_i = U_i(\dot{\mathbf{r}}_i; t), \quad U_{ik} = U_{ik}(|\mathbf{r}_i - \mathbf{r}_k|, \dot{\mathbf{r}}_i, \dot{\mathbf{r}}_k; t), \quad i,k = 1,2,\dots,n,$$

hence if the external forces vanish (isolated mechanical system), while the internal forces depend on the distances between two particles ($\mathbf{F}_{ik} = \mathbf{F}_{ik}\left(\overrightarrow{P_i P_k}; t\right)$, verifying the principle of action and reaction).

As well, in case of a transformation of the form ($\boldsymbol{\alpha}$ is a constant tensor of second order)

$$\mathbf{r}_i' = \boldsymbol{\alpha}\mathbf{r}_i, \quad \dot{\mathbf{r}}_i' = \boldsymbol{\alpha}\dot{\mathbf{r}}_i, \quad i = 1,2,\dots,n, \quad t' = t, \tag{11.1.70'}$$

which belongs to the group SO(3) of finite rotations with three parameters (special orthogonal group in E_3), we notice that the relations

$$\mathbf{r}_i' \cdot \mathbf{r}_k' = (\boldsymbol{\alpha}\mathbf{r}_i) \cdot (\boldsymbol{\alpha}\mathbf{r}_k) = \left(\alpha_{jl} x_l^{(i)} \mathbf{i}_j\right) \cdot \left(\alpha_{mp} x_p^{(k)} \mathbf{i}_m\right) = \alpha_{jl}\alpha_{mp}\delta_{jm} x_l^{(i)} x_p^{(k)}$$
$$= \alpha_{jl}\alpha_{jp} x_l^{(i)} x_p^{(k)} = \delta_{lp} x_l^{(i)} x_p^{(k)} = x_l^{(i)} x_l^{(k)} = \mathbf{r}_i \cdot \mathbf{r}_k,$$
$$\dot{\mathbf{r}}_i' \cdot \dot{\mathbf{r}}_k' = \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_k$$

and, in particular, the relations $\mathbf{r}_i'^2 = \mathbf{r}_i^2$, $\dot{\mathbf{r}}_i'^2 = \dot{\mathbf{r}}_i^2$, $i,k = 1,2,\dots,n$, take place; one can also show that these relations occur only in case of a transformation of the form (11.1.70'). In such conditions, a theorem of Cauchy allows to state that a relation of the form (11.1.71) takes place only and only if (the dependence on velocities has not been mentioned)

$$U = U\left(\mathbf{r}_1^2, \mathbf{r}_1 \cdot \mathbf{r}_2, \mathbf{r}_1 \cdot \mathbf{r}_3, \dots, \mathbf{r}_1 \cdot \mathbf{r}_n, \mathbf{r}_2 \cdot \mathbf{r}_1, \mathbf{r}_2^2, \mathbf{r}_2 \cdot \mathbf{r}_3, \dots, \mathbf{r}_{n-1} \cdot \mathbf{r}_n, \mathbf{r}_n^2\right).$$

We can write

$$\sum_{i=1}^{n} \mathbf{r}_i \times \nabla_i U = \in_{jkl} \sum_{i=1}^{n} x_j^{(i)} \frac{\partial U}{\partial x_k^{(i)}} \mathbf{i}_l = \in_{jkl} \sum_{i=1}^{n}\sum_{m=1}^{n} x_j^{(i)} \frac{\partial U}{\partial\left(x_p^{(i)} x_p^{(m)}\right)} x_k^{(m)} \mathbf{i}_l = \mathbf{0}$$

as a product of two tensors, one symmetric and the other skew-symmetric with respect to the indices j and k, the resultant moment of the given forces being thus equal to zero. We state

Theorem 11.1.34 *If the simple potential* (*quasi-potential*) *of the given conservative* (*quasi-conservative*) *forces which act upon a free mechanical system is invariant with respect to the group* SO(3) *of finite rotations, then the moment of momentum of this system is conserved in time.*

Noting that

$$|\mathbf{r}_i' - \mathbf{r}_k'|^2 = |\mathbf{r}_i - \mathbf{r}_k|^2, \quad |\dot{\mathbf{r}}_i' - \dot{\mathbf{r}}_k'|^2 = |\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_k|^2, \quad |\mathbf{r}_i'| = |\mathbf{r}_i|, \quad |\dot{\mathbf{r}}_i'| = |\dot{\mathbf{r}}_i|,$$

we can state that the relation (11.1.71) takes place if (we have $\mathbf{F}_{ik} = \mathbf{F}_{ik}\left(\overrightarrow{P_iP_k};t\right)$, verifying the principle of action and reaction)

$$U_i = U_i(|\mathbf{r}_i|, |\dot{\mathbf{r}}_i|; t), \quad U_{ik} = U_{ik}(|\mathbf{r}_i - \mathbf{r}_k|, |\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_k|; t), \quad i,k = 1,2,...,n.$$

By a transformation of the form

$$\mathbf{r}_i' = \mathbf{r}_i, \quad \dot{\mathbf{r}}_i' = \dot{\mathbf{r}}_i, \quad i = 1,2,...,n, \quad t' = t + t_0, \quad t_0 = \text{const}, \tag{11.1.70''}$$

of the temporal variable, which belongs to the Abelian group of time translations $\mathscr{T}$ with one parameter, we can write $U(t') - U(t) = t_0 \partial U / \partial t = t_0 \dot{U}$, using Lagrange's theorem. If $U(t') = U(t)$, then it results $\dot{U}(t) = 0$ too, so that one cannot have a quasi-potential (hence, nor a quasi-conservative force). If $U = U(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n)$, non-depending on velocities, hence if $U_i = U_i(\mathbf{r}_i)$, $U_{ik} = U_{ik}(\mathbf{r}_i, \mathbf{r}_k)$, $i,k = 1,2,...,n$ (in fact, $U_{ik} = U_{ik}(\mathbf{r}_i - \mathbf{r}_k)$, to can verify the basic principle of the internal forces), then $\mathrm{d}U = \sum_{i=1}^{n} \nabla_i U \cdot \mathrm{d}\mathbf{r}_i$ and we may state

Theorem 11.1.35 *If the simple potential of the given conservative forces which act upon a free mechanical system is invariant with respect to the group of time translations* $\mathscr{T}$, *then the mechanical energy of this system is conserved in time.*

Finally, let be a transformation of the form

$$\mathbf{r}_i' = \mathbf{r}_i + \mathbf{v}_0 t, \quad \dot{\mathbf{r}}_i' = \dot{\mathbf{r}}_i + \mathbf{v}_0, \quad \mathbf{v}_0 = \overrightarrow{\text{const}}, \quad i = 1,2,...,n, \quad t' = t, \tag{11.1.70'''}$$

which belongs to the Abelian group Γ of Galileo, with three parameters. We notice that the acceleration $\ddot{\mathbf{r}}_i$, hence the dynamic resultant **A**, given by (11.1.66), is invariant to such a transformation too. In the conditions of the Theorem 11.1.33, it results $\mathbf{A} = \mathbf{0}$, while the potential (quasi-potential) U is invariant to a transformation of the form (11.1.70); a relation of the form (11.1.71) takes place also with respect to the transformation (11.1.70''') if $U_i = U_i(t)$, $U_{ik} = U_{ik}(|\mathbf{r}_i - \mathbf{r}_k|, |\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_k|; t)$, as a particular case (the mechanical system is isolated, while the internal forces verify the principle of action and reaction). Taking into account the theorem of motion of the mass centre, we may state

Theorem 11.1.36 *If the simple potential* (*quasi-potential*) *of the given conservative* (*quasi-conservative*) *forces which act upon a free discrete mechanical system is invariant with respect to the Galileo group, then the mass centre of this system has a uniform rectilinear motion with respect to an inertial frame of reference.*

The four groups of transformations considered are subgroups of the Galileo-Newton group $\mathscr{G}$, with ten parameters, leading to the three vector and one scalar first integrals (hence, to the ten scalar first integrals) mentioned above.

We will use the group properties to determine the first integrals of the equations of motion in a unitary theory, in the frame of Lagrangian and Hamiltonian mechanics.

11.2 Dynamics of Discrete Mechanical Systems with Respect to a Non-inertial Frame of Reference

In what follows, we present the form taken by the general theorems of mechanics with respect to a non-inertial frame of reference; to this goal, we compute first of all the mechanical quantities previously introduced (momentum, moment of momentum, kinetic energy, work, power) with respect to such a frame. Especially, if the frame has the pole at the mass centre of the considered discrete mechanical system, then the results obtained have a remarkable form.

11.2.1 Motion of a Discrete Mechanical System with Respect to a Koenig Frame of Reference

We introduce a Koenig frame of reference with respect to the mass centre of a discrete mechanical system and present the general and conservation theorems with respect to such a frame. We make also some considerations concerning the problem of n bodies.

11.2.1.1 Koenig's Frame of Reference. Koenig's Theorems

By a Koenig frame of reference we mean a non-inertial frame $\mathscr{R}$, of axes Cx_i, $i = 1,2,3$ (movable with respect to the inertial frame $\mathscr{R}'$, of axes $O'x'_j$, $j = 1,2,3$, considered fixed), which has its pole at the centre of mass C (of position vector $\boldsymbol{\rho}'$) of the discrete mechanical system $\mathscr{S}$ and which does not rotate (it moves with the axes parallel to themselves) with respect to the frame $\mathscr{R}'$ (Fig. 11.6). One passes from the frame $\mathscr{R}'$ to the frame $\mathscr{R}$ by relations of the form

$$\mathbf{r}'_i = \mathbf{r}_i + \boldsymbol{\rho}', \quad x'^{(i)}_j = x^{(i)}_j + \rho_j, \quad i = 1,2,\dots,n, \quad j = 1,2,3; \tag{11.2.1}$$

because $\boldsymbol{\omega} = \mathbf{0}$, in conformity to the hypothesis made above, it results that

$$\dot{\mathbf{r}}'_i = \dot{\mathbf{r}}_i + \dot{\boldsymbol{\rho}}', \quad i = 1,2,\dots,n. \tag{11.2.1'}$$

In this case, the momentum is given by (we denote by "prime" the quantities calculated with respect to the fixed frame $\mathscr{R}'$)

$$\mathbf{H}' = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i' = \sum_{i=1}^{n} m_i \left(\dot{\mathbf{r}}_i + \dot{\boldsymbol{\rho}}' \right) = M \dot{\boldsymbol{\rho}}', \tag{11.2.2}$$

where we took into account the relations (properties of the mass centre)

$$\sum_{i=1}^{n} m_i = M, \quad \sum_{i=1}^{n} m_i \mathbf{r}_i = \mathbf{0}, \quad \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \mathbf{0}; \tag{11.2.1"}$$

this result is known and may be obtained by differentiating the relation (3.1.9) with respect to time.

The moment of momentum is expressed in the form

$$\mathbf{K}_{O'}' = \sum_{i=1}^{n} \mathbf{r}_i' \times \left(m_i \dot{\mathbf{r}}_i' \right) = \sum_{i=1}^{n} m_i \left(\mathbf{r}_i + \boldsymbol{\rho}' \right) \times \left(\dot{\mathbf{r}}_i + \dot{\boldsymbol{\rho}}' \right)$$

$$= \sum_{i=1}^{n} \mathbf{r}_i \times \left(m_i \dot{\mathbf{r}}_i \right) + \left(\sum_{i=1}^{n} m_i \mathbf{r}_i \right) \times \dot{\boldsymbol{\rho}}' + \boldsymbol{\rho}' \times \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i + \boldsymbol{\rho}' \times \dot{\boldsymbol{\rho}}' \sum_{i=1}^{n} m_i ,$$

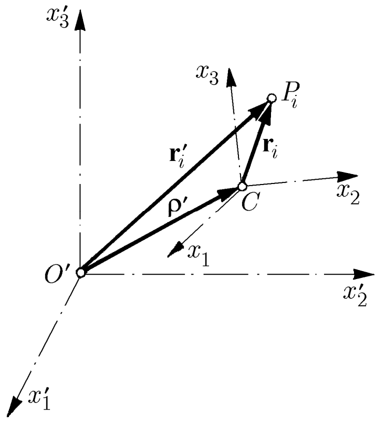

Fig. 11.6 Motion with respect to a Koenig frame of reference

wherefrom one gets the relation

$$\mathbf{K}_{O'}' = \mathbf{K}_C^{(C)} + \boldsymbol{\rho}' \times \left(M \dot{\boldsymbol{\rho}}' \right), \tag{11.2.3}$$

$\mathbf{K}_C^{(C)}$ being the moment of momentum of the system with respect to the centre of mass (in the relative motion, with respect to a Koenig frame with the pole at C); we can thus state

Theorem 11.2.1 (*S. Koenig*). *The moment of momentum of a discrete mechanical system with respect to a fixed pole is equal to the sum of the moment of momentum of the same system with respect to the pole of a Koenig frame of reference and the moment of momentum of its mass centre at which the whole mass of the mechanical system is concentrated, with respect to the fixed pole.*

Analogously, the kinetic energy is given by

$$T' = \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{\mathbf{r}}_i + \dot{\boldsymbol{\rho}}'\right)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2 + \dot{\boldsymbol{\rho}}' \cdot \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i + \frac{1}{2}\dot{\rho}'^2 \sum_{i=1}^{n} m_i ,$$

so that we may write

$$T' = T^{(C)} + \frac{1}{2} M \dot{\rho}'^2 , \tag{11.2.4}$$

$T^{(C)}$ being the kinetic energy of the system in relative motion (with respect to the mass centre); we thus state

Theorem 11.2.2 (*S. Koenig*). *The kinetic energy of a discrete mechanical system with respect to a fixed frame of reference is equal to the sum of the kinetic energy of the same system with respect to a Koenig frame of reference and the kinetic energy of the mass centre at which the whole mass of the mechanical system is concentrated, with respect to the fixed frame.*

The elementary work of the given and constraint, external and internal forces is given by

$$\begin{aligned} \mathrm{d}W' + \mathrm{d}W'_{\mathrm{int}} + \mathrm{d}W'_R + \mathrm{d}W'_{R\mathrm{int}} &= \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) \cdot \mathrm{d}\mathbf{r}'_i + \sum_{i=1}^{n} \sum_{k=1}^{n}{}' (\mathbf{F}_{ik} + \mathbf{R}_{ik}) \cdot \mathrm{d}\mathbf{r}'_i \\ &= \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) \cdot (\mathrm{d}\mathbf{r}_i + \mathrm{d}\boldsymbol{\rho}') + \sum_{i=1}^{n} \sum_{k=1}^{n}{}' (\mathbf{F}_{ik} + \mathbf{R}_{ik}) \cdot (\mathrm{d}\mathbf{r}_i + \mathrm{d}\boldsymbol{\rho}') ; \end{aligned}$$

hence,

$$\mathrm{d}W' + \mathrm{d}W'_{\mathrm{int}} + \mathrm{d}W'_R + \mathrm{d}W'_{R\mathrm{int}} = \mathrm{d}W^{(C)} + \mathrm{d}W^{(C)}_{\mathrm{int}} + \mathrm{d}W^{(C)}_R + \mathrm{d}W^{(C)}_{R\mathrm{int}} + (\mathbf{R} + \overline{\mathbf{R}}) \cdot \mathrm{d}\boldsymbol{\rho}' \tag{11.2.5}$$

and we can state (we denote by upper index the elementary work with respect to a Koenig frame of reference)

Theorem 11.2.3 (*of Koenig type*). *The elementary work of the given and constraint, external and internal forces which act upon a discrete mechanical system, with respect to a fixed frame of reference, is equal to the sum of the elementary work of the same forces, with respect to a Koenig frame, and the elementary work of the given and constraint external forces, considered to be applied at the mass centre, with respect to the fixed frame.*

From (11.2.5) we notice that we can write

$$\begin{gathered} \mathrm{d}W' = \mathrm{d}W^{(C)} + \mathbf{R} \cdot \mathbf{v}'_C \mathrm{d}t , \quad \mathrm{d}W'_R = \mathrm{d}W^{(C)}_R + \overline{\mathbf{R}} \cdot \mathbf{v}'_C \mathrm{d}t , \\ \mathrm{d}W'_{\mathrm{int}} = \mathrm{d}W^{(C)}_{\mathrm{int}} , \quad \mathrm{d}W'_{R\mathrm{int}} = \mathrm{d}W^{(C)}_{R\mathrm{int}} . \end{gathered} \tag{11.2.5'}$$

Hence, the elementary work of the external (given and constraint) forces is not the same with respect to an inertial (fixed) frame of reference and to a non-inertial (movable) Koenig frame, its variation depending on the resultant of the external (given and constraint) forces; this elementary work (e.g., of the given external forces) remains invariant if and only if the scalar product $\mathbf{R} \cdot \mathbf{v}'_C = 0$, hence if the resultant $\mathbf{R} = \mathbf{0}$ or if $\mathbf{v}'_C = \mathbf{0}$ (the centre of mass is rigidly linked – at rest – with respect to the fixed frame) or if $\mathbf{v}'_C \perp \mathbf{R}$ (the resultant of the given external forces is contained, at any moment, in a plane normal to the trajectory of the centre of mass). In case of catastatic external constraints, the elementary work of the constraint external forces with respect to an inertial frame (in the absolute motion) vanishes ($\mathrm{d}W'_R = 0$); from (11.2.5') it results that the elementary work with respect to a non-inertial Koenig frame (in the relative motion) is, in general, non-zero. This elementary work remains invariant only if $\mathbf{R} \cdot \mathbf{v}'_C = 0$ (which has an analogous mechanical interpretation as above). In exchange, the elementary work of the internal forces (for which the resultant vanishes) remains always invariant by passing from an inertial frame to a non-inertial Koenig frame.

We notice that the Theorem 3.1.4 (Huygens-Steiner) is of the same type as Koenig's theorems. As well, the formulae (3.1.9) and (11.2.2) are of the same type, the corresponding quantities being equal to zero with respect to a Koenig frame.

11.2.1.2 General Theorems with Respect to a Koenig Frame. Conservation Theorems

Passing from a given frame of reference $\mathscr{R}'$ to a Koenig frame $\mathscr{R}$, the sum of the given and constraint external forces

$$\mathbf{R}' + \bar{\mathbf{R}}' = \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i) = \mathbf{R} + \bar{\mathbf{R}} \tag{11.2.6}$$

remains invariant; assuming that the frame $\mathscr{R}'$ is inertial and using the formula (11.2.2), the Theorem 11.1.21 of the momentum allows to find again the Theorem 11.1.22 of motion of the centre of mass.

As well, the resultant moment of the given and constraint external forces is expressed in the form

$$\begin{aligned}\mathbf{M}'_{O'} + \bar{\mathbf{M}}'_{O'} &= \sum_{i=1}^{n} \mathbf{r}'_i \times (\mathbf{F}_i + \mathbf{R}_i) = \sum_{i=1}^{n} (\mathbf{r}_i + \boldsymbol{\rho}') \times (\mathbf{F}_i + \mathbf{R}_i) \\ &= \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{F}_i + \mathbf{R}_i) + \boldsymbol{\rho}' \times \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i),\end{aligned}$$

so that

$$\mathbf{M}'_{O'} + \bar{\mathbf{M}}'_{O'} = \mathbf{M}_C + \bar{\mathbf{M}}_C + \boldsymbol{\rho}' \times \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i), \tag{11.2.6'}$$

hence a formula of the form (2.2.27) (a result of the same form as the theorems of Koenig type). In this case, assuming – further – that the frame $\mathscr{R}'$ is inertial and taking into account the Theorem 11.2.1, the Theorem 11.1.23 of the moment of momentum leads to (without distinction between the absolute and the relative derivatives, because $\boldsymbol{\omega} = \mathbf{0}$)

$$\dot{\mathbf{K}}'_{O'} = \dot{\mathbf{K}}_C^{(C)} + \dot{\boldsymbol{\rho}}' \times (M\dot{\boldsymbol{\rho}}') + \boldsymbol{\rho}' \times (M\ddot{\boldsymbol{\rho}}') = \mathbf{M}_C + \bar{\mathbf{M}}_C + \boldsymbol{\rho}' \times \sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i);$$

using the theorem of motion of the centre of mass, we obtain, finally,

$$\dot{\mathbf{K}}_C^{(C)} = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.7}$$

so that we state

Theorem 11.2.4 (*S. Koenig*). *The theorem of moment of momentum remains invariant by passing from an inertial* (*fixed*) *frame of reference to a non-inertial* (*movable*) *Koenig frame.*

Taking into account the Theorems 11.2.2 and 11.2.3 and assuming that the frame $\mathscr{R}'$ is inertial, the Theorem 11.1.25 of the kinetic energy allows to write (we have, as well, $\mathrm{d}T^{(C)}/\mathrm{d}t = \partial T^{(C)}/\partial t$; an upper index specifies the kinetic energy with respect to a Koenig frame)

$$\begin{aligned}\mathrm{d}T' &= \mathrm{d}T^{(C)} + M\dot{\boldsymbol{\rho}}' \cdot \mathrm{d}\dot{\boldsymbol{\rho}}' = \mathrm{d}T^{(C)} + M\ddot{\boldsymbol{\rho}}' \cdot \mathrm{d}\boldsymbol{\rho}' \\ &= \mathrm{d}W^{(C)} + \mathrm{d}W_{\mathrm{int}}^{(C)} + \mathrm{d}W_R^{(C)} + \mathrm{d}W_{R\mathrm{int}}^{(C)} + \left[\sum_{i=1}^{n} (\mathbf{F}_i + \mathbf{R}_i)\right] \cdot \mathrm{d}\boldsymbol{\rho}',\end{aligned}$$

so that (we use, further, the theorem of motion of the centre of mass)

$$\mathrm{d}T^{(C)} = \mathrm{d}W^{(C)} + \mathrm{d}W_{\mathrm{int}}^{(C)} + \mathrm{d}W_R^{(C)} + \mathrm{d}W_{R\mathrm{int}}^{(C)}, \tag{11.2.8}$$

and we can state

Theorem 11.2.5 (*S. Koenig*). *The theorem of kinetic energy remains invariant by passing from an inertial* (*fixed*) *frame of reference to a non-inertial* (*movable*) *Koenig frame.*

The Theorems 11.2.4 and 11.2.5 represent a vector and a scalar condition, respectively, hence four necessary scalar conditions which must be verified in the motion of a discrete mechanical system $\mathscr{S}$ with respect to a frame of Koenig; these conditions are not independent of the seven conditions (the three general theorems), written with respect to an inertial frame. Obviously, in this case too, all considerations made for the motion with respect to an inertial frame hold. We can thus obtain a vector first integral (a conservation theorem of moment of momentum) and a scalar first integral (a conservation theorem of mechanical energy) in the motion with respect to the centre of mass, hence four scalar first integrals.

We notice that we can state a theorem of areal velocities and a theorem of areas with respect to a Koenig frame; as well, one can write a formula of the form (11.1.32"). One can thus explain the jump of a swimmer from the jumping board (to attain the water in vertical position, he is varying – by adequate motions – his moment of inertia, hence his angular velocity with respect to a horizontal axis – the axis of a Koenig frame – which passes through his centre of mass), the jump of the skier from the spring-board (to reach the ground in the desired position), the jump of a gymnast on the ground, the fact that a cat attains the ground on its feet, anyhow it falls, the motions which instinctively we make by the hands to straighten ourselves in case of a wrong step etc.; in all these cases, the weight of the considered body is a force passing through the mass centre. We mention also the manual manoeuvres made by the cosmonauts to direct conveniently the space vehicles; in this case, the Newtonian force of attraction passes also through the mass centre of the cabin. As a matter of fact, a general problem of the type schematized in Fig.11.2 can be considered with respect to a Koenig frame too.

If a mechanical system $\mathscr{S}$ is subjected to the action of a uniform gravitational field (e.g., a heavy bar, homogeneous or not, launched in the vicinity of the Earth, neglecting the resistance of the air), then its centre of mass describes a parabola; it remains to study the relative motion of the system $\mathscr{S}$ (with respect to its mass centre). We can write a conservation theorem of the moment of momentum ($\mathbf{K}_C^{(C)} = \mathbf{C}$); assuming that the system $\mathscr{S}$ is non-deformable, a relation of the form (11.1.32") with respect to a fixed axis through the centre of mass, along the direction of the constant $\mathbf{C}$, takes place, this system having a motion of rotation about the respective axis (in particular, the bar, considered to be rigid, is rotating in the plane of maximum of areas, which is normal to $\mathbf{C}$ and which passes through the centre of mass). Indeed, if the direction of the bar is of unit vector $\mathbf{u}$, then the velocity of each point of it is along that unit vector, so that $I\mathbf{u} \times \dot{\mathbf{u}} = \mathbf{C}$, where I is the moment of inertia with respect to the fixed axis, resulting $\mathbf{C} \cdot \mathbf{u} = 0$. If, corresponding to the initial conditions, we have $\mathbf{C} = \mathbf{0}$, then the mechanical system $\mathscr{S}$ has a motion of translation with respect to the inertial frame, being at rest with respect to a Koenig frame. We notice that the external work of the gravity forces with respect to a Koenig frame vanishes

$$\mathrm{d}W^{(C)} = -\sum_{i=1}^{n} m_i g \mathrm{d}x_3^{(i)} = -g\sum_{i=1}^{n} m_i \mathrm{d}x_3^{(i)} = -g\mathrm{d}\sum_{i=1}^{n} m_i x_3^{(i)} = 0\,,$$

assuming that the Cx_3-axis is along the ascendent vertical; in this case, the relation (11.2.8) reads (the discrete mechanical system $\mathscr{S}$ being free, we have $\mathrm{d}W_R^{(C)} = \mathrm{d}W_{R\mathrm{int}}^{(C)} = 0$)

$$\mathrm{d}T^{(C)} = \mathrm{d}W_{\mathrm{int}}^{(C)}\,. \tag{11.2.8'}$$

If the mechanical system is non-deformable, it results $T^{(C)} = \mathrm{const}$, hence a conservation theorem of the kinetic energy with respect to a Koenig frame.

Assuming that the solar system is isolated, it results that its moment of momentum with respect to the mass centre, situated in the neighbourhood of the mass centre of the Sun, is constant in time ($\mathbf{K}_C^{(C)} = \mathbf{C}$); the plane of maximum of areas, normal to the constant $\mathbf{C}$, is an *invariable plane* for the motions in the interior of this system. Laplace determined this plane, calculating the components of the moment of momentum $\mathbf{K}_C^{(C)}$. He modelled the planets as particles reduced to their centres of mass; Poinsot completed this computation, introducing also the influence – in fact, negligible – of the terms provided by the proper rotation of each planet. These conclusions remain valid even if the influences external to the solar system are not neglected. Indeed, the distances from other stars to various particles which form the solar system are very great so that the external forces of Newtonian attraction which act upon these particles and are in direct proportion to their masses form, with a good approximation, a system of parallel forces, their resultant moment with respect to the centre of mass vanishing.

11.2.1.3 Problem of n Particles

Returning to the problem of n particles (see Sect. 1.2.8 too), we will consider the motion of a discrete mechanical system with respect to the mass centre C. For $n = 2$, the velocity of the centre of mass with respect to an inertial frame of reference is given by

$$\mathbf{v}'_C = \frac{\mathbf{H}'}{M + m} = \frac{M\mathbf{V}' + m\mathbf{v}'}{M + m} = \frac{M\mathbf{V}'_0 + m\mathbf{v}'_0}{M + m},$$

where $\mathbf{V}'$ is the velocity of the Sun S, of mass M, while $\mathbf{v}'$ is the velocity of a planet P, of mass m, with respect to the same frame; taking into account the conservation theorem of momentum, we have put in evidence the velocities at the initial moment. In this case, the velocities of the two particles with respect to a Koenig frame are given by

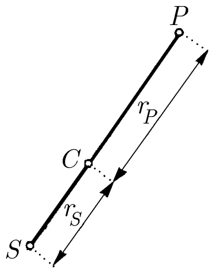

Fig. 11.7 Problem of two particles

$$\mathbf{V} = \mathbf{V}' - \mathbf{v}'_C = \frac{m(\mathbf{V}' - \mathbf{v}')}{M + m}, \quad \mathbf{v} = \mathbf{v}' - \mathbf{v}'_C = \frac{M(\mathbf{v}' - \mathbf{V}')}{M + m}; \tag{11.2.9}$$

these velocities have opposite directions at any moment (inclusive at the initial moment). We notice that the mass centre C is on the segment of a line SP of length $\overline{SP} = r$, at the distances r_S and r_P from the centres of the particles S and P, respectively, so that (Fig. 11.7)

$$Mr_S = mr_P, \quad r_S + r_P = r = \frac{M + m}{m} r_S = \frac{M + m}{M} r_P. \tag{11.2.9'}$$

The equation of relative motion of the particle P reads

$$m\ddot{\mathbf{r}}_P = -f\frac{mM}{r^2}\operatorname{vers}\mathbf{r}_P = -f\frac{mM^3}{(M+m)^2 r_P^2}\mathbf{r}_P\,;$$

hence, the particle P moves with respect to the Koenig frame as if it would be gravitationally attracted by the mass centre C, at which would be an attractive mass equal to $M^3/(M+m)^2 = M/(1+m/M)^2$, describing a conic $\mathscr{C}_P^{(C)}$ after the law of areas, the centre C being at one of the foci. The genus of the conic is specified by the relations

$$r_P^0\frac{M^2}{(M+m)^2}(\mathbf{V}_0 - \mathbf{v}_0)^2 \lesseqqgtr 2f\frac{M^3}{(M+m)^2}\,,$$

where we have put in evidence the position of the particle P at the initial moment; taking into account (11.2.9') and the relation $\mathbf{v}_S - \mathbf{v}_P = \mathbf{V} - \mathbf{v}$, we find again the conditions which specified the genus of the conics $\mathscr{C}_P$ and $\mathscr{C}_S$ in Sect. 1.2.8. Analogously, one can also show that the particle S describes a conic $\mathscr{C}_S^{(C)}$ too, after the law of areas, with respect to the mass centre, that one being at one of the foci, its genus being given by the same relations. These conics are situated in the plane determined by the straight line P_0S_0 and by the vector $\mathbf{V} - \mathbf{v}$, being thus coplanar with the conics $\mathscr{C}_P$ and $\mathscr{C}_S$. Taking into account the first relation (11.2.9'), we notice that the two conics $\mathscr{C}_P^{(C)}$ and $\mathscr{C}_S^{(C)}$ are obtained one of the other by a transformation of similitude (the ratio $r_S/r_P = \text{const}$), the centre of similitude being C. We notice that the areal velocity is constant ($\mathbf{\Omega}_P = \mathbf{r}_P \times \mathbf{v}/2 = \mathbf{c}, \mathbf{\Omega}_S = \mathbf{r}_S \times \mathbf{V}/2 = \mathbf{C}$) in the motion of each particle; the plane in which are both trajectories is normal to a fixed direction, specified by $\mathbf{c} \parallel \mathbf{C}$, and is – in fact – the plane of the maximum of areas. Finally, the two particles S and P move so that their mass centre C describes uniformly a fixed straight line, while the line SP is rotating about C in a plane of fixed orientation; the particles S and P describe conics in this plane, obtained one of the other by similitude and having one of the foci at C.

The period of revolution of a particle P in motion with respect to a Koenig frame is given by

$$T = 2\pi a\sqrt{\frac{a}{f}}\frac{M+m}{M\sqrt{M}} = 2\pi a\sqrt{\frac{a}{fM}}\left(1+\frac{m}{M}\right) = k\left(1+\frac{m}{M}\right), \quad k = \text{const}\,. \tag{11.2.9''}$$

Let us consider, at the atomic level, the motion of an electron about the nucleus; in case of an atom of hydrogen we have $m/M = 1/1850$, while in case of an atom of ionized helium $m/M = 1/7400$. Observing that the wave length is in direct proportion to the period of revolution T, we can write for the corresponding wave lengths

$$\frac{\lambda_{\mathrm{He}}}{\lambda_{\mathrm{H}}} = \left(1 + \frac{1}{7400}\right) \Big/ \left(1 + \frac{1}{1850}\right) = 1 - \frac{1}{2468} \cong 1 - \frac{1}{2500};$$

experimentally, one obtains $\lambda_{\mathrm{He}} = 6560.2$ Å and $\lambda_{\mathrm{H}} = 6562.8$ Å, so that $\lambda_{\mathrm{He}} / \lambda_{\mathrm{H}} = 6560.2/6562.8 \cong 1 - 1/2524 \cong 1 - 1/2500$, this result being in good concordance with the theoretical one.

If $n > 3$, then one can follow the considerations in Sect. 1.2.8.

11.2.2 Motion of a Discrete Mechanical System with Respect to an Arbitrary Non-inertial Frame of Reference

The results obtained in Sects. 11.2.1.1 and 11.2.1.2 will be generalized, assuming that the movable frame of reference $\mathscr{R}$ has a pole other than the centre of mass of the free discrete mechanical system $\mathscr{S}$ or/and assuming that the movable frame has a motion of rotation ($\boldsymbol{\omega} \neq \mathbf{0}$) with respect to the fixed frame $\mathscr{R}'$. The quantities calculated with respect to the inertial frame $\mathscr{R}'$ will be denoted by "prime", while to the quantities calculated with respect to the non-inertial frame $\mathscr{R}$ will not be given such a specification (Fig. 11.8).

11.2.2.1 Momentum and Moment of Momentum with Respect to an Arbitrary Frame of Reference

We can write

$$\mathbf{r}_i' = \mathbf{r}_O' + \mathbf{r}_i, \quad i = 1,2,\ldots,n, \tag{11.2.10}$$

for a particle P_i, wherefrom, differentiating with respect to the frame of reference $\mathscr{R}'$, we have

$$\mathbf{v}_i' = \mathbf{v}_O' + \mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i, \quad i = 1,2,\ldots,n. \tag{11.2.10'}$$

Multiplying by the mass m_i and summing for all the particles of the system $\mathscr{S}$, we obtain

$$\mathbf{H}' = \mathbf{H} + M(\mathbf{v}_O' + \boldsymbol{\omega} \times \boldsymbol{\rho}), \tag{11.2.11}$$

$\boldsymbol{\rho}$ being the position vector of the mass centre in the frame $\mathscr{R}$; we state thus

Theorem 11.2.6 (*V. Vâlcovici*). *The momentum of a discrete mechanical system with respect to a fixed frame of reference is equal to the sum of the momentum of this system with respect to an arbitrary frame and the momentum of the mass centre of the system, considered to be at rest with respect to the latter frame, assuming that its whole mass is concentrated at this centre, with respect to the fixed frame.*

Applying the theorem of momentum with respect to the inertial frame and the formula (A.2.37), we can write

$$\mathbf{A}' = \frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} = \frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\omega}\times\mathbf{H} + M\mathbf{a}'_O + M\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} + M\boldsymbol{\omega}\times\frac{\partial\boldsymbol{\rho}}{\partial t}$$
$$+ M\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) = \mathbf{R}' + \bar{\mathbf{R}}', \qquad (11.2.11')$$

where $\mathbf{R}' = \mathbf{R}$ and $\bar{\mathbf{R}}' = \bar{\mathbf{R}}$ are the resultants of the given and constraint external forces, respectively; both resultants are invariant by a change of frame. We introduce the dynamic resultant (11.1.66) with respect to the non-inertial frame $\mathscr{R}$ ($\mathbf{H} = M\partial\boldsymbol{\rho}/\partial t$)

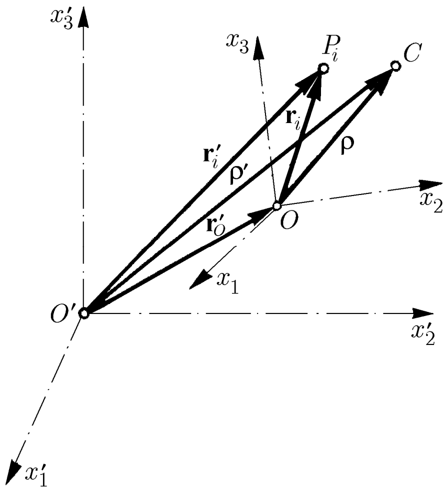

Fig. 11.8 Motion with respect to an arbitrary non-inertial frame of reference

$$\mathbf{A} = \frac{\partial \mathbf{H}}{\partial t} = \mathbf{R} + \bar{\mathbf{R}} + \mathbf{F}_t^{(C)} + \mathbf{F}_C^{(C)}, \qquad (11.2.12)$$

where the complementary forces (the transportation force and the Coriolis force) are given by

$$\mathbf{F}_t^{(C)} = -M[\mathbf{a}'_O + \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})], \quad \mathbf{F}_C^{(C)} = -2M\boldsymbol{\omega}\times\frac{\partial\boldsymbol{\rho}}{\partial t}, \qquad (11.2.12')$$

corresponding to the centre of mass (which plays thus an important rôle), where we consider that the whole mass of the mechanical system $\mathscr{S}$ is concentrated. One can obtain this result starting from the equations of motion written for a non-inertial frame (in relative motion)

$$m_i\frac{\partial \mathbf{H}_i}{\partial t} = m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{R}_i + \mathbf{F}_t^{(i)} + \mathbf{F}_C^{(i)}, \quad i = 1,2,...,n, \qquad (11.2.13)$$

where

$$\mathbf{F}_t^{(i)} = -m_i\left[\mathbf{a}'_O + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)\right], \quad \mathbf{F}_C^{(i)} = -2m_i \boldsymbol{\omega} \times \mathbf{v}_i, \tag{11.2.13'}$$

and summing for all the particles of the mechanical system $\mathscr{S}$.

In the particular case in which the frame $\mathscr{R}$ does not rotate (moves with the axes parallel to themselves) we have $\boldsymbol{\omega} = \mathbf{0}$ and it results

$$\mathbf{H}' = \mathbf{H} + M\mathbf{v}'_O, \quad \mathbf{F}_t = -M\mathbf{a}'_O, \quad \mathbf{F}_C = \mathbf{0}; \tag{11.2.11''}$$

as well, in the case in which the frame $\mathscr{R}$ has its pole at the centre of mass ($O \equiv C$, $\boldsymbol{\rho} = \mathbf{0}$), we can write

$$\mathbf{H}' = M\mathbf{v}'_O = M\mathbf{v}'_C, \quad \mathbf{F}_t = -M\mathbf{a}'_O, \quad \mathbf{F}_C = \mathbf{0}, \tag{11.2.11'''}$$

even if $\boldsymbol{\omega} \neq \mathbf{0}$.

Taking into account the relation (11.2.10'), we have

$$\mathbf{v}'_C = \mathbf{v}'_O + \mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\rho}, \tag{11.2.14}$$

wherefrom we obtain

$$\mathbf{a}'_C = \mathbf{a}'_O + \mathbf{a}_C + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + 2\boldsymbol{\omega} \times \mathbf{v}_C; \tag{11.2.14'}$$

observing that $\partial \mathbf{H} / \partial t = M\mathbf{a}_C$, the equation (11.2.12), (11.2.12') takes the remarkable form

$$M\mathbf{a}'_C = \mathbf{R} + \overline{\mathbf{R}}, \tag{11.2.15}$$

corresponding to the motion of the centre of mass.

A vector product at the left of the relation (11.2.10') by $\mathbf{r}'_i$ leads to

$$\begin{aligned}\mathbf{r}'_i \times \mathbf{v}'_i &= (\mathbf{r}'_O + \mathbf{r}_i) \times (\mathbf{v}'_O + \mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}'_O \times \mathbf{v}'_O + \mathbf{r}'_O \times \mathbf{v}_i \\ &+ \mathbf{r}'_O \times (\boldsymbol{\omega} \times \mathbf{r}_i) - \mathbf{v}'_O \times \mathbf{r}_i + \mathbf{r}_i \times \mathbf{v}_i + \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i),\end{aligned}$$

where we took into account (11.2.10); multiplying by m_i and summing for all the particles of the discrete mechanical system $\mathscr{S}$, it results

$$\mathbf{K}'_{O'} = \overline{\mathbf{K}}_O + \mathbf{r}'_O \times (M\mathbf{v}'_C) + \boldsymbol{\rho} \times (M\mathbf{v}'_O), \tag{11.2.16}$$

where we considered (11.2.14) and have introduced a quantity of the nature of a moment of momentum

$$\overline{\mathbf{K}}_O = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}. \tag{11.2.16'}$$

We notice that $\overline{\mathbf{K}}_O$ does not represent the moment of momentum of the discrete mechanical system $\mathscr{S}$ with respect to the pole O of the non-inertial frame $\mathscr{R}$, taken with respect to the inertial frame $\mathscr{R}'$. But $\overline{\mathbf{K}}_O$ represents the moment of momentum of the discrete mechanical system $\mathscr{S}$ with respect to the pole O, in a non-inertial frame $\overline{\mathscr{R}}$, with the pole at the same point O and the axes $O\overline{x}_i$ parallel to the axes $O'x_i'$, $i = 1,2,3$, of the inertial frame $\mathscr{R}'$ (because the frame $\overline{\mathscr{R}}$ does not rotate with respect to the frame $\mathscr{R}'$, the derivative with respect to time remains invariant and $\mathbf{v}_i = \mathrm{d}\overline{\mathbf{r}}_i / \mathrm{d}t$); this quantity takes a remarkable form by introducing the tensor of inertia. Thus, observing that

$$\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_i)\mathbf{r}_i = \left(x_l^{(i)} x_l^{(i)} \delta_{jk} - x_j^{(i)} x_k^{(i)} \right) \omega_k \mathbf{i}_j ,$$

multiplying by m_i, summing for all the particles of the discrete mechanical system $\mathscr{S}$ and introducing the tensor of inertia defined by the relation (3.1.81), we find a relation of the form (3.1.83) for a quantity $\mathbf{K}^O$ of the nature of a moment of momentum, which will be called *pseudomoment of momentum* of the discrete mechanical system $\mathscr{S}$ with respect to the pole O of the non-inertial frame $\mathscr{R}$, i.e.

$$\mathbf{K}^O = \mathbf{I}_O \boldsymbol{\omega} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) , \tag{11.2.17}$$

where we have introduced the contracted product of the tensor of inertia at the pole O by the vector angular velocity; it results

$$\overline{\mathbf{K}}_O = \mathbf{K}_O + \mathbf{K}^O , \tag{11.2.17'}$$

where

$$\mathbf{K}_O = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \mathbf{v}_i \tag{11.2.17''}$$

is the moment of momentum of the discrete mechanical system $\mathscr{S}$ with respect to the pole O of the non-inertial frame $\mathscr{R}$. We state

Theorem 11.2.7 (*V. Vâlcovici, C. Iacob*). *The moment of momentum of a discrete mechanical system with respect to a pole O' of a given inertial frame of reference $\mathscr{R}'$, in this frame, is equal to the sum of the moment of momentum of this system with respect to an arbitrary pole O of a non-inertial frame $\overline{\mathscr{R}}$* (*which does not rotate with respect to the frame $\mathscr{R}'$*), *in the latter frame* (*the sum of the moment of momentum of the system with respect to the pole O in a non-inertial frame $\mathscr{R}$, with the pole at the very same point, and the contracted product of the tensor of inertia with respect to the* *same pole by the angular velocity vector of the non-inertial frame $\mathscr{R}$ with respect to the inertial frame $\mathscr{R}'$), the moment of momentum of the mass centre, translated at the pole O, at which is considered to be concentrated the whole mass of the system $\mathscr{S}$, with respect to the pole O', in the frame $\mathscr{R}'$, and the moment of momentum of the pole O, translated at the centre of mass at which is assumed to be concentrated the whole mass of the system $\mathscr{S}$, calculated with respect to the pole O, in the same inertial frame $\mathscr{R}'$.*

Differentiating the relation (11.2.16) with respect to time in the fixed frame, we can write

$$\frac{\mathrm{d}\mathbf{K}'_{O'}}{\mathrm{d}t} = \frac{\mathrm{d}\bar{\mathbf{K}}_O}{\mathrm{d}t} + M[\mathbf{v}'_O \times \mathbf{v}'_C + \mathbf{r}'_O \times \mathbf{a}'_C + \mathbf{v}_C \times \mathbf{v}'_O + (\boldsymbol{\omega} \times \boldsymbol{\rho}) \times \mathbf{v}'_O + \boldsymbol{\rho} \times \mathbf{a}'_O];$$

observing that

$$\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{r}'_O \times \mathbf{R}, \quad \bar{\mathbf{M}}_{O'} = \bar{\mathbf{M}}_O + \mathbf{r}'_O \times \bar{\mathbf{R}}$$

for the moments of the given and constraint external forces and taking into account (11.2.14), (11.2.15), V. Vâlcovici showed that the theorem of moment of momentum becomes

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) + \frac{\mathrm{d}\bar{\mathbf{K}}_O}{\mathrm{d}t} = \mathbf{M}_O + \bar{\mathbf{M}}_O, \tag{11.2.18}$$

where

$$\frac{\mathrm{d}\bar{\mathbf{K}}_O}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{K}_O}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O\boldsymbol{\omega}) = \frac{\partial \mathbf{K}_O}{\partial t} + \boldsymbol{\omega} \times \mathbf{K}_O + \frac{\partial \mathbf{I}_O}{\partial t}\boldsymbol{\omega} + \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}). \tag{11.2.18'}$$

Observing that

$$\begin{aligned}&\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) + \mathbf{v}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = 2(\mathbf{r}_i \cdot \mathbf{v}_i)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{v}_i)\mathbf{r}_i - (\boldsymbol{\omega} \cdot \mathbf{r}_i)\mathbf{v}_i \\ &= \left(2x_l^{(i)}\dot{x}_l^{(i)}\delta_{jk} - x_j^{(i)}\dot{x}_k^{(i)} - \dot{x}_j^{(i)}x_k^{(i)}\right)\omega_k\mathbf{i}_j = \frac{\partial}{\partial t}\left(x_l^{(i)}x_l^{(i)}\delta_{jk} - x_j^{(i)}x_k^{(i)}\right)\omega_k\mathbf{i}_j,\end{aligned}$$

multiplying by m_i, summing for all the particles of the discrete mechanical system $\mathscr{S}$ and introducing the tensor of inertia, we find the remarkable relation

$$\sum_{i=1}^{n} m_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) + \sum_{i=1}^{n} m_i\mathbf{v}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \frac{\partial \mathbf{I}_O}{\partial t}\boldsymbol{\omega}, \tag{11.2.18''}$$

corresponding to the relation (11.2.17).

Starting from the equations of motion (11.2.13), (11.2.13'), we effect a vector product at the left by $\mathbf{r}_i$ and sum for all the particles of the system $\mathscr{S}$; we get thus

$$\frac{\partial \mathbf{K}_O}{\partial t} = \mathbf{M}_O + \bar{\mathbf{M}}_O + \sum_{i=1}^{n} \mathbf{r}_i \times \left(\mathbf{F}_t^{(i)} + \mathbf{F}_C^{(i)} \right). \tag{11.2.18'''}$$

We notice that

$$\mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}) + \boldsymbol{\rho} \times (M \mathbf{a}'_O) = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_i)$$
$$+ \boldsymbol{\omega} \times \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \sum_{i=1}^{n} m_i \mathbf{r}_i \times \mathbf{a}'_O = -\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_t^{(i)},$$

because

$$\boldsymbol{\omega} \times [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] = -(\boldsymbol{\omega} \cdot \mathbf{r}_i)(\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)],$$

as well as (we use the relation of definition (3.1.81))

$$\boldsymbol{\omega} \times \mathbf{K}_O + \frac{\partial \mathbf{I}_O}{\partial t} \boldsymbol{\omega} = \boldsymbol{\omega} \times \sum_{i=1}^{n} m_i \mathbf{r}_i \times \mathbf{v}_i + \frac{\partial I_{jk}}{\partial t} \omega_k \mathbf{i}_j = \in_{jkl} \in_{lmn} \omega_k \sum_{i=1}^{n} m_i x_m^{(i)} v_n^{(i)} \mathbf{i}_j$$
$$+ \omega_k \frac{\partial}{\partial t} \sum_{i=1}^{n} m_i \left(x_p^{(i)} x_p^{(i)} \delta_{jk} - x_j^{(i)} x_k^{(i)} \right) \mathbf{i}_j = \left(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right) \omega_k \sum_{i=1}^{n} m_i x_m^{(i)} v_n^{(i)} \mathbf{i}_j$$
$$+ \omega_k \sum_{i=1}^{n} m_i \left(2 x_p^{(i)} v_p^{(i)} \delta_{jk} - v_j^{(i)} x_k^{(i)} - x_j^{(i)} v_k^{(i)} \right) \mathbf{i}_j = 2 \sum_{i=1}^{n} m_i \left(x_p^{(i)} v_p^{(i)} \omega_j - x_k^{(i)} v_j^{(i)} \omega_k \right) \mathbf{i}_j$$
$$= 2 \sum_{i=1}^{n} m_i [(\mathbf{r}_i \cdot \mathbf{v}_i) \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega}) \mathbf{v}_i] = 2 \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{v}_i) = -\sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_C^{(i)};$$

one can thus state that the relations (11.2.18) and (11.2.18''') are equivalent.

One can easily see that the theorem of moment of momentum (11.2.18) reads

$$\boldsymbol{\rho} \times (M \mathbf{a}'_O) + \frac{\mathrm{d} \mathbf{K}_O}{\mathrm{d} t} = \mathbf{M}_O + \bar{\mathbf{M}}_O \tag{11.2.18$^{\text{iv}}$}$$

if we have $\mathbf{K}^O = \mathbf{I}_O \boldsymbol{\omega} = \overrightarrow{\text{const}}$ with respect to the inertial frame $\mathscr{R}'$.

In the particular case in which the frame $\mathscr{R}$ does not rotate ($\boldsymbol{\omega} = \mathbf{0}$, hence $\bar{\mathbf{K}}_O = \mathbf{K}_O$), it results the formula

$$\mathbf{K}'_{O'} = \mathbf{K}_O + \mathbf{r}'_O \times (M \mathbf{v}'_C) + \boldsymbol{\rho} \times (M \mathbf{v}'_O), \tag{11.2.19}$$

obtained by V. Vâlcovici as a particular case of the Theorem 11.2.7; the theorem of moment of momentum becomes

$$\boldsymbol{\rho} \times (M \mathbf{a}'_O) + \frac{\partial \mathbf{K}^O}{\partial t} = \mathbf{M}_O + \bar{\mathbf{M}}_O \tag{11.2.19'}$$

and we can state

Theorem 11.2.8 (*V. Vâlcovici*). *The theorem of moment of momentum of a free discrete mechanical system maintains its form with respect to a non-inertial frame of reference which does not rotate* (*it moves with the axes parallel to themselves*) *if, from the moment of the given and constraint external forces, we subtract the dynamic moment* (*with respect to an inertial frame*) *of the pole of the non-inertial considered frame, translated at the centre of mass of the system, at which is assumed that is concentrated the whole mass of the system, with respect to this pole.*

One can thus see that the theorem of moment of momentum of a free or constraint discrete mechanical system maintains its form with respect to a non-inertial frame of reference which does not rotate if and only if $\boldsymbol{\rho} \times \mathbf{a}'_O = \mathbf{0}$, hence in one of the following cases: (i) the pole of the non-inertial frame has a uniform and rectilinear motion with respect to an inertial frame, the movable frame being thus inertial too; (ii) the non-inertial frame is a Koenig frame ($\boldsymbol{\rho} = \mathbf{0}$, hence $O \equiv C$, Theorem 11.2.4); (iii) the support of the acceleration $\mathbf{a}'_O$ of the pole of the non-inertial frame, with respect to an inertial frame, passes always through the centre of mass C. As a matter of fact, the case (ii) is a particular case of (iii).

We say that a non-inertial frame $\mathscr{R}$ is a *frame of Koenig type* if $\boldsymbol{\omega} = \mathbf{0}$ but its pole is not the mass centre; in this case, the frame $\mathscr{R}$ is just such a frame. As it was shown by V. Vâlcovici, to can write a *formula of Koenig type*

$$\mathbf{K}'_{O'} = \mathbf{K}_O + \mathbf{r}'_O \times (M\mathbf{v}'_O) \tag{11.2.20}$$

it is necessary and sufficient that (we notice $\mathbf{v}'_C = \mathbf{v}'_O + \mathbf{v}_C$)

$$\mathbf{r}'_O \times (M\mathbf{v}_C) + \boldsymbol{\rho} \times (M\mathbf{v}'_O) = \mathbf{0}\,. \tag{11.2.20'}$$

In the particular case in which the pole of the non-inertial frame coincides with the centre of mass of the system ($O \equiv C$, $\boldsymbol{\rho} = \mathbf{0}$), we obtain a formula of Koenig type

$$\mathbf{K}'_{O'} = \bar{\mathbf{K}}_C + \boldsymbol{\rho}' \times (M\mathbf{v}'_C)\,, \quad \bar{\mathbf{K}}_C = \mathbf{K}_C + \mathbf{K}^C\,, \tag{11.2.21}$$

and the theorem of moment of momentum reads

$$\frac{\mathrm{d}\bar{\mathbf{K}}_C}{\mathrm{d}t} = \mathbf{M}_C + \bar{\mathbf{M}}_C\,; \tag{11.2.21'}$$

if we have $\boldsymbol{\omega} = \mathbf{0}$ too, then the non-inertial frame $\mathscr{R}$ is a Koenig one, so that

$$\mathbf{K}'_{O'} = \mathbf{K}_C + \boldsymbol{\rho}' \times (M\mathbf{v}'_C)\,, \quad \frac{\partial \mathbf{K}_C}{\partial t} = \mathbf{M}_C + \bar{\mathbf{M}}_C\,. \tag{11.2.22}$$

Taking into account the results in Sect. 11.1.2.3, we notice that we may write $\mathbf{K}'_O = \mathbf{K}'_{O'} - \mathbf{r}'_O \times (M\mathbf{v}'_C)$; the formula (11.2.16) takes thus the form

$$\mathbf{K}'_O = \bar{\mathbf{K}}_O + \boldsymbol{\rho} \times (M\mathbf{v}'_O)\,, \tag{11.2.23}$$

the moment of momentum $\mathbf{K}'_O$ being calculated with respect to the inertial frame $\mathscr{R}'$. Analogously, we can write

$$\mathbf{K}'_O = \mathbf{K}'_C + \boldsymbol{\rho}' \times (M\mathbf{v}'_C) - \mathbf{r}'_O \times (M\mathbf{v}'_C) = \mathbf{K}'_C + \boldsymbol{\rho} \times (M\mathbf{v}'_C),$$

with respect to the inertial frame, as well as $\mathbf{K}_O = \mathbf{K}_C + \boldsymbol{\rho} \times (M\mathbf{v}_C)$, with respect to a non-inertial frame. Taking into account (3.1.116'), (11.2.14) and (11.2.17') and observing that $\mathbf{I}_O(\mathscr{S}_C) = \boldsymbol{\rho} \times (M\boldsymbol{\omega} \times \boldsymbol{\rho})$, we can write the relation (11.2.23) in the form

$$\mathbf{K}'_C = \bar{\mathbf{K}}_C = \mathbf{K}_C + \mathbf{K}^C = \mathbf{K}_C + \mathbf{I}_C(\mathscr{S})\boldsymbol{\omega}, \tag{11.2.23'}$$

stating

Theorem 11.2.9 (*C. Iacob*). *The moment of momentum of a discrete mechanical system with respect to the centre of mass, in an inertial frame of reference $\mathscr{R}'$, is equal to the moment of momentum of this system with respect to the centre of mass, calculated in a non-inertial frame $\overline{\mathscr{R}}$ of Koenig type* (*the sum of the moment of momentum of the system with respect to the centre of mass, in an arbitrary non-inertial frame $\mathscr{R}$, with the same pole as $\overline{\mathscr{R}}$, and the contracted product of the tensor of inertia, with respect to the same pole, by the angular velocity vector of the frame $\mathscr{R}$, with respect to the frame $\mathscr{R}'$*).

The privileged rôle of the centre of mass is thus put into evidence. If $\boldsymbol{\omega} = \mathbf{0}$, the arbitrary non-inertial frame $\mathscr{R}$ being in motion of translation with respect to the inertial frame, then we get

$$\mathbf{K}'_C = \mathbf{K}_C; \tag{11.2.23''}$$

this result corresponds to the relation (11.2.19) of V. Vâlcovici. If, in particular, the pole of the non-inertial frame coincides with the centre of mass, then we can write the relation

$$\mathbf{K}'_C = \mathbf{K}_C^{(C)}, \tag{11.2.23'''}$$

which corresponds to the Theorem 11.2.1 of Koenig.

If in the formula (11.1.23) we take $Q \equiv C$, then the theorem of moment of momentum reads

$$\frac{\mathrm{d}\mathbf{K}'_C}{\mathrm{d}t} = \frac{\mathrm{d}\bar{\mathbf{K}}_C}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{K}_C}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_C(\mathscr{S})\boldsymbol{\omega}) = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.24}$$

where we took into account (11.2.23'); we mention that the derivatives are calculated with respect to the inertial frame $\mathscr{R}'$. If $\boldsymbol{\omega} = \mathbf{0}$, then it results

$$\frac{\partial \mathbf{K}_C}{\partial t} = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.24'}$$

the derivative being calculated with respect to the non-inertial frame $\mathscr{R}$ with the pole at O, while if, in particular, $O \equiv C$, then we obtain

$$\frac{\partial \mathbf{K}_C^{(C)}}{\partial t} = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.24''}$$

the non-inertial frame being with the pole at the centre of mass; this last result corresponds to the Theorem 11.2.4 of Koenig.

Introducing the dynamic moment (11.1.67) and taking into account (11.1.23) and (11.1.67'), we get

$$\mathbf{D}'_O = \frac{\mathrm{d}\mathbf{K}'_O}{\mathrm{d}t} + \mathbf{v}'_O \times (M\mathbf{v}'_C). \tag{11.2.25}$$

Using the relation (11.2.23), we find again the theorem of dynamic moment in the inertial frame $\mathscr{R}'$, with respect to an arbitrary pole O, in the form

$$\mathbf{D}'_O = \mathbf{M}_O + \bar{\mathbf{M}}_O. \tag{11.2.25'}$$

Taking into account the relations (11.2.18), (11.2.18'), we can write this theorem in the non-inertial frame of reference $\mathscr{R}$ too; we obtain

$$\mathbf{D}_O + \boldsymbol{\omega} \times \mathbf{K}_O + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O \boldsymbol{\omega}) + \boldsymbol{\rho} \times (M\mathbf{a}'_O) = \mathbf{M}_O + \bar{\mathbf{M}}_O. \tag{11.2.25''}$$

If $\boldsymbol{\omega} = \mathbf{0}$, then it results

$$\mathbf{D}_O + \boldsymbol{\rho} \times (M\mathbf{a}'_O) = \mathbf{M}_O + \bar{\mathbf{M}}_O, \tag{11.2.25'''}$$

corresponding to the Theorem 11.2.8. If we have $O \equiv C$ too, then we can write

$$\mathbf{D}_C^{(C)} = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.26}$$

which corresponds to the relation (11.2.24''); starting from the relation (11.2.24'), it can be easily verified that

$$\mathbf{D}_C = \mathbf{M}_C + \bar{\mathbf{M}}_C, \tag{11.2.26'}$$

with respect to an arbitrary non-inertial frame with the pole at O. From (11.2.11') to (11.2.25') it results also a theorem of dynamic torsor in an inertial frame with respect to an arbitrary pole

$$\tau_O\{\mathbf{A}'_i\} = \tau_O\{\mathbf{F}_i\} + \tau_O\{\mathbf{R}_i\}. \tag{11.2.27}$$

We notice that, starting from the theorem of momentum, one can obtain conservation theorems (hence, first integrals) only with respect to the inertial frame $\mathscr{R}'$. Thus, if $\mathbf{R} + \bar{\mathbf{R}} = \mathbf{0}$, then from (11.2.15) it results that $\mathbf{a}'_C = \mathbf{0}$, wherefrom $\mathbf{v}'_C = \mathbf{C}$, $\boldsymbol{\rho}' = \mathbf{C}t + \mathbf{C}'$, $\mathbf{C}, \mathbf{C}' = \overrightarrow{\text{const}}$, with respect to the frame $\mathscr{R}'$.

In what concerns the theorem of moment of momentum with respect to the pole of a non-inertial frame of reference $\mathscr{R}$, one can make some interesting considerations. Thus, if $\boldsymbol{\rho} \times \mathbf{a}'_O = \mathbf{0}$, hence if the support of the acceleration of the pole O with respect to an inertial frame passes through the centre of mass of the discrete mechanical system $\mathscr{S}$ (we eliminate the trivial case in which the pole O has a uniform and rectilinear motion with respect to a frame $\mathscr{R}'$, hence the case in which the frame $\mathscr{R}$ is inertial too, as well as the case in which $O \equiv C$), and if the sum $\mathbf{M}_O + \bar{\mathbf{M}}_O = \mathbf{0}$, then $\mathbf{K}_O = \overrightarrow{\text{const}}$, the moment of momentum $\mathbf{K}_O$ (calculated with respect to the non-inertial frame $\mathscr{R}$) being conserved in time with respect to the inertial frame $\mathscr{R}'$; one obtains thus a vector first integral. If we have $\boldsymbol{\omega} = \mathbf{0}$ too (the non-inertial frame $\mathscr{R}$ does not rotate with respect to the inertial frame $\mathscr{R}'$), then $\mathbf{K}_O = \overrightarrow{\text{const}}$ (the moment of momentum with respect to the pole O, in the frame $\overline{\mathscr{R}}$, is reduced to the moment of momentum with respect to the same pole, in the frame $\mathscr{R}$) with respect to the non-inertial frame $\mathscr{R}$.

If in the relation (11.2.24) we make $\mathbf{M}_C + \bar{\mathbf{M}}_C = \mathbf{0}$, then it results $\mathbf{K}'_C = \overrightarrow{\text{const}}$, the moment of momentum $\mathbf{K}'_C$ (calculated with respect to the non-inertial frame $\mathscr{R}$) being conserved in time with respect to the inertial frame $\mathscr{R}'$; we notice that this result cannot be obtained from the above one, making $O \equiv C$, because it takes place with respect to an arbitrary non-inertial frame $\mathscr{R}$. If we have $\boldsymbol{\omega} = \mathbf{0}$ too, then $\mathbf{K}_C = \overrightarrow{\text{const}}$ with respect to the non-inertial frame $\mathscr{R}$, and if, in particular, $O \equiv C$, then $\mathbf{K}_C^{(C)} = \overrightarrow{\text{const}}$ with respect to a non-inertial frame with the pole at the centre of mass (this last first integral can be obtained from the Theorem 11.2.4 of Koenig).

As a conclusion, one can obtain a vector first integral (equivalent to three scalar first integrals) with respect to a non-inertial frame too, but which is not independent of that which can be obtained with respect to an inertial frame.

11.2.2.2 Kinetic Energy and Work with Respect to an Arbitrary Non-inertial Frame of Reference. Comoment of Two Torsors

Squaring the relation (11.2.10'), we can write

$$(\mathbf{v}'_O + \mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i)^2 = \mathbf{v}'^2_O + 2\mathbf{v}'_O \cdot (\mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i) + (\mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i)^2 .$$

Multiplying by m_i and summing for all the particles of the discrete mechanical system $\mathscr{S}$, it results (we take into account (11.2.14))

$$T' = \bar{T} + \frac{1}{2} M v_O'^2 + M \mathbf{v}_O' \cdot \mathbf{v}_C + M(\mathbf{v}_O', \boldsymbol{\omega}, \boldsymbol{\rho}) = \bar{T} - \frac{1}{2} M v_O'^2 + M \mathbf{v}_O' \cdot \mathbf{v}_C', \tag{11.2.28}$$

where we have introduced a quantity of the nature of a kinetic energy

$$\bar{T} = \frac{1}{2} \sum_{i=1}^{n} m_i (\mathbf{v}_i + \boldsymbol{\omega} \times \mathbf{r}_i)^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} \right)^2, \tag{11.2.28'}$$

which, as in the case of the moment of momentum, does not represent the kinetic energy of the discrete mechanical system $\mathscr{S}$ with respect to the non-inertial frame; indeed, the derivative of the position vector $\mathbf{r}_i$, in the non-inertial frame $\mathscr{R}$, is taken with respect to the inertial frame $\mathscr{R}'$. This quantity, which is – in fact – the kinetic energy of the discrete mechanical system $\mathscr{S}$ with respect to the non-inertial frame $\overline{\mathscr{R}}$, becomes

$$\bar{T} = T + \frac{1}{2} \boldsymbol{\omega} \cdot (\bar{\mathbf{K}}_O + \mathbf{K}_O) = T + \boldsymbol{\omega} \cdot \mathbf{K}_O + T^O, \tag{11.2.28''}$$

where

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \tag{11.2.28'''}$$

is the kinetic energy of the discrete mechanical system $\mathscr{S}$ with respect to the non-inertial frame $\mathscr{R}$, while

$$T^O = \frac{1}{2} \boldsymbol{\omega} \cdot (\mathbf{I}_O \boldsymbol{\omega}) = \frac{1}{2} I_{jk} \omega_j \omega_k = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right); \tag{11.2.29}$$

we have put thus in evidence also the representation with respect to the principal axes of inertia, that is a quantity of the nature of a kinetic energy, which we call *pseudokinetic energy*, of the discrete mechanical system $\mathscr{S}$ with respect to the non-inertial frame $\mathscr{R}$ having the pole at O. We state

Theorem 11.2.10 (*V. Vâlcovici*). *The kinetic energy of a discrete mechanical system with respect to a given inertial frame of reference $\mathscr{R}'$ is equal to the sum of the kinetic energy of this system with respect to an arbitrary non-inertial frame $\overline{\mathscr{R}}$, which does not rotate with respect to the frame $\mathscr{R}'$* (*the sum of the kinetic energy of the system with respect to a non-inertial frame $\mathscr{R}$ with the same pole O as the frame $\overline{\mathscr{R}}$, the scalar product of the angular velocity vector by the moment of momentum of the system with respect to the same pole, in the same frame $\mathscr{R}$, and the semi-scalar product of the angular velocity vector by the contracted product of the tensor of inertia with respect to the pole O by the angular velocity vector*) *and the scalar product of the velocity of the pole O with respect to the frame $\mathscr{R}'$ by the momentum of the system with respect to* *the same frame $\mathscr{R}'$, from which the kinetic energy of the pole O, in the frame $\mathscr{R}'$, where the whole mass of the discrete mechanical system is considered to be concentrated, is subtracted.*

If $\boldsymbol{\omega}(t)$ is along the direction Δ of unit vector $\mathbf{n}$ at the moment t, then we can write

$$\frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{I}_O\boldsymbol{\omega})=\frac{1}{2}\mathbf{n}\cdot(\mathbf{I}_O\mathbf{n})\omega^2=\frac{1}{2}I_\Delta\omega^2, \tag{11.2.29'}$$

where we have taken into consideration (3.1.82'), $\omega=\omega(t)$ being the magnitude of the angular velocity of instantaneous rotation; the axial moment of inertia I_Δ plays thus the rôle of a mass for the instantaneous motion of rotation.

Taking into account (11.1.16), we find also the remarkable solution

$$T'=T+\frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{K}'_{O'}+\mathbf{K}_O)+\frac{1}{2}M\mathbf{v}'_C\cdot(\mathbf{v}'_O-\boldsymbol{\omega}\times\mathbf{r}'_O)+\frac{1}{2}M\mathbf{v}'_O\cdot\mathbf{v}_C. \tag{11.2.30}$$

The elementary work of the given and constraint external and internal forces which act upon the discrete mechanical system $\mathscr{S}$ is given by

$$\begin{aligned}\mathrm{d}W'+\mathrm{d}W'_R+\mathrm{d}W'_{\mathrm{int}}+\mathrm{d}W'_{R\mathrm{int}}&=\sum_{i=1}^n(\mathbf{F}_i+\mathbf{R}_i)\cdot\mathrm{d}\mathbf{r}'_i+\sum_{i=1}^n\sum_{k=1}^n{}'(\mathbf{F}_{ik}+\mathbf{R}_{ik})\cdot\mathrm{d}\mathbf{r}'_i\\&=\left[\sum_{i=1}^n(\mathbf{F}_i+\mathbf{R}_i)+\sum_{i=1}^n\sum_{k=1}^n{}'(\mathbf{F}_{ik}+\mathbf{R}_{ik})\right]\cdot\mathrm{d}\mathbf{r}'_O+\sum_{i=1}^n(\mathbf{F}_i+\mathbf{R}_i)\cdot\mathrm{d}\mathbf{r}_i\\&+\sum_{i=1}^n\sum_{k=1}^n{}'(\mathbf{F}_{ik}+\mathbf{R}_{ik})\cdot\mathrm{d}\mathbf{r}_i+\boldsymbol{\omega}\times\sum_{i=1}^n\mathbf{r}_i\cdot\left[\mathbf{F}_i+\mathbf{R}_i+\sum_{k=1}^n{}'(\mathbf{F}_{ik}+\mathbf{R}_{ik})\right]\mathrm{d}t,\end{aligned}$$

where we have used the formula (11.2.10'); introducing the torsor of the given and constraint external forces at the pole of the non-inertial frame ($\tau_O\{\mathbf{F}_i+\mathbf{R}_i\}=\{\mathbf{R},\mathbf{M}_O\}+\{\bar{\mathbf{R}},\bar{\mathbf{M}}_O\}$, the torsor of the given and constraint internal forces being equal to zero) and taking into account the properties of the triple scalar product, we read

$$\begin{aligned}\mathrm{d}W'+\mathrm{d}W'_R+\mathrm{d}W'_{\mathrm{int}}+\mathrm{d}W'_{R\mathrm{int}}&=\mathrm{d}W+\mathrm{d}W_R+\mathrm{d}W_{\mathrm{int}}+\mathrm{d}W_{R\mathrm{int}}\\&+(\mathbf{R}+\bar{\mathbf{R}})\cdot\mathrm{d}\mathbf{r}'_O+(\mathbf{M}_O+\bar{\mathbf{M}}_O)\cdot\boldsymbol{\omega}\,\mathrm{d}t,\end{aligned} \tag{11.2.31}$$

finding thus the relation between the elementary work of the given and constraint forces in the two frames: inertial and non-inertial. For the power of the given and constraint forces we have

$$P'+P'_R+P'_{\mathrm{int}}+P'_{R\mathrm{int}}=P+P_R+P_{\mathrm{int}}+P_{R\mathrm{int}}+(\mathbf{R}+\bar{\mathbf{R}})\cdot\mathbf{v}'_O+(\mathbf{M}_O+\bar{\mathbf{M}}_O)\cdot\boldsymbol{\omega}, \tag{11.2.31'}$$

so that we can state

Theorem 11.2.11 *The power of the given and constraint external and internal forces which act upon a discrete mechanical system, with respect to an inertial frame of reference, is equal to the sum of the power of the same forces with respect to an arbitrary non-inertial frame and the power of the given and constraint external forces applied at the pole of the latter frame with respect to the inertial frame.*

Let $\{\mathbf{V}\} \equiv \{\mathbf{V}_i, i = 1,2,...,n\}$ and $\{\mathbf{V}'\} \equiv \{\mathbf{V}'_j, j = 1,2,...,m\}$ be two systems of (bound or sliding) vectors of torsors, at the pole O, $\tau_O\{\mathbf{V}\} \equiv \{\mathbf{R}, \mathbf{M}_O\}$ and $\tau_O\{\mathbf{V}'\} \equiv \{\mathbf{R}', \mathbf{M}'_O\}$, respectively. We call *torsor product* (or *comoment*) of the two torsors the scalar quantity

$$(\tau\{\mathbf{V}\}, \tau\{\mathbf{V}'\}) = \mathbf{R} \cdot \mathbf{M}'_O + \mathbf{R}' \cdot \mathbf{M}_O, \tag{11.2.32}$$

where O is an arbitrary pole. Observing that the resultants $\mathbf{R}$ and $\mathbf{R}'$ are invariant by a change of pole and that $\mathbf{M}_{O'} = \mathbf{M}_O + \overrightarrow{O'O} \times \mathbf{R}$, $\mathbf{M}'_{O'} = \mathbf{M}'_O + \overrightarrow{O'O} \times \mathbf{R}'$, we can write

$$\begin{aligned} \mathbf{R} \cdot \mathbf{M}'_{O'} + \mathbf{R}' \cdot \mathbf{M}_{O'} &= \mathbf{R} \cdot \mathbf{M}'_O + \left(\mathbf{R}, \overrightarrow{O'O}, \mathbf{R}'\right) + \mathbf{R}' \cdot \mathbf{M}_O \\ &+ \left(\mathbf{R}', \overrightarrow{O'O}, \mathbf{R}\right) = \mathbf{R} \cdot \mathbf{M}'_O + \mathbf{R}' \cdot \mathbf{M}_O; \end{aligned}$$

hence, the comoment of two torsors does not depend on the pole with respect to which it is calculated, the definition being thus consistent.

Introducing *the kinematic torsor* at the pole O

$$\mathscr{T}'_O \equiv \{\boldsymbol{\omega}, \mathbf{v}'_O\}, \tag{11.2.33}$$

for which $\mathbf{v}'_{O'} = \mathbf{v}'_O + \overrightarrow{O'O} \times \boldsymbol{\omega}$, we can write the relation (11.2.31') in the form

$$P' + P'_R + P'_{\mathrm{int}} + P'_{R\mathrm{int}} = P + P_R + P_{\mathrm{int}} + P_{R\mathrm{int}} + (\mathscr{T}', \tau\{\mathbf{F}_i\} + \tau\{\mathbf{R}_i\}), \tag{11.2.31''}$$

where we have put in evidence the torsor of the given and constraint external forces.

Applying the theorem of kinetic energy with respect to an inertial frame of reference and starting from (11.2.28), (11.2.28"), (11.2.31'), we can write

$$\begin{aligned} \frac{\mathrm{d}T'}{\mathrm{d}t} = \frac{\mathrm{d}\overline{T}}{\mathrm{d}t} - M\mathbf{v}'_O \cdot \mathbf{a}'_O + M\mathbf{a}'_O \cdot \mathbf{v}'_C + M\mathbf{v}'_O \cdot \mathbf{a}'_C &= P' + P'_R + P'_{\mathrm{int}} + P'_{R\mathrm{int}} \\ &= P + P_R + P_{\mathrm{int}} + P_{R\mathrm{int}} + (\mathbf{R} + \overline{\mathbf{R}}) \cdot \mathbf{v}'_O + (\mathbf{M}_O + \overline{\mathbf{M}}_O) \cdot \boldsymbol{\omega}; \end{aligned}$$

taking into account (11.2.14), (11.2.15) and (11.2.17), we get

$$\frac{\mathrm{d}\overline{T}}{\mathrm{d}t} + M(\mathbf{a}'_O, \boldsymbol{\omega}, \boldsymbol{\rho}) + \mathbf{a}'_O \cdot (M\mathbf{v}_C) = P + P_R + P_{\mathrm{int}} + P_{R\mathrm{int}} + (\mathbf{M}_O + \overline{\mathbf{M}}_O) \cdot \boldsymbol{\omega}, \tag{11.2.34}$$

where

$$\frac{\mathrm{d}\bar{T}}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \dot{\boldsymbol{\omega}} \cdot \mathbf{K}^O + \boldsymbol{\omega} \cdot \frac{\partial \mathbf{K}_O}{\partial t} + \frac{1}{2}\boldsymbol{\omega} \cdot \left(\frac{\partial \mathbf{I}_O}{\partial t} \boldsymbol{\omega} \right), \tag{11.2.34'}$$

because

$$\frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_O \dot{\boldsymbol{\omega}}) + \frac{1}{2}\dot{\boldsymbol{\omega}} \cdot (\mathbf{I}_O \boldsymbol{\omega}) = \frac{1}{2} I_{ij} \dot{\omega}_j \omega_i + \frac{1}{2} I_{ij} \omega_j \dot{\omega}_i$$
$$= \frac{1}{2}(I_{ij} + I_{ji})\omega_i \dot{\omega}_j = I_{(ij)} \omega_i \dot{\omega}_j = \dot{\boldsymbol{\omega}} \cdot (\mathbf{I}_O \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot (\mathbf{I}_O \dot{\boldsymbol{\omega}}) .$$

A scalar product of the equation (11.2.18), (11.2.18') by $\boldsymbol{\omega}$ leads to

$$\frac{\partial T}{\partial t} + \dot{\boldsymbol{\omega}} \cdot \mathbf{K}_O - \frac{1}{2}\boldsymbol{\omega} \cdot \left(\frac{\partial \mathbf{I}_O}{\partial t} \boldsymbol{\omega} \right) + \mathbf{a}'_O \cdot (M \mathbf{v}_C) = P + P_R + P_{\text{int}} + P_{R\text{int}}, \tag{11.2.34''}$$

where we have used the equation (11.2.34), (11.2.34').

The theorem of kinetic energy for the system of equations of motion (11.2.13) has the form

$$\frac{\partial T}{\partial t} = P + P_R + P_{\text{int}} + P_{R\text{int}} + \sum_{i=1}^{n} \mathbf{F}_t^{(i)} \cdot \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{F}_C^{(i)} \cdot \mathbf{v}_i \,; \tag{11.2.34'''}$$

taking into account (11.2.13') and (11.2.18'') and observing that the last power vanishes, we find again the relation (11.2.34'').

As V. Vâlcovici has shown, the theorem of kinetic energy with respect to an inertial frame of reference maintains its form with respect to a non-inertial frame if and only if

$$\dot{\boldsymbol{\omega}} \cdot \mathbf{K}_O + \mathbf{a}'_O \cdot (M \mathbf{v}_C) = \frac{1}{2}\boldsymbol{\omega} \cdot \left(\frac{\partial \mathbf{I}_O}{\partial t} \boldsymbol{\omega} \right). \tag{11.2.35}$$

If we have $\boldsymbol{\omega} = \mathbf{0}$, hence $\bar{T} = T$, then the non-inertial frame $\mathscr{R}$ does not rotate and we obtain

$$T' = T + \frac{1}{2} M v_O'^2 + \mathbf{v}'_O \cdot (M \mathbf{v}_C) = T - \frac{1}{2} M v_O'^2 + \mathbf{v}'_O \cdot (M \mathbf{v}'_C), \tag{11.2.36}$$

so that the theorem of kinetic energy takes the form

$$\mathbf{a}'_O \cdot (M \mathbf{v}_C) + \frac{\partial T}{\partial t} = P + P_R + P_{\text{int}} + P_{R\text{int}} . \tag{11.2.36'}$$

From here (or from (11.2.35)) we deduce that the theorem of kinetic energy of a free or constraint discrete mechanical system maintains its form with respect to a non-inertial frame which does not rotate ($\boldsymbol{\omega} = \mathbf{0}$) if $\mathbf{a}'_O \cdot \mathbf{v}_C = \mathbf{a}'_O \cdot (\mathbf{v}'_C - \mathbf{v}'_O) = 0$, hence in one of the following cases: (i) the pole of the non-inertial frame has a uniform and rectilinear motion with respect to an inertial frame, the movable frame being inertial too; (ii) the non-inertial frame is a Koenig frame ($\mathbf{v}_C = \mathbf{0}$ and $\boldsymbol{\rho} = \overrightarrow{\text{const}}$, hence one can have $O \equiv C$, Theorem 11.2.5); (iii) the projection of the velocity of the centre of mass with respect to an inertial frame, on the direction of the acceleration of the pole of a non-inertial frame is equal to the projection of the velocity of this pole on the same direction, as it was shown by O. Bonnet. As a matter of fact, the case (ii) is a particular case of the case (iii).

As well, from (11.2.28) we obtain a result of O. Bonnet, in conformity to which a non-inertial frame of reference which is not rotating is a frame of Koenig type for the kinetic energy if $\mathbf{v}'_O \cdot \mathbf{v}_C = \mathbf{v}'_O \cdot (\mathbf{v}'_C - \mathbf{v}'_O) = 0$, hence if the projection of the velocity of the mass centre with respect to an inertial frame on the direction of the velocity of the pole of a non-inertial frame is equal to the magnitude of the velocity of that pole.

If the pole of the non-inertial frame of reference coincides with the mass centre of the discrete mechanical system ($O \equiv C$, hence $\boldsymbol{\rho} = \mathbf{0}$), then we obtain a formula of Koenig type

$$T' = \bar{T}^{(C)} + \frac{1}{2} M v_C'^2 \tag{11.2.37}$$

and the theorem of kinetic energy reads

$$\frac{\mathrm{d}\bar{T}^{(C)}}{\mathrm{d}t} = P^{(C)} + P_R^{(C)} + P_{\text{int}}^{(C)} + P_{R\text{int}}^{(C)} + (\mathbf{M}_C + \bar{\mathbf{M}}_C) \cdot \boldsymbol{\omega}\,; \tag{11.2.37'}$$

if we have $\boldsymbol{\omega} = \mathbf{0}$ too, then the non-inertial frame $\mathscr{R}$ is a Koenig frame, so that

$$T' = T^{(C)} + \frac{1}{2} M v_C'^2\,, \quad \frac{\partial T^{(C)}}{\partial t} = P^{(C)} + P_R^{(C)} + P_{\text{int}}^{(C)} + P_{R\text{int}}^{(C)}\,. \tag{11.2.38}$$

From (11.2.28") we notice that the first formula (11.2.38) of Koenig can take place also for $\boldsymbol{\omega} \neq \mathbf{0}$ if

$$\boldsymbol{\omega} \cdot \bar{\mathbf{K}}_C - \frac{1}{2} \boldsymbol{\omega} \cdot (\mathbf{I}_C \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \mathbf{K}_C + \frac{1}{2} \boldsymbol{\omega} \cdot (\mathbf{I}_C \boldsymbol{\omega}) = 0\,; \tag{11.2.39}$$

we can also write

$$\omega = -\frac{2}{I_C} \mathrm{pr}_{\boldsymbol{\omega}} \mathbf{K}_C\,, \tag{11.2.39'}$$

where I_C is the moment of inertia of the discrete mechanical system with respect to the instantaneous axis of rotation of the non-inertial frame of reference with the pole at the centre of mass. We find thus again a result of V. Vâlcovici, according to which the first formula (11.2.38) of Koenig takes place only and only if the angular velocity vector of the non-inertial frame with respect to the inertial one is directed in an opposite sense to that of the projection of the moment of momentum of the discrete mechanical system on the instantaneous axis of rotation, having a magnitude equal to the double of the magnitude of this projection divided by the moment of inertia of the system with respect to the instantaneous axis.

As well, using the condition (11.2.35), we notice that the second formula (11.2.38) of Koenig takes place for $\omega \neq 0$ too if

$$\dot{\boldsymbol{\omega}} \cdot \mathbf{K}_C = \frac{1}{2} \boldsymbol{\omega} \cdot \left(\frac{\partial \mathbf{I}_C}{\partial t} \boldsymbol{\omega} \right). \tag{11.2.40}$$

Thus, as it has been stated by V. Vâlcovici, the theorem of kinetic energy remains invariant in its form with respect to a non-inertial frame of reference with the pole at the mass centre if and only if the scalar product of the angular acceleration vector of the non-inertial frame with respect to an inertial one by the moment of momentum of the discrete mechanical system, relative to the non-inertial frame, is equal to the half of the scalar product of the angular velocity vector of the same frame with respect to the inertial one by the contracted product of the derivative with respect to time of the tensor of inertia with respect to the centre of mass of the system, in the non-inertial frame, by the considered angular velocity vector.

Equating to zero the relative motion with respect to the non-inertial frame of reference, we find – from (11.2.28), (11.2.28") – the transportation kinetic energy $T'(\mathrm{tr}\, O)$ of the discrete mechanical system $\mathscr{S}$, by the non-inertial frame $\mathscr{R}$, with respect to the inertial frame $\mathscr{R}'$ (we make $\mathbf{v}_i = \mathbf{0}$ in the formula (11.2.10'), so that $\mathbf{v}_i'(\mathrm{tr}\, O) = \mathbf{v}_O' + \boldsymbol{\omega} \times \mathbf{r}_i$). To calculate the transportation kinetic energy $T(\mathrm{tr}\, O')$ of the discrete mechanical system $\mathscr{S}$ by the inertial frame $\mathscr{R}'$, with respect to the non-inertial frame $\mathscr{R}$, we introduce the kinematic torsor $\mathscr{T}_{O'} \equiv \{-\boldsymbol{\omega}, \mathbf{v}_{O'}\}$; observing that

$$\mathbf{v}_O' = \frac{\partial \mathbf{r}_O'}{\partial t} + \boldsymbol{\omega} \times \mathbf{r}_O' = -\frac{\partial \mathbf{r}_{O'}}{\partial t} + \boldsymbol{\omega} \times \mathbf{r}_O' = -\mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}_O',$$

we have

$$\mathbf{v}_i(\mathrm{tr}\, O') = \mathbf{v}_{O'} + (-\boldsymbol{\omega}) \times (\mathbf{r}_O' + \mathbf{r}_i) = -\mathbf{v}_O - \boldsymbol{\omega} \times \mathbf{r}_i = -\mathbf{v}_i(\mathrm{tr}\, O).$$

Thus, C. Iacob showed that

$$T'(\mathrm{tr}\, O) = T(\mathrm{tr}\, O') = \frac{1}{2} M v_O'^2 + M(\mathbf{v}_O', \boldsymbol{\omega}, \boldsymbol{\rho}) + \frac{1}{2} \boldsymbol{\omega} \cdot (\mathbf{I}_O \boldsymbol{\omega}). \tag{11.2.41}$$

With this result, the relations (11.2.28), (11.2.28") may be written also in the form

$$T' = T + T'(\mathrm{tr}\, O) + M \mathbf{v}_O' \cdot \mathbf{v}_C + \boldsymbol{\omega} \cdot \mathbf{K}_O, \tag{11.2.42}$$

$$T' = T - T'(\mathrm{tr}\, O) + M \mathbf{v}_O' \cdot \mathbf{v}_C' + \boldsymbol{\omega} \cdot \mathbf{K}_O', \tag{11.2.42'}$$

where we have invert the rôle of the frames $\mathscr{R}'$ and $\mathscr{R}$, taking into account the previous results and the relation $\mathbf{K}_{O'}' = \mathbf{K}_O' + \mathbf{r}_O' \times (M \mathbf{v}_C')$.

Summing the relations (11.2.42), (11.2.42'), observing that $\mathbf{H} = M\mathbf{v}_C$, $\mathbf{H}' = M\mathbf{v}'_C$, introducing the kinematic torsor (11.2.33) and the kinetic torsors $\tau_O = \{\mathbf{H}, \mathbf{K}_O\}$, $\tau'_O = \{\mathbf{H}', \mathbf{K}'_O\}$, we get

$$T' = T + \frac{1}{2}(\mathscr{T}', \tau' + \tau).$$

Let be also the kinematic torsor $\mathscr{T}'_O = \{-\boldsymbol{\omega}, \mathbf{v}'_O\}$, which corresponds to the motion of the inertial frame of reference with respect to the non-inertial one; taking into account the relation between $\mathbf{v}_{O'}$ and $\mathbf{v}'_O$ and the relation $\mathbf{K}_O = \mathbf{K}_{O'} + \mathbf{r}_{O'} \times (M\mathbf{v}_C)$, we obtain the reciprocity relation

$$T' - \frac{1}{2}(\mathscr{T}', \tau') = T - \frac{1}{2}(\mathscr{T}, \tau), \tag{11.2.43}$$

due to C. Iacob.

In particular, if $\boldsymbol{\omega} = \mathbf{0}$, then we have

$$T'(\operatorname{tr} O) = T(\operatorname{tr} O') = \frac{1}{2} M v_O'^2. \tag{11.2.41'}$$

From (11.2.36') we notice that for $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{a}'_O \cdot \mathbf{v}_C = 0$, hence for the conditions found by O. Bonnet in case of scleronomic constraints (for which $P_R = P_{R\mathrm{int}} = 0$), assuming that the given internal forces derive from a simple or from a generalized potential, we can write a theorem of mechanical energy of a discrete mechanical system, free or with scleronomic constraints, with respect to a non-inertial frame which does not rotate with respect to an inertial one, in the form

$$\frac{\partial E}{\partial t} = P, \quad E = T + V, \tag{11.2.44}$$

where V is the potential energy of the system with respect to the pole O of the movable frame. If $P = 0$, in a certain interval of time, we get $E = \mathrm{const}$ for this interval, hence a conservation theorem of mechanical energy (a scalar first integral), which is not independent from that which could be obtained with respect to an inertial frame.

11.2.2.3 Problem of n Particles

We will consider now the problem of n particles with respect to a non-inertial frame of reference $\mathscr{R}$, which does not rotate with respect to an inertial frame $\mathscr{R}'$, but which is not a Koenig frame (see Sects. 11.1.2.8 and 11.2.1.3 too). If the origin O of the non-inertial frame coincides with one of the particles, e.g., with the particle P_1, of position vector $\mathbf{r}'_1$, with respect to the frame $\mathscr{R}'$, we can write $\mathbf{r}'_i = \mathbf{r}'_1 + \mathbf{r}_i$, $i = 2,3,...,n$, $\mathbf{r}_1 = \mathbf{0}$. The equations of relative motion with respect to the frame $\mathscr{R}$ are

$$m_i \ddot{\mathbf{r}}_i = \sum_{k=1}^{n}{}' \mathbf{F}_{ik} + \mathbf{F}_t^{(i)}, \quad \mathbf{F}_t^{(i)} = -m_i \ddot{\mathbf{r}}_i', \quad i = 2,3,...,n;$$

noting that

$$\frac{1}{m_i} \sum_{k=1}^{n}{}' \mathbf{F}_{ik} = f \sum_{k=1}^{n}{}' m_k \frac{\mathbf{r}_{ik}}{r_{ik}^3} = -f \sum_{k=1}^{n}{}' m_k \frac{\mathbf{r}_i - \mathbf{r}_k}{|\mathbf{r}_i - \mathbf{r}_k|^3} = -f m_1 \frac{\mathbf{r}_i}{r_i^3} + f \sum_{k=2}^{n} m_k \frac{\mathbf{r}_{ik}}{r_{ik}^3},$$

$$\begin{aligned} \frac{1}{m_1} \sum_{k=2}^{n} \mathbf{F}_{1k} &= f \sum_{k=2}^{n} m_k \frac{\mathbf{r}_{1k}}{r_{1k}^3} = -f \sum_{k=2}^{n} m_k \frac{\mathbf{r}_1 - \mathbf{r}_k}{|\mathbf{r}_1 - \mathbf{r}_k|^3} = f \sum_{k=2}^{n} m_k \frac{\mathbf{r}_k}{r_k^3} \\ &= f \left(m_i \frac{\mathbf{r}_i}{r_i^3} + \sum_{k=2}^{n}{}' m_k \frac{\mathbf{r}_k}{r_k^3} \right), \end{aligned}$$

we can write these equations in the form (the operator ∇_i is considered with respect to the non-inertial frame)

$$\ddot{\mathbf{r}}_i + f(m_1 + m_i)\frac{\mathbf{r}_i}{r_i^3} = \nabla_i R_i, \quad R_i = f \sum_{k=2}^{n}{}' m_k \left(\frac{1}{r_{ik}} - \frac{\mathbf{r}_i \cdot \mathbf{r}_k}{r_k^3} \right), \quad i = 2,3,..,n. \tag{11.2.45}$$

In the particular case $n = 2$, it results $R_i = 0$ (we have only $i = 2$); the equation (11.2.45) is reduced to an equation of the form (11.1.51), hence to the equation of motion corresponding to the problem of two particles. Thus, the influence of the other particles upon the motion of the particle P_i is given by the perturbing function R_i. These equations are used in the study of the motion of the planets with respect to the Sun, in the frame of the solar system. In this case, the Sun is considered to be the particle P_1 (the centre of mass of the Sun is chosen as origin O of the frame $\mathscr{R}$), having the mass m_1, which does not intervene in R_i; we may thus state that the values of the perturbing function are very small.

Another possibility to study the problem, due to Jacobi, is based on the introduction for the discrete mechanical subsystems $\mathscr{S}_i \equiv \{P_1, P_2, ..., P_i\}$, $i = 1,2,...,n-1$, of $n-1$ non-inertial frames $\mathscr{R}_i$, with the poles at the mass centres C_i of position vectors

$$\boldsymbol{\rho}_i' = \frac{1}{M_i} \sum_{j=1}^{i} m_j \mathbf{r}_j', \quad M_i = \sum_{j=1}^{i} m_j, \quad i = 1,2,...,n-1, \tag{11.2.46}$$

with respect to the inertial frame $\mathscr{R}'$, and which do not rotate about the latter frame; the motion of the particle P_1 is thus considered with respect to the inertial frame $\mathscr{R}'$, while the motion of the particle P_{i+1} is considered with respect to the non-inertial frame $\mathscr{R}_i$, having the position vector $\mathbf{r}_{i+1}$ with respect to that frame. Observing that $\mathbf{r}_{i+1} = \mathbf{r}_{i+1}' - \boldsymbol{\rho}_i'$, it results

$$M_i \mathbf{r}_{i+1} = M_i \mathbf{r}'_{i+1} - \sum_{j=1}^{i} M_j \mathbf{r}'_j \, ; \tag{11.2.46'}$$

taking into account (11.1.48'), we can write the equation of motion with respect to the frame $\mathscr{R}_i$ in the form

$$M_i \ddot{\mathbf{r}}_{i+1} = \frac{M}{m_{i+1}} \nabla'_{i+1} U - \sum_{j=1}^{i} \nabla'_j U \, , \quad i = 1,2,\dots,n-1 \, , \tag{11.2.47}$$

the operator $\nabla'_i = \mathbf{i}'_k \partial / \partial x'^{(i)}_k$ being taken with respect to the inertial frame $\mathscr{R}'$. By means of the relation (11.2.46') and using the notation (11.2.46), we can express the operators ∇'_i, taken with respect to the frame $\mathscr{R}'$, by the operators ∇_{i+1}, taken with respect to the frames $\mathscr{R}_i$; we obtain thus

$$\nabla'_1 = -m_1 \sum_{k=1}^{n-1} \frac{1}{M_k} \nabla_{k+1} \, , \quad \nabla'_{i+1} = \nabla_{i+1} - m_{i+1} \sum_{k=i+1}^{n-1} \frac{1}{M_k} \nabla_{k+1} \, ,$$
$$i = 1,2,\dots,n-2 \, , \quad \nabla'_n = \nabla_n \, . \tag{11.2.46''}$$

Replacing in (11.2.47) and observing that

$$\sum_{j=1}^{i} \nabla'_j = \sum_{j=2}^{i} \nabla_j - \sum_{j=1}^{i} \sum_{k=j}^{n-1} \frac{m_j}{M_k} \nabla_{k+1} = -M_i \sum_{k=i}^{n-1} \frac{1}{M_k} \nabla_{k+1} \, , \tag{11.2.46'''}$$

we can write the equation of motion (11.2.47) of the particle P_{i+1} with respect to the frame $\mathscr{R}_i$ in the form

$$\overline{m}_{i+1} \ddot{\mathbf{r}}_{i+1} = \nabla_{i+1} \overline{U} \, , \quad i = 1,2,\dots,n-1 \, , \tag{11.2.47'}$$

where we have introduced the notation $\overline{m}_{i+1} = (M_i / M_{i+1}) m_{i+1}$ and where $U(|\mathbf{r}'_i - \mathbf{r}'_k|) = \overline{U}(|\mathbf{r}_i - \mathbf{r}_k|)$. To can calculate the last form $\overline{U}$ of the potential, we start from the relation (11.2.46), which leads to

$$M_{i-1} \boldsymbol{\rho}_{i-1} - M_{i-2} \boldsymbol{\rho}_{i-2} = M_{i-1} (\mathbf{r}'_i - \mathbf{r}_i) - M_{i-2} (\mathbf{r}'_{i-1} - \mathbf{r}_{i-1})$$
$$= m_{i-1} \mathbf{r}'_{i-1} = (M_{i-1} - M_{i-2}) \mathbf{r}'_{i-1} \, ,$$

wherefrom

$$\mathbf{r}'_i - \mathbf{r}'_{i-1} = \mathbf{r}_i - \frac{M_{i-2}}{M_{i-1}} \mathbf{r}_{i-1} \, ; \tag{11.2.48}$$

writing these relations for successive values of the index i and summing for all the indices, we obtain the relation

$$\mathbf{r}_i' - \mathbf{r}_j' = \mathbf{r}_i - \mathbf{r}_j + \sum_{k=j}^{i-1} \frac{m_k}{M_k} \mathbf{r}_k \,, \tag{11.2.48'}$$

useful for the goal had in view.

Introducing *the canonical co-ordinates* (*the generalized co-ordinates* $q_k = x_j^{(i)}$ and *the generalized momenta* $p_k = \overline{m}_i \dot{x}_j^{(i)} = \overline{m}_k \dot{q}_k$, $k = 3(i-1)+j$, $i = 1,2,...,n-1$, $j = 1,2,3$), we can write the equations of motion (11.2.47') in *the canonical form*

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad k = 1,2,...,3(n-1), \tag{11.2.49}$$

where we have introduced *Hamilton's function*

$$H = \frac{1}{2} \sum_{k=1}^{3(n-1)} \frac{1}{\overline{m}_k} p_k^2 - U = \frac{1}{2} \sum_{k=1}^{3(n-1)} p_k \dot{q}_k - U \,. \tag{11.2.49'}$$

These equations are particularly useful for the study of the motion of satellites if these ones are influenced by only one celestial body, excepting that one with respect to which one considers the motion (e.g., the study of the motion of the Moon, taking into account the influence of the Sun and of the Earth).

Chapter 12

Dynamics of Continuous Mechanical Systems

The motion of a continuous mechanical system with respect to an inertial (Galilean) frame of reference is studied in this chapter, considering free and constraint systems; one passes then to the case of a non-inertial (non-Galilean) frame of reference. In particular, some one-dimensional mechanical systems are dealt with.

12.1 General Considerations

A *continuous mechanical system* (*continuous medium*, *continuous body*, *continuous material*) $\mathscr{S}$ represents the mathematical model of a body which is entirely immersed in a domain $\mathscr{D}$ of the space E_3; this domain is *the geometric support* Ω of the considered system. After some notions with introductory character and after presenting the general principles which allow to set up the mathematical model of such a mechanical system, one passes to the corresponding general theorems and to the conservation ones.

12.1.1 Introductory Notions. General Principles

To pass from a discrete mechanical system to a continuous one represents – in fact – to pass from a mechanical system with a finite number of degrees of freedom to a mechanical system with an infinite number of degrees of freedom. The mathematical model used in the first case must be completed and Newton's principles must be consequently adapted.

12.1.1.1 Introductory Notions

As it was shown in Chap. 1, Sect. 1.1.8, a *continuous mechanical system* constitutes a mathematical model formed by a domain $\mathscr{D}$ to which is associated a mass; the quantities in connection with such a system are, in general, represented by continuous functions. In computations, we will consider a *point* $\mathscr{P}$ *of a continuous mechanical system* $\mathscr{S}$; but that one is not a particle in the sense considered till now, having not a finite mass (however, we will use also the denomination of *particle of the continuous mechanical system*). The topology of the Euclidean domain $\mathscr{D}$ is the topology of the mechanical system $\mathscr{S}$ too, while the distance between two points of that system is the Euclidean distance between their positions at the same moment t; these positions will be considered, in general, with respect to an inertial frame of reference. If the distances between all pairs of points of the continuous mechanical system $\mathscr{S}$ remain invariable in

P.P. Teodorescu, *Mechanical Systems, Classical Models*,

time, then we have to do with a *rigid solid*; otherwise, this *mechanical system* is *deformable* (*deformable continuous medium*). In Chap. 1, Sect. 1.1.10 has been made a classification of such media, i.e.: *deformable solids* and *fluids* (*liquids*, *gases*, *plasma*); as well, the properties of *elasticity*, *plasticity* and *viscosity* of those mechanical systems have been put in evidence.

Under the action of *external charges* (which can be: concentrated or distributed forces, other concentrated charges (applied on the external surface of the body, hence on the frontier of the domain in which is immersed the body or on its interior), volumic (or massic) forces or moments, forces of inertia, charges produced by a thermic or by an electromagnetic field, charges produced by radioactive radiations, deformations provoked by various causes, imposed displacements etc.), the particles (infinitesimal elements) which form a solid body change (eventually, in time) the position (with respect to an inertial frame of reference, considered fixed) which they had before the action of those charges. If, after a translation and a rotation, all the particles of the body, subjected to the action of charges, have the same mutual positions as before the application of those charges, then we say that we have to do with a *rigid solid motion*; otherwise, the body is subjected to a *deformation*. The totality of the deformations of a particle of the body forms *the state of deformation at a point* (the point is the geometric support of the considered particle). The totality of the states of deformation corresponding to all points (particles) of the solid body constitutes *the state of deformation of the body*. Together with the notion of deformation, the notion of *displacement* is put in evidence too. The totality of the displacements corresponding to all the points of the solid body constitutes *the state of displacement* of the body. Corresponding to what was specified before, the bodies which allow only displacements of rigid body are called *rigid solids*; the other solid bodies are *deformable solids*. Due to deformations, the (static or dynamic) equilibrium of the constraint forces which act between the particles of the body does no more hold, so that supplementary internal forces arise; the totality of those internal forces (called *efforts*, if they act upon an arbitrary section of the body, or *stresses*, if they correspond to the efforts acting on a unit area), which correspond to a particle, form *the state of stress at a point* (geometric support of the considered particle). The totality of the states of stress corresponding to all the points (particles) of the solid body forms *the state of stress of the body*.

In case of a *fluid* (which changes much its form under the action of the external causes), the deformation at a point is replaced by *the velocity of deformation* at that point, while the displacement of a point of the mechanical system is replaced by its *velocity*; thus, we have to do with a *state of velocity of deformation* and with a *state of velocity*, respectively.

The mathematical model of continuous deformable media (solids or fluids) must be completed by a *constitutive law* (of theoretical and experimental nature), which represents a relation between the state of deformation (of velocity of deformation) and the state of stress of the respective continuous medium. The results of theoretical nature (the geometrical-kinematical and mechanical aspects) mentioned above, valid for an arbitrary continuous deformable mechanical system, are thus specified for a certain continuous deformable medium.

In general, the motion of a particle $\mathscr{P}$ of the continuous mechanical system $\mathscr{S}$ is given by an equation of the form

$$\mathbf{r} = \mathbf{r}(\mathscr{P};t), \tag{12.1.1}$$

which specifies thus the position of that particle at any moment $t \in \mathscr{T} = [t_0, t_1]$, with respect to an inertial frame of reference, considered fixed. The velocity and the acceleration of the particle are thus defined by the relations

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(\mathscr{P};t), \quad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(\mathscr{P};t). \tag{12.1.1'}$$

The totality of motions of each particle defines *the motion of the continuous mechanical system* $\mathscr{S}$. In this case, to the continuous deformable medium corresponds a domain $\mathscr{D} = \mathscr{D}(t)$, by the mapping $\mathbf{r}(\mathscr{S};t)$, that is the geometric support Ω of the medium at a moment t. This domain represents *the configuration of the continuous mechanical system* $\mathscr{S}$ at the respective moment; hence, the motion of the mechanical system $\mathscr{S}$ is a succession of configurations. The configuration in which the particles $\mathscr{P}$ of the continuous mechanical system $\mathscr{S}$ are identified with their positions (the locations P_0 occupied by them, specified by the position vectors $\mathbf{r}_0$) is called *reference configuration*. In case of a deformable solid, this configuration is called *the non-deformed configuration* (*state*), while in case of a fluid it will be *the initial configuration* (*state*) $\mathscr{D}_0$, corresponding to a moment $t = t_0$; the configuration at an arbitrary moment t, specified by the point P of position vector $\mathbf{r}$, is called *actual configuration* (*state*). Corresponding to Chap. 1, Sect. 1.1.9, the motion of the particle $\mathscr{P}$ is described by the equation

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_0;t), \quad x_i = x_i\left(x_1^0, x_2^0, x_3^0;t\right), \quad i = 1,2,3, \tag{12.1.2}$$

defined on $\mathscr{D}_0 \times \mathscr{T}$. The vector $\mathbf{r}$ travels through the configuration $\mathscr{D}$ if the vector $\mathbf{r}_0$ travels through the configuration $\mathscr{D}_0$; we may write

$$\mathscr{D} = \mathscr{D}(\mathscr{D}_0;t), \tag{12.1.2'}$$

defining thus a mapping of the space E_3 in itself. The velocity and the acceleration at a certain moment t are given by (the position vector $\mathbf{r}_0$ is considered to be constant)

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(\mathbf{r}_0;t), \quad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(\mathbf{r}_0;t). \tag{12.1.2''}$$

The co-ordinates x_i^0, $i = 1,2,3$, are called *material co-ordinates* (identifying the particle $\mathscr{P}$ by the point P_0 of position vector $\mathbf{r}_0$, one follows the motion of that one in time), corresponding to a *material* (*Lagrangian*) *description* of the motion; they are called *Lagrangian co-ordinates* too (although they have been introduced by Euler) or *reference co-ordinates* (corresponding to a reference configuration). In this case, x_i, $i = 1,2,3$, are the unknown functions of the problem. If the displacements are great (e.g., in case of a fluid), it is convenient to choose as independent variables the co-ordinates x_i, $i = 1,2,3$, which will be called *spatial co-ordinates* (of the place P in space, through which passes a particle $\mathscr{P}$ at the moment t), corresponding to a *spatial* (*Eulerian*) *description* of the motion; they are called *Eulerian co-ordinates* too (in fact, they have been introduced by d'Alembert). In this case, $v_i(x_1, x_2, x_3; t)$, $i = 1,2,3$, are the unknown functions of the problem. We assume that the Jacobian J, defined by the relation (1.1.75'), is non-zero in $\mathscr{D}_0$ (eventually, excepting some singular points, lines or surfaces), so that – from the relation between the elements of volume ($\mathrm{d}V = J\mathrm{d}V_0$) – we deduce that the geometric support Ω cannot vanish or become infinite; to *the axiom of continuity* (the particles preserve their individuality) correspond thus *the indestructibility* and *the impenetrability of the matter*, respectively. From a mathematical point of view, the mapping (12.1.2) is a bijection between $\mathscr{D}_0$ and $\mathscr{D}$. The theorem of implicit functions allows to write

$$\mathbf{r}_0 = \mathbf{r}_0(\mathbf{r}; t), \quad x_i^0 = x_i^0(x_1, x_2, x_3; t), \quad i = 1,2,3, \tag{12.1.2'''}$$

on $\mathscr{D} \times \mathscr{T}$; these functions are univocally determinate (in particular, $x_i^0 = x_i(x_1^0, x_2^0, x_3^0; t)$, $i = 1,2,3$) if (necessary and sufficient condition), at least in a neighbourhood of the considered point P_0, the functions (12.1.2) are of class $C^1(\mathscr{D}_0)$ (as a matter of fact, in this case also the functions (12.1.2''') are of class $C^1(\mathscr{D})$). If the functions (12.1.2) are of class $C^2(\mathscr{D}_0)$, then the points which are initially neighbouring remain neighbouring also at the moment t; in this case, the functions (12.1.2''') are of a class $C^2(\mathscr{D})$ too.

In a material description (convenient in case of deformable solids) we can express the equation (12.1.2) in the form

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}(\mathbf{r}_0; t) \tag{12.1.3}$$

on $\mathscr{D}_0 \times \mathscr{T}$, where $\mathbf{u}$ is *the displacement vector* of the particle $\mathscr{P}$ from the non-deformed configuration $\mathscr{D}_0$ to the deformed configuration $\mathscr{D}$ (Fig. 12.1); the velocity and the acceleration of that particle will be expressed in the form

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{u}}{\partial t} = \dot{\mathbf{u}}, \quad \mathbf{a} = \ddot{\mathbf{r}} = \frac{\partial^2 \mathbf{u}}{\partial t^2} = \ddot{\mathbf{u}}, \tag{12.1.3'}$$

the vector $\mathbf{r}_0$ being constant with respect to time. In general, in case of a (scalar or vector) field $\Phi(\mathbf{r}_0; t)$ or $\mathbf{\Psi}(\mathbf{r}_0; t)$, respectively, defined on $\mathscr{S}$, the derivative with respect to the time t is reduced to the corresponding partial derivative ($\dot{\Phi} = \partial \Phi / \partial t$ or $\dot{\mathbf{\Psi}} = \partial \mathbf{\Psi} / \partial t$, respectively); this derivative is called *material derivative*. We notice that one can use the relation (12.1.3) also in the form

$$\mathbf{r}_0 = \mathbf{r} - \mathbf{u}(\mathbf{r};t) \tag{12.1.3''}$$

on $\mathscr{D} \times \mathscr{T}$.

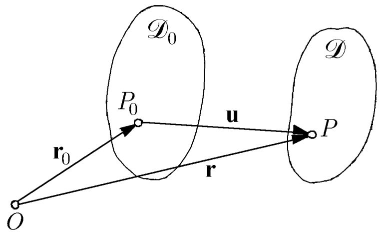

Fig. 12.1 Material and spatial co-ordinates

In a spatial description (convenient in case of fluids) we define the characteristic quantities (velocity, density, pressure etc.) as functions of $\mathbf{r}$ and t; e.g., to have $\mathbf{v}(\mathbf{r};t)$ means to know the velocities of all particles (at any moment) which pass through all the points P (of the domain $\mathscr{D}$). From the equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i(\mathbf{r};t), \quad i = 1,2,3, \tag{12.1.4}$$

we obtain the trajectories of motion; indeed, if the solution exists and is unique, then we can write the independent first integrals $f_i(x_1, x_2, x_3; t) = C_i$, $C_i = \text{const}$, $i = 1,2,3$, which, on the basis of the theorem of implicit functions, lead to the relations (12.1.2) (we assume that for $t = t_0$ we have $x_i = x_i^0$, hence $C_i = f_i(x_1^0, x_2^0, x_3^0; t_0)$). But, in general, it is sufficient to determine fields (e.g., scalar fields) of the form $f = f(\mathbf{r};t)$; the derivative of such a quantity with respect to time (*the material derivative*) is given by the formula

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \dot{f} + v_i f_{,i} = \dot{f} + \mathbf{v} \cdot \nabla f, \tag{12.1.5}$$

on $\mathscr{D} \times \mathscr{T}$. Analogously, one obtains the material derivative of a vector field too; the acceleration is given by

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{12.1.6}$$

on $\mathscr{D} \times \mathscr{T}$. If we take into account the relation (12.1.2), then we notice that a quantity $f(\mathbf{r};t)$, defined in spatial co-ordinates, may be studied in a material description in the form $f(\mathbf{r};t) = f(\mathbf{r}(\mathbf{r}_0;t);t)$.

We call *material variety* (point, curve, surface, three-dimensional domain) a variety formed of particles; often, the use of such a variety can very much simplify the study of the motion of the considered mechanical system $\mathscr{S}$. In conformity to the conservation theorem of material varieties, their images are varieties of the same order. Taking into account the conditions (3.2.18) imposed to a particle subjected to finite constraints, one can show that (*Euler-Lagrange criterion*) a *surface* of equation $\Phi(\mathbf{r};t) = 0$ is *material* (is the image of a material surface) if (necessary and sufficient condition)

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \dot{\Phi} + \mathbf{v} \cdot \operatorname{grad} \Phi = 0 . \tag{12.1.7}$$

In case of a vector field $\mathbf{\Psi}(\mathbf{r};t)$, its lines are *material curves* if and only if the formula (*Helmholtz-Zorawski criterion*)

$$\mathbf{\Psi} \times \left[\dot{\mathbf{\Psi}} + \operatorname{curl}(\mathbf{\Psi} \times \mathbf{v}) + \mathbf{v} \operatorname{div} \mathbf{\Psi} \right] = \mathbf{0} , \tag{12.1.8}$$

which can be put in connection with the formula (A.2.81'), holds. In particular, for $\mathbf{\Psi} = \mathbf{v}$ it results

$$\mathbf{v} \times \dot{\mathbf{v}} = \mathbf{0} , \tag{12.1.8'}$$

that is the necessary and sufficient condition for the field lines (current lines) of the velocity to be *trajectories*.

The mass $m(\mathscr{S})$ of the mechanical system $\mathscr{S}$ (including a continuous mechanical system) has been introduced in Chap. 1, Sect. 1.1.6, its mathematical model having certain properties: i) $m(\mathscr{S}) > 0$; ii) the property of additivity; iii) $\dot{m} = 0$. Differentiating in the sense of the theory of distributions, we find *the density* $\mu = \mu(\mathbf{r};t)$, which is given by the relation (1.1.71).

Besides the formulae given in Chap. 1, Sect. 1.1.9 and in Ann., Sect. 2.3.5, it is useful to establish other two formulae of differentiation for the integrals which depend on a parameter. Let be such an integral of the form

$$I_{D(t)} \equiv \iiint_{D(t)} F(x_1, x_2, x_3; t) \mu(x_1, x_2, x_3; t) \mathrm{d}\tau , \tag{12.1.9}$$

where $\mathrm{d}\tau$ is the element of volume. Observing that $\operatorname{div}(F\mu\mathbf{v}) = F \operatorname{div}(\mu\mathbf{v})$ $+ \mu\mathbf{v} \cdot \operatorname{grad} F$, we may write

$$\frac{\mathrm{d}I_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} F \left[\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu\mathbf{v}) \right] \mathrm{d}\tau + \iiint_{D(t)} \mu \left(\frac{\partial F}{\partial t} + \mathbf{v} \cdot \operatorname{grad} F \right) \mathrm{d}\tau ;$$

according to the relation (1.1.79'), the first integral vanishes, while the relation (12.1.5) allows to write

$$\frac{\mathrm{d}I_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} \mu \frac{\mathrm{d}F}{\mathrm{d}t} \mathrm{d}\tau . \tag{12.1.9'}$$

Analogously, for the vector integral

$$\mathbf{I}_{D(t)} \equiv \iiint_{D(t)} \mathbf{V}(\mathbf{r};t)\mu(\mathbf{r};t)\mathrm{d}\tau\,, \tag{12.1.10}$$

we obtain

$$\frac{\mathrm{d}\mathbf{I}_{D(t)}}{\mathrm{d}t} = \iiint_{D(t)} \mu \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\mathrm{d}\tau\,. \tag{12.1.10'}$$

We notice that the physical properties of material bodies, which are modelled as mechanical systems, do not depend neither on the frame of reference, nor if that one is at rest or in motion; hence, these properties are characterized by *objective quantities* (independent on the frame), which satisfy *the principle of objectivity* (*the principle of material indifference*, *the principle of frame independence*). The most general relation between co-ordinates and time with respect to a frame $\mathscr{R}$ and a chronology $\mathscr{C}$ or a frame $\mathscr{R}'$ and a chronology $\mathscr{C}'$, respectively, is of the form

$$\begin{gathered} x_i'(t') = Q_{ij}(t)\left(x_j - x_j^0\right) + c_i(t)\,, \quad i = 1,2,3\,, \\ t' = t + t_0\,, \quad x_j^0, t_0 = \mathrm{const}\,, \end{gathered} \tag{12.1.11}$$

where $\mathbf{Q}$ is a proper orthogonal matrix ($\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, $\det\mathbf{Q} = 1$), the frames being right-handed orthonormed, while $\mathbf{c}(t)$ is a vector. A scalar quantity s is objective if

$$s'(\mathbf{r}';t) = s(\mathbf{r};t)\,, \tag{12.1.12}$$

a vector quantity $\mathbf{V}$ is objective if

$$\mathbf{V}'(\mathbf{r}';t') = \mathbf{Q}(t)\mathbf{V}(\mathbf{r};t)\,, \quad V_i' = Q_{ij}V_j\,, \quad i = 1,2,3\,, \tag{12.1.12'}$$

while a tensor quantity $\mathbf{T}$ of second order is objective if

$$\mathbf{T}'(\mathbf{r}';t') = \mathbf{Q}(t)\mathbf{T}(\mathbf{r};t)\mathbf{Q}^{\mathrm{T}}(t)\,, \quad T_{ij}' = Q_{ik}Q_{jl}T_{kl}\,, \quad i,j = 1,2,3\,. \tag{12.1.12''}$$

Because the considered frames are movable ($\mathbf{Q} = \mathbf{Q}(t)$ and $\mathbf{c} = \mathbf{c}(t)$), these relations are different from those which define a scalar, a vector or a tensor of second order. One can easily see that the position vector, the velocity and the acceleration are not objective vectors. The unit mass is an objective scalar.

12.1.1.2 General Principles

The principles of mechanics, so as they have been enounced in Chap. 1, Sect. 1.2.1 for a particle and as they have been used in Chap. 11 for a discrete mechanical system (a finite number of particles), cannot satisfactorily describe the evolution of a continuous mechanical system $\mathscr{S}$. To pass from discrete to continuum, from a finite to an infinite number of particles, it is necessary much rigour; as a matter of fact, this represents the passing from a finite number to an infinite number of degrees of freedom (excepting the case of the rigid solid). Indeed, the notion of particle (of finite positive mass) loses its sense, because it would lead to an infinite mass for the mechanical system $\mathscr{S}$. The notions of velocity and acceleration are linked to a point (hence to a particle), so that Newton's law (1.1.89) can no more be applied in this form; as a matter of fact, all principles of mechanics must be formulated of new and consequently adapted. The principles which allow the mathematical modelling of a continuous mechanical system must be formulated so as to can reduce them to Galileo-Newton principles in case of a discrete mechanical system.

To express these principles, we denote by S a subsystem of the system $\mathscr{S}$, $D \subset \mathscr{D}$ being the domain occupied by that one, while $\omega \subset \Omega$ is its geometric support. *The principles of motion* of the continuous deformable media are expressed by *balance relations* of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\omega} \mathbf{Q}\,\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{D} \mu\mathbf{Q}\,\mathrm{d}V = \iint_{\partial D} \mathbf{R}\mathbf{n}\,\mathrm{d}S + \iiint_{D} \mu\mathbf{S}\,\mathrm{d}V\,, \quad \forall D \subset \mathscr{D}\,, \qquad (12.1.13)$$

where $\mathbf{Q}$ and $\mathbf{S}$ are tensors of the same order defined on $D \times \mathscr{T}$, while $\mathbf{R}$ is a tensor of an order greater with a unity, defined on $\overline{D} \times \mathscr{T}$, $\overline{D} = D \cup \partial D$; hence, at any moment t, the increasing (or decreasing) of a quantity of density μ for any subsystem S is due to a *flux* of entering (or exit) through ∂D and of positive (or negative) *internal sources*, $\mu\mathbf{S}$ being their intensity on the unit volume. Applying the formula (12.1.10') to the integral at the left, as well as a formula of flux-divergence type (of the type of the formula (A.2.67)), we get

$$\iiint_{D} \mu\dot{\mathbf{Q}}\,\mathrm{d}V = \iiint_{D} (\operatorname{div}\mathbf{R} + \mu\mathbf{S})\,\mathrm{d}V\,, \quad \forall D \subset \mathscr{D}\,;$$

assuming that $\mathbf{R}$ are fields of class $C^1(D)$ and $\mu\dot{\mathbf{Q}}$ and $\mu\mathbf{S}$ are fields of class $C^0(D)$ and observing that D is an arbitrary subdomain, we obtain the local form of the balance equations (for continuous motions)

$$\operatorname{div}\mathbf{R} + \mu\mathbf{S} = \mu\dot{\mathbf{Q}}\,. \qquad (12.1.13')$$

We can enounce thus four important principles, i.e.:

i) **Conservation principle of mass.** *The mass* $m(S)$ *of any subsystem* $S \subset \mathscr{S}$ *is conserved during the motion.*

This principle, which synthesizes all the three axioms at the basis of the definition of mass, may be expressed in the form

$$\dot{m}(S) = 0\,, \quad \forall S \subset \mathscr{S}\,, \qquad (12.1.14)$$

and may be obtained from (12.1.13) if one makes $\mathbf{R} = \mathbf{S} = \mathbf{0}$ and $\mathbf{Q} = \mathbf{I}$. In a material description, one gets *d'Alembert's condition of mass continuity* (1.1.76), while, in a spatial description, it results *Euler's condition of mass continuity* in the form (1.1.79) or in the form (1.1.79'). The latter relation is equivalent to

$$\iiint_D \left[\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu \mathbf{v}) \right] \mathrm{d}V = 0 .$$

Applying the Gauss-Ostrogradskiĭ formula (A.2.67), we obtain

$$\iiint_D \frac{\partial \mu}{\partial t} \mathrm{d}V = -\int_{\partial D} \mu \mathbf{v} \cdot \mathbf{n} \mathrm{d}S , \quad \forall D \subset \mathscr{D} ; \tag{12.1.15}$$

this equation (called *transportation equation of mass*) shows that the mass variation of D in a unity of time (the left member) is due to the flux of matter through the frontier ∂D (we are led to a decrease of mass if $\mathbf{v} \cdot \mathbf{n} > 0$ or to an increase of mass if $\mathbf{v} \cdot \mathbf{n} < 0$).

ii) **Principle of internal forces** (*Cauchy*). *For any subsystem* $S \subset \mathscr{S}$ *which occupies the domain* $D \subset \mathscr{D}$ *there exists a distribution of internal forces* $\mathbf{p}$ *on the frontier* ∂D*, the action of which upon the subsystem* S *is equivalent to the action of the subsystem* $\mathscr{S} \setminus S$ *upon the same subsystem.*

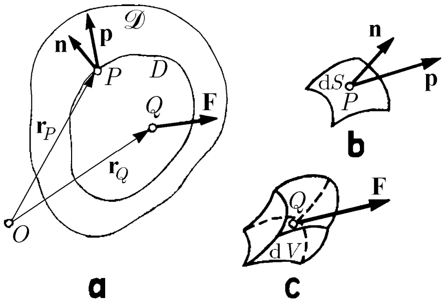

Fig. 12.2 Continuous mechanical system. Internal forces (**a**); stress vector (**b**); body force (**c**)

This principle is, in fact, a *postulate of existence*. It corresponds to the Theorem 11.1.27 of dynamic equilibrium of parts for discrete mechanical systems, being an extension of it for a continuous case (Fig. 12.2a). In case of a discrete mechanical system, upon a subsystem of it may act external forces on a part of the frontier (which can be also zero) and internal forces on the rest of it (which can be the whole frontier too); analogously, a particle subjected to constraints may be considered as being a subsystem of the system formed by that particle and the subsystem which generates the constraints. In case of a continuous mechanical system arises just such a situation, where the internal forces $\mathbf{p}$ are not known a priori; they must be determined using the equations of motion (equilibrium). The superficial forces $\mathbf{p}$ represent a density and form – obviously – an absolutely continuous surface field. Corresponding to Cauchy's mathematical modelling, the internal force **p** at a point P of position vector **r** on ∂D is the same for all surfaces of same *external normal* **n** and of same tangent plane at P; this dependence is expressed in the form (Fig. 12.2b)

$$\mathbf{p} = \mathbf{p}(\mathbf{n}, \mathbf{r}; t), \tag{12.1.16}$$

the vector **p** (denoted sometimes by $\overset{n}{\mathbf{p}} = \overset{n}{\mathbf{p}}(\mathbf{r};t)$) being called *stress vector*. In case of a fluid, *the internal normal* (of unit vector $-\mathbf{n}$) is used, because the stress vector corresponds to a predominant phenomenon of compression (in case of a deformable solid, both compression and stretching appear).

Besides the stress vectors **p**, which represent *contact actions*, we will consider *actions at distance* too, expressed by a *field of body* (*mass*) *forces* **F**, referred to a unit mass, which we assume to be absolutely continuous functions of volume. Unlike the stress vectors, which depend on the configuration of the continuous mechanical system, the body forces do not depend on this configuration ($\mathbf{F} = \mathbf{F}(\mathbf{r};t)$, Fig. 12.2c). As the contact actions, the actions at distance are represented by objective quantities; such quantities are the body force **F** ($\mathbf{F}' = \mathbf{QF}$) and the stress vector **p** ($\mathbf{p}' = \mathbf{Qp}$).

We have seen that the principle ii) corresponds to the theorem of dynamic equilibrium of parts. In the mathematical modelling of a continuous mechanical system, we use – further – the results obtained for discrete mechanical systems, adapting them consequently. Thus, the theorem of rigidity, which – applied to all subsystems of a discrete mechanical system – gives sufficient equations to describe the motion of that system, may be extended to a continuum, enouncing

iii) **Principle of variation of kinetic torsor.** *The derivative with respect to time of the kinetic torsor of any subsystem* $S \subset \mathscr{S}$, *in any of its configurations, with respect to a fixed pole, is equal to the torsor of the forces which act upon that subsystem, with respect to the same pole.*

Obviously, the forces which act upon the considered subsystem S are body forces **F** (given forces) and internal forces **p** (forces linking with the subsystem $\mathscr{S} \setminus S$), postulated by Cauchy. As well, we assume the existence of an inertial frame of reference and of a chronology with respect to which we may enounce this principle. In fact, this principle contains two parts:

iii$_1$) **Principle of variation of momentum**. *The derivative with respect to time of the momentum of any subsystem* $S \subset \mathscr{S}$, *in any of its configurations, is equal to the resultant of the forces which act upon that subsystem.*

Starting from the relation of definition (11.1.1) of the momentum of an arbitrary mechanical system $\mathscr{S}$, we can write

$$\mathbf{H}(S) = \iiint_D \mu(\mathbf{r};t)\mathbf{v}(\mathbf{r};t)\,\mathrm{d}V \tag{12.1.17}$$

for a subsystem $S \subset \mathscr{S}$ of a continuous mechanical system, where D is the domain occupied by that subsystem in the actual state, at the moment t. Introducing the actions of contact and at distance mentioned above, we may express this principle in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \mu\mathbf{v}\,\mathrm{d}V = \iint_{\partial D}\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mu\mathbf{F}\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,. \tag{12.1.17'}$$

iii$_2$) **Principle of variation of moment of momentum.** *The derivative with respect to time of the moment of momentum of any subsystem* $S \subset \mathscr{S}$*, in any of its configurations, with respect to a fixed pole, is equal to the moment of the forces which act upon that subsystem, with respect to the same pole.*

The relation of definition (11.1.2) of the moment of momentum of an arbitrary mechanical system $\mathscr{S}$ with respect to a given pole O leads to

$$\mathbf{K}_O(S) = \iiint_D \mathbf{r}\times[\mu(\mathbf{r};t)\mathbf{v}(\mathbf{r};t)]\,\mathrm{d}V \tag{12.1.18}$$

for the subsystem $S \subset \mathscr{S}$ of a continuous mechanical system. Putting in evidence the moment of the forces of contact or at distance considered above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \mathbf{r}\times(\mu\mathbf{v})\,\mathrm{d}V = \iint_{\partial D}\mathbf{r}\times\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mathbf{r}\times(\mu\mathbf{F})\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,. \tag{12.1.18'}$$

In the case in which $S \equiv \mathscr{S}$, it results $D \equiv \mathscr{D}$, while the stress vectors $\overset{n}{\mathbf{p}}$ become *given external superficial forces*. The two equations (12.1.17'), (12.1.18') have, in this case, the advantage to contain only given external forces, hence known ones, but represent only necessary conditions to describe the motion of the continuous mechanical system $\mathscr{S}$.

The formula (12.1.10') allows to express the equations (12.1.17'), (12.1.18') also in the form

$$\iiint_D \mu\mathbf{a}\,\mathrm{d}V = \iint_{\partial D}\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mu\mathbf{F}\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.19}$$

$$\iiint_D \mathbf{r}\times(\mu\mathbf{a})\,\mathrm{d}V = \iint_{\partial D}\mathbf{r}\times\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mathbf{r}\times(\mu\mathbf{F})\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.19'}$$

being thus led to another principle, equivalent to the principle iii); we may enounce

iii') **Principle of variation of the dynamic torsor.** *The dynamic torsor of any subsystem* $S \subset \mathscr{S}$*, in any of its configurations, with respect to a fixed pole, is equal to the torsor of the forces which act upon that subsystem, with respect to the same pole.*

This principle contains *the principle of variation of the dynamic resultant* (12.1.19) and *the principle of variation of the dynamic moment* (12.1.19').

The conditions of static equilibrium are obtained writing that the position vector of each particle remains constant in time ($\mathbf{v} = \mathbf{0}$); $\forall t \geq 0$, these conditions have the form

$$\iint_{\partial D}\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mu\mathbf{F}\,\mathrm{d}V = \mathbf{0}\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.20}$$

$$\iint_{\partial D}\mathbf{r}\times\overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mathbf{r}\times(\mu\mathbf{F})\,\mathrm{d}V = \mathbf{0}\,,\quad \forall D \subset \mathscr{D}\,. \tag{12.1.20'}$$

Applying the principle of variation of momentum to a subdomain of the form of a cylinder the height of which tends to zero, we may state

Theorem 12.1.1 (*Cauchy*). *At the same point of a continuous mechanical system and at the same moment, the stress vector verifies the relation*

$$\mathbf{p}(\mathbf{n},\mathbf{r};t) = -\mathbf{p}(-\mathbf{n},\mathbf{r};t)\,. \tag{12.1.21}$$

Considering a domain in the form of a three-orthogonal tetrahedron, for which the height relative to the inclined face tends to zero, and applying the principle of variation of momentum, we find the relation

$$\mathbf{p}(\mathbf{n},\mathbf{r};t) = \mathbf{p}(\mathbf{i}_j,\mathbf{r};t)n_j\,. \tag{12.1.22}$$

where n_j are the components of the unit vector $\mathbf{n}$ of the external normal to the element of surface which passes through the point P of position vector $\mathbf{r}$. Projecting this relation on the co-ordinate axis of unit vector $\mathbf{i}_j$, we can state

Theorem 12.1.2 (*Cauchy's basic theorem*). *The state of stress* (*the stress vector* $\mathbf{p}$) *around a point of a continuous mechanical system may be linearly expressed by means of a tensor of second order* (*the stress tensor* $\boldsymbol{\sigma}$) *in the form*

$$\mathbf{p}(\mathbf{n},\mathbf{r};t) = \boldsymbol{\sigma}^{\mathrm{T}}(\mathbf{r};t)\mathbf{n}\,,\quad \overset{n}{p}_i = \sigma_{ji}n_j\,,\quad \sigma_{ij} = p_j(\mathbf{i}_i,\mathbf{r};t)\,,\quad i,j = 1,2,3\,. \tag{12.1.22'}$$

Applying also the principle of variation of moment of momentum to the same domain, one can see that the stress tensor is symmetric (the relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$, $\sigma_{ij} = \sigma_{ji}$, $i,j = 1,2,3$, takes place). We notice that the tensor $\boldsymbol{\sigma}$ defined on D is an objective one.

Taking into account the Theorem 12.1.2, we can write the principles iii$_1$) and iii$_2$) by means of the *balance equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \mu\mathbf{v}\,\mathrm{d}V = \iint_{\partial D}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}\,\mathrm{d}S + \iiint_D \mu\mathbf{F}\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_D \mathbf{r}\times(\mu\mathbf{v})\,\mathrm{d}V = \iint_{\partial D}\mathbf{r}\times(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n})\,\mathrm{d}S + \iiint_D \mathbf{r}\times(\mu\mathbf{F})\,\mathrm{d}V\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.23'}$$

which are of the form (12.1.13); the local form of those balance equations is

$$\operatorname{div}\boldsymbol{\sigma}^{\mathrm{T}} + \mu\mathbf{F} = \mu\mathbf{a}\,,\quad \sigma_{ji,j} + \mu F_i = \mu a_i\,,\quad i = 1,2,3\,. \tag{12.1.23''}$$

Assuming that $\mathbf{p}(\mathbf{n},\mathbf{r};t)$ is the given superficial force which acts upon the continuous mechanical system $\mathscr{S}$ (the jump of the stress vector by passing through the frontier ∂D is equal to zero), the basic theorem of Cauchy, expressed in the form (12.1.22'), allows to put the boundary conditions in stresses at any moment $t \geq t_0$. For a complete formulation of the boundary value problem of a continuous mechanical system we enounce also

iv) **Principle of initial conditions.** *The evolution of a continuous mechanical system* $\mathscr{S}$ *may be determined* $\forall t > t_0$ *if the state* (*positions and velocities*) *of that system at the initial moment* t_0 *is known.*

This principle puts in evidence *the deterministic aspect* of classical models of the continuous mechanical systems (in general, of any mechanical system).

Corresponding to the ideas expressed in Chap. 11, Sect. 1.2.7, we can enounce the first principle of thermodynamics (see relation (11.1.47''')) in the form

$$\mathrm{d}E = \mathrm{d}(T + V) + \mathrm{d}E_{\mathrm{int}} = \mathrm{d}W + \mathrm{d}Q + \mathrm{d}\overline{W}\,, \quad \forall D \subset \mathscr{D}\,, \tag{12.1.24}$$

where V is the potential energy, T is the kinetic energy given by (11.1.6) and expressed in the form

$$T(S) = \frac{1}{2}\iiint_D \mu(\mathbf{r};t)v^2(\mathbf{r};t)\mathrm{d}V\,, \tag{12.1.25}$$

D being the domain occupied by the subsystem $S \subset \mathscr{S}$ in the actual state at the moment t, E_{int} is *the internal energy* (an objective scalar state quantity) of the subsystem S, due to the internal non-conservative forces and postulated as an absolutely continuous function of mass, having the form (e is *the unit internal energy*, an objective quantity too)

$$E_{\mathrm{int}}(S) = \iiint_\omega e\mathrm{d}m = \iiint_D \mu(\mathbf{r};t)e(\mathbf{r};t)\mathrm{d}V\,, \tag{12.1.26}$$

while $\mathrm{d}\overline{W}$ is an *elementary work of non-mechanical and non-calorical nature* (e.g., of electromagnetic nature); the sum $E = T + V + E_{\mathrm{int}}$ is called *total energy*. In case of a continuous mechanical system $\mathscr{S}$, *adiabatically non-isolated*, the quantity of energy may be increased by a *flux of heat* $\mathrm{d}Q$, even without the intervention of an external mechanical work (if the mechanical system $\mathscr{S}$ is *adiabatically isolated*, then we have $\mathrm{d}Q = 0$); we can express $\mathrm{d}Q/\mathrm{d}t$ as a sum of a function q, absolutely continuous of area (an action of contact, due to the phenomenon of conduction), and a function r, absolutely continuous of volume (an action at distance, due to the phenomenon of radiation or due to thermal sources)

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \iint_{\partial D} q(\mathbf{n},\mathbf{r};t)\mathrm{d}S + \iiint_D \mu(\mathbf{r};t)r(\mathbf{r};t)\mathrm{d}V\,, \tag{12.1.27}$$

where r is a unit quantity (with respect to the unit mass), while q is a quantity given by

$$q(\mathbf{n},\mathbf{r};t) = -\mathbf{n}\cdot\mathbf{q}(\mathbf{r};t)\,, \tag{12.1.27'}$$

$\mathbf{q}$ being the *heat current density vector* (the sign – puts in evidence the internal normal, corresponding to the heat received). The relation (12.1.27') represents *the Fourier-Stokes principle of heat flux*, corresponding to the basic theorem of Cauchy. The elementary work of the forces which act upon the subsystem $S \subset \mathscr{S}$, in the actual configuration at the moment t, is expressed in the form

$$\mathrm{d}W = \iint_{\partial D} \mathbf{p}(\mathbf{n},\mathbf{r};t)\cdot \mathrm{d}\mathbf{r}\mathrm{d}S + \iiint_D \mu(\mathbf{r};t)\mathbf{F}(\mathbf{r};t)\cdot \mathrm{d}\mathbf{r}\mathrm{d}V\,, \tag{12.1.28}$$

the power of those forces being given by

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \iint_{\partial D} \mathbf{p}(\mathbf{n},\mathbf{r};t)\cdot \mathbf{v}(\mathbf{r};t)\mathrm{d}S + \iiint_D \mu(\mathbf{r};t)\mathbf{F}(\mathbf{r};t)\cdot \mathbf{v}(\mathbf{r};t)\mathrm{d}V\,. \tag{12.1.28'}$$

Analogously, *the power of actions of non-mechanical and non-calorical nature* will be

$$\overline{P} = \frac{\mathrm{d}\overline{W}}{\mathrm{d}t}. \tag{12.1.29}$$

The principle of energy variation can be written in the form

$$\frac{\mathrm{d}(T+V)}{\mathrm{d}t} + \frac{\mathrm{d}E_{\mathrm{int}}}{\mathrm{d}t} = P + \frac{\mathrm{d}Q}{\mathrm{d}t} + \overline{P}\,,\quad \forall D \subset \mathscr{D}\,, \tag{12.1.24'}$$

corresponding to the first principle of thermodynamics (12.1.24). Taking into account (12.1.25) and (12.1.28'), as well as the differentiation formula (12.1.10'), we can write this principle in the form

$$\begin{aligned} \iiint_D \mu\mathbf{v}\cdot\dot{\mathbf{v}}\mathrm{d}V + \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}E_{\mathrm{int}}}{\mathrm{d}t} &= \iint_{\partial D}\mathbf{p}\cdot\mathbf{v}\mathrm{d}S \\ &+ \iiint_D \mu\mathbf{F}\cdot\mathbf{v}\mathrm{d}V + \frac{\mathrm{d}Q}{\mathrm{d}t} + \overline{P}\,,\quad \forall D\subset\mathscr{D}\,. \end{aligned} \tag{12.1.24''}$$

Putting the condition that the principle of energy variation be invariant to rigid displacements, A.E. Green and R.S. Rivlin have shown that, starting from this principle, one can obtain the principle of torsor variation. For instance, if the relation (12.1.24") takes place for any field of velocities, hence for the field of velocities $\mathbf{v}+\mathbf{c}$, $\mathbf{c} = \overrightarrow{\mathrm{const}}$, too, then we may write

$$\begin{aligned} \iiint_D \mu(\mathbf{v}+\mathbf{c})\cdot\dot{\mathbf{v}}\mathrm{d}V + \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}E_{\mathrm{int}}}{\mathrm{d}t} &= \iint_{\partial D}\mathbf{p}\cdot(\mathbf{v}+\mathbf{c})\mathrm{d}S \\ &+ \iiint_D \mu\mathbf{F}\cdot(\mathbf{v}+\mathbf{c})\mathrm{d}V + \frac{\mathrm{d}Q}{\mathrm{d}t} + \overline{P}\,,\quad \forall D\subset\mathscr{D}\,; \end{aligned}$$

taking into account (12.1.24"), there results

$$\mathbf{c}\cdot\left(\iiint_D \mu\dot{\mathbf{v}}\mathrm{d}V - \iint_{\partial D}\mathbf{p}\mathrm{d}S - \iiint_D \mu\mathbf{F}\mathrm{d}V\right) = 0\,,\quad \forall D\subset\mathscr{D}\,.$$

But the velocity $\mathbf{c}$ is arbitrary, hence the parenthesis vanishes; thus, we find again the principle of variation of the dynamic resultant, equivalent to the principle of variation of momentum.

Assuming that the mechanical system $\mathscr{S}$ is adiabatically isolated ($\mathrm{d}Q/\mathrm{d}t = 0$) and that the power $\overline{P}$ vanishes ($\overline{P} = 0$), we write the power P in the form (we take into account the basic theorem of Cauchy and use the Gauss-Ostrogradskiĭ formula (A.2.67), observing that $(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}) \cdot \mathbf{v} = (\sigma_{ji} n_j) v_i = (\sigma_{ji} v_i) n_j = (\boldsymbol{\sigma}\mathbf{v}) \cdot \mathbf{n}$,

$$P = \iint_{\partial D} (\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}) \cdot \mathbf{v}\,\mathrm{d}S + \iiint_D \mu\mathbf{F} \cdot \mathbf{v}\,\mathrm{d}V = \iiint_D [\operatorname{div}(\boldsymbol{\sigma}\mathbf{v}) + \mu\mathbf{F} \cdot \mathbf{v}]\,\mathrm{d}V$$
$$= \iiint_D [(\sigma_{ji} v_i)_{,j} + \mu F_i v_i]\,\mathrm{d}V = \iiint_D [(\sigma_{ji,j} + \mu F_i) v_i + \sigma_{ji} v_{i,j}]\,\mathrm{d}V$$
$$= \iiint_D (\mu v_i a_i + \sigma_{ij} a_{ij})\,\mathrm{d}V\,,$$

where we have used the decomposition $v_{i,j} = v_{(i,j)} + v_{[i,j]}$, we have noticed that $\sigma_{ij} v_{[i,j]} = 0$ and we have introduced the *deformation velocity tensor*

$$a_{ij} = v_{(i,j)} = \frac{1}{2}(v_{i,j} + v_{j,i})\,; \tag{12.1.30}$$

finally, we can write (a *generalization of Clapeyron's principle,* corresponding to the static case)

$$P = \frac{\mathrm{d}T}{\mathrm{d}t} + P', \quad \forall D \subset \mathscr{D}\,, \tag{12.1.31}$$

where

$$P' = \iiint_D \sigma_{ij} a_{ij}\,\mathrm{d}V \tag{12.1.32}$$

represents *the variation of deformation energy*.

We mention that, in the static case, a_{ij} is replaced by *the strain tensor*

$$\varepsilon_{ij} = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i})\,, \tag{12.1.30'}$$

the velocity $\mathbf{v}$ being replaced by the displacement $\mathbf{u}$.

In case of conservative internal forces, we have $P' = \mathrm{d}V / \mathrm{d}t$, the contribution of eventual non-conservative internal forces (or of a non-conservative part of them) being contained in $\mathrm{d}E_{\mathrm{int}} / \mathrm{d}t$.

Obviously, the first principle of thermodynamics (12.1.24), as well as the principle of variation of energy (12.1.24') (or (12.1.24")) or the generalization (12.1.31) of Clapeyron's principle may be written for a continuous mechanical system $\mathscr{S}$ (corresponding to the domain $\mathscr{D}$) too; but these relations represent only necessary conditions to describe the motion.

12.1.2 General Theorems. Conservation Theorems

In what follows, we present the universal theorems for a continuous mechanical system, both for an inertial and a non-inertial frame of reference; starting from these theorems, we put in evidence some conservation theorems.

12.1.2.1 General and Conservation Theorems with Respect to an Inertial Frame of Reference

Starting from the principle of variation of the torsor, enounced in the preceding subsection, we can write (corresponding to the relations (11.1.53) and (11.1.53") too)

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{R} + \bar{\mathbf{R}}\,, \quad \mathbf{H} = \iiint_{\mathscr{D}} \mu(\mathbf{r};t)\mathbf{v}(\mathbf{r};t)\,\mathrm{d}V\,,$$

$$\mathbf{R} = \sum_{i=1}^{n} \mathbf{F}_i\,, \quad \bar{\mathbf{R}} = \sum_{i=1}^{n} \mathbf{R}_i\,, \tag{12.1.33}$$

$$\frac{\mathrm{d}\mathbf{K}_O}{\mathrm{d}t} = \mathbf{M}_O + \overline{\mathbf{M}}_O\,, \quad \mathbf{K}_O = \iiint_{\mathscr{D}} \mathbf{r} \times [\mu(\mathbf{r};t)\mathbf{v}(\mathbf{r};t)]\,\mathrm{d}V\,,$$

$$\mathbf{M}_O = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i\,, \quad \overline{\mathbf{M}}_O = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{R}_i\,, \tag{12.1.33'}$$

for the continuous mechanical system $\mathscr{S}$, where $\mathbf{F}_i$ and $\mathbf{R}_i$, $i = 1,2,...,n$, are given and constraint external forces which act upon this system; we can add to these forces also absolute continuous body forces, as in the formulae (12.1.17') and (12.1.18'). We thus state:

Theorem 12.1.3 (*theorem of momentum*). *The derivative with respect to time of the momentum of a continuous mechanical system subjected to constraints is equal to the resultant of the given and constraint external forces which act upon that system.*

Theorem 12.1.4 (*theorem of moment of momentum*). *The derivative with respect to time of the moment of momentum of a continuous mechanical system subjected to constraints, with respect to a fixed pole, is equal to the resultant moment of the given and constraint external forces which act upon that system, with respect to the same pole.*

These theorems take place for both holonomic and non-holonomic constraints; as well, we can assume the existence of unilateral constraints. The above mentioned theorems can be included in

Theorem 12.1.5 (*theorem of kinetic torsor*). *The derivative with respect to time of the kinetic torsor of a continuous mechanical system subjected to constraints, with respect to a fixed pole, is equal to the torsor of the given and constraint external forces which act upon that system, with respect to the same pole.*

Introducing the impulse of the resultant of the given and constraint external forces, as well as the impulse of the resultant moment of the same forces in a given interval of time, one can write relations of the form (11.1.54 to 11.1.54") concerning the finite variation of the quantities considered above. Analogously, we can state

Theorem 12.1.6 (*theorem of dynamic torsor; Newton-Euler*). *The dynamic torsor of a continuous mechanical system subjected to constraints, with respect to a fixed pole, is* *equal to the torsor of the given and constraint external forces which act upon that system, with respect to the same pole.*

This theorem includes the theorems of dynamic resultant and of dynamic moment.

Let be a continuous mechanical system $\mathscr{S}$, the support of which is the domain $\mathscr{D}$, separated into two subsystems $\mathscr{S}_1$ and $\mathscr{S}_2$ ($\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{S}$, $\mathscr{S}_1 \cap \mathscr{S}_2 = \emptyset$), of supports $\mathscr{D}_1$ and $\mathscr{D}_2$, and let be $\{\mathbf{R}_1, \mathbf{M}_1\}$ and $\{\mathbf{R}_2, \mathbf{M}_2\}$ the torsors of the external forces acting upon the subsystems $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, with respect to the same pole O; corresponding to the principle ii), we denote by $\{\mathbf{R}_{12}, \mathbf{M}_{12}\}$ and $\{\mathbf{R}_{21}, \mathbf{M}_{21}\}$ the torsors of the internal forces with which the subsystem $\mathscr{S}_2$ acts upon the subsystem $\mathscr{S}_1$ and with which the subsystem $\mathscr{S}_1$ acts upon the subsystem $\mathscr{S}_2$, respectively, with respect to the mentioned pole O. The theorem of torsor (in one of the two forms) allows to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}_1} \mu \mathbf{v} \mathrm{d}V = \mathbf{R}_1 + \mathbf{R}_{12}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}_2} \mu \mathbf{v} \mathrm{d}V = \mathbf{R}_2 + \mathbf{R}_{21},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}} \mu \mathbf{v} \mathrm{d}V = \mathbf{R}_1 + \mathbf{R}_2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}_1} \mathbf{r} \times (\mu \mathbf{v}) \mathrm{d}V = \mathbf{M}_1 + \mathbf{M}_{12}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}_2} \mathbf{r} \times (\mu \mathbf{v}) \mathrm{d}V = \mathbf{M}_2 + \mathbf{M}_{21},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{D}} \mathbf{r} \times (\mu \mathbf{v}) \mathrm{d}V = \mathbf{M}_1 + \mathbf{M}_2;$$

taking into account the property of additivity of the integral, we get

$$\mathbf{R}_{12} + \mathbf{R}_{21} = \mathbf{0}, \quad \mathbf{M}_{12} + \mathbf{M}_{21} = \mathbf{0}, \tag{12.1.34}$$

so that we can state

Theorem 12.1.7 (*theorem of action and reaction*). *Be given a mechanical system $\mathscr{S}$, the torsor of the actions exerted by a subsystem $\mathscr{S}_1 \subset \mathscr{S}$ upon another subsystem $\mathscr{S}_2 \subset \mathscr{S}$ ($\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{S}$, $\mathscr{S}_1 \cap \mathscr{S}_2 = \emptyset$) equilibrates the torsor of the actions* (*reactions*) *exerted by the subsystem $\mathscr{S}_2$ upon the subsystem $\mathscr{S}_1$.*

If, in particular, both subsystems are reduced to two particles P_1 and P_2, then the relations (12.1.34) become

$$\mathbf{R}_{12} + \mathbf{R}_{21} = \mathbf{0}, \quad \mathbf{R}_{12} = \lambda \overrightarrow{P_1 P_2}, \quad \lambda \text{ scalar}, \tag{12.1.34'}$$

finding again the principle of action and reaction in Newton's formulation.

Starting from the relation (3.1.3), we can write the position vector of the centre of mass of the subsystem $S \subset \mathscr{S}$ in the form

$$\boldsymbol{\rho}(S) = \frac{1}{m(S)}\iiint_D \mu(\mathbf{r};t)\mathbf{r}\mathrm{d}V, \quad \forall D \subset \mathscr{D}; \tag{12.1.35}$$

taking into account the differentiation formula (12.1.10') and the formula (12.1.17), we get

$$\mathbf{H}(S) = m(S)\mathbf{v}_C(S), \quad \forall S \subset \mathscr{S}. \tag{12.1.35'}$$

Differentiating once more with respect to time and taking into account the principle iii$_1$) of variation of the impulse, expressed in the form (12.1.17'), we can write

$$m(S)\mathbf{a}_C(S) = \iint_{\partial D} \overset{n}{\mathbf{p}}\,\mathrm{d}S + \iiint_D \mu\mathbf{F}\,\mathrm{d}V, \quad \forall D \subset \mathscr{D}; \tag{12.1.36}$$

we enounce thus (a principle equivalent to the principle iii$_1$) or a theorem, considered as a consequence of the latter principle)

iii'$_1$) **Principle of motion of the centre of mass.** *The centre of mass of any subsystem $S \subset \mathscr{S}$, in any configuration of it, is moving as a free particle, where it is supposed to be concentrated the whole mass of that subsystem and which is acted upon by the resultant of the forces which act upon it.*

For the continuous mechanical system $\mathscr{S}$ of mass M it results

$$M\mathbf{a}_C = \mathbf{R} + \overline{\mathbf{R}}, \tag{12.1.36'}$$

so that we can state

Theorem 12.1.8 (*theorem of motion of the centre of mass*). *The centre of mass of a continuous mechanical system subjected to constraints is moving as a free particle at which would be concentrated the whole mass of that system, being acted upon by the resultant of the given and constraint external forces.*

If the moment of momentum is calculated with respect to a pole Q rigidly linked to the inertial frame $\mathscr{R}$, then both the principle of moment of momentum and the theorem of moment of momentum remain, further, valid. If the pole Q is movable, but the computation is effected with respect to the same inertial frame $\mathscr{R}$, then $\mathbf{K}_O = \mathbf{K}_Q + \overrightarrow{OQ} \times (\mathbf{R} + \overline{\mathbf{R}})$ and

$$\frac{\mathrm{d}\mathbf{K}_Q}{\mathrm{d}t} = \mathbf{M}_Q + \overline{\mathbf{M}}_Q - \mathbf{v}_Q \times \mathbf{H}. \tag{12.1.37}$$

In the case in which $\mathrm{d}Q = \mathrm{d}\overline{W} = 0$, we can state (we use the same formula (11.1.55))

Theorem 12.1.9 (*theorem of kinetic energy*). *The differential of the kinetic energy of a continuous mechanical system subjected to constraints is equal to the elementary work of the given and constraint external and internal forces which act upon that system.*

In case of scleronomic constraints we have $\mathrm{d}W_R = \mathrm{d}W_{R\mathrm{int}} = 0$; the theorem of kinetic energy takes a simpler form ($\mathrm{d}T = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}}$), eventually the form (12.1.31), written for the whole domain $\mathscr{D}$.

As in the case of a discrete mechanical system (see Sect. 11.1.2.9), we can obtain, in certain conditions, conservation theorems also for the continuous mechanical system $\mathscr{S}$.

Thus, if $\mathbf{R} + \overline{\mathbf{R}} = \mathbf{0}$ (necessary condition for statical equilibrium), then we can state a conservation theorem of momentum, the mass centre having a rectilinear and uniform motion, while if $\mathbf{M}_O + \overline{\mathbf{M}}_O = \mathbf{0}$ (necessary condition for statical equilibrium), then we can state a conservation theorem of moment of momentum.

In case of scleronomic constraints and of internal conservative forces (the stress tensor derives from a potential) we obtain a relation of the form (corresponding to the relation (12.1.31))

$$\mathrm{d}W = \mathrm{d}T + \mathrm{d}W_{\mathrm{int}}\,, \tag{12.1.38}$$

where *the external work* (given by *loading* of the mechanical system $\mathscr{S}$ by external loads), *the internal work* (corresponding to the *unloading* of the system $\mathscr{S}$) and *the kinetic energy* are given by

$$\begin{gathered}\frac{\mathrm{d}W}{\mathrm{d}t} = \iint_{\partial\mathscr{D}} \overset{n}{\mathbf{p}}(\mathbf{r};t)\cdot\mathbf{v}(\mathbf{r};t)\mathrm{d}S + \iiint_{\mathscr{D}} \mu(\mathbf{r};t)\mathbf{F}(\mathbf{r};t)\cdot\mathbf{v}(\mathbf{r};t)\mathrm{d}V\,,\\ \frac{\mathrm{d}W_{\mathrm{int}}}{\mathrm{d}t} = \iiint_{\mathscr{D}} \sigma_{ij}a_{ij}\mathrm{d}V\,,\quad T = \frac{1}{2}\iiint_{\mathscr{D}} \mu(\mathbf{r};t)v^2(\mathbf{r};t)\mathrm{d}V\,,\end{gathered} \tag{12.1.39}$$

respectively.

Assuming, in case of a deformable solid, that *the natural state of stress* (the state of stress – the existence of which is supposed – for which all the stresses vanish) and *the initial state of deformation* (for which all the quantities which characterize the deformation are – by definition – equal to zero) correspond to the initial moment $t = 0$, then one obtains a *generalization of Clapeyron's theorem* of the statical case

$$W = T + W_{\mathrm{int}}\,; \tag{12.1.38'}$$

in fact, that is a conservation theorem of energy. In the statical case ($T = 0$), it results *Clapeyron's theorem* in the form

$$W = W_{\mathrm{int}}\,, \tag{12.1.38''}$$

with

$$W = \frac{1}{2}\iint_{\partial\mathscr{D}} \overset{n}{\mathbf{p}}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r})\mathrm{d}S + \frac{1}{2}\iiint_{\mathscr{D}} \mu(\mathbf{r})\mathbf{F}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r})\mathrm{d}V\,, \tag{12.1.39'}$$

where $\mathbf{u}(\mathbf{r})$ is the displacement vector.

12.1.2.2 General and Conservation Theorems with Respect to a Non-inertial Frame of Reference

We refer the continuous mechanical system $\mathscr{S}$ to an inertial (fixed) frame of reference $\mathscr{R}'$ and to a non-inertial (movable) frame $\mathscr{R}$ (Fig. 12.3), using the notations in Sect. 11.2.2. We can write

$$\mathbf{r}' = \mathbf{r}'_O + \mathbf{r}\,, \quad \mathbf{v}' = \mathbf{v}'_O + \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \tag{12.1.40}$$

for a particle P. Defining the momentum of this system in the form

$$\mathbf{H}' = \iiint_{\mathscr{D}} \mu \mathbf{v}' \mathrm{d}V \tag{12.1.41}$$

and taking into account

$$M = \iiint_{\mathscr{D}} \mu \mathrm{d}V\,, \quad M\boldsymbol{\rho} = \iiint_{\mathscr{D}} \mu \mathbf{r} \mathrm{d}V\,, \tag{12.1.40'}$$

we obtain the formula (11.2.11) too, the Theorem 11.2.6 remaining valid also for a continuous mechanical system. As well, applying the theorem of momentum, we find again the equation (11.2.15), corresponding to the motion of the mass centre.

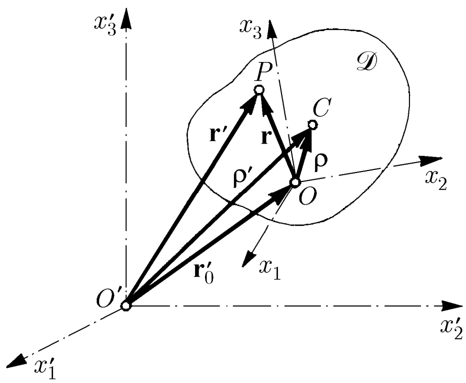

Fig. 12.3 Inertial frame of reference $\mathscr{R}'$ and non-inertial frame of reference $\mathscr{R}$

The moment of momentum is defined in the form

$$\mathbf{K}'_{O'} = \iiint_{\mathscr{D}} \mathbf{r}' \times (\mu \mathbf{v}') \mathrm{d}V\,; \tag{12.1.42}$$

taking into account (12.1.40), (12.1.40'), we find again the formula (11.2.16) of the moment of momentum, where the velocity $\mathbf{v}'_C$ is given by (11.2.14) and where the moment of momentum with respect to a non-inertial frame $\overline{\mathscr{R}}$, which does not rotate about the inertial frame $\mathscr{R}'$, is expressed in the form

$$\overline{\mathbf{K}}_O = \iiint_{\mathscr{D}} \mu \mathbf{r} \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}V = \iiint_{\mathscr{D}} \mu \mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \mathrm{d}V\,. \tag{12.1.43}$$

Introducing at the pole O the tensor of inertia defined by the relation (3.1.81) and using the contracted product of this tensor by the angular velocity rotation vector, we get

$$\iiint_{\mathscr{D}} \mu \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}V = \mathbf{I}_O \boldsymbol{\omega}\,, \tag{12.1.44}$$

being thus lead to the relation (11.2.17') too, where the moment of momentum of the continuous mechanical system $\mathscr{S}$ with respect to the pole O of the non-inertial frame $\mathscr{R}$ is given by

$$\mathbf{K}_O = \iiint_{\mathscr{D}} \mu \mathbf{r} \times \mathbf{v} \mathrm{d} V . \tag{12.1.42'}$$

The Theorem 11.2.7 may be thus stated for a continuous mechanical system $\mathscr{S}$ too, while the theorem of moment of momentum maintains its form (11.2.18), (11.2.18'), observing that the relation

$$\iiint_{\mathscr{D}} \mu \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) \mathrm{d} V + \iiint_{\mathscr{D}} \mu \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r}) \mathrm{d} V = \frac{\partial \mathbf{I}_O}{\partial t} \boldsymbol{\omega} \tag{12.1.44'}$$

takes place.

If, in particular, the frame $\mathscr{R}$ does not rotate ($\boldsymbol{\omega} = \mathbf{0}$, hence $\bar{\mathbf{K}}_O = \mathbf{K}_O$ too) or the pole of the frame $\mathscr{R}$ coincides with the mass centre ($O \equiv C$, hence $\boldsymbol{\rho} = \mathbf{0}$), then one obtains again the formulae (11.2.19–11.2.21'); we remark, especially, the frames and the formulae of Koenig type. Finally, if both conditions mentioned above hold simultaneously, then the frame $\mathscr{R}$ is a Koenig frame and Koenig's theorems hold too.

For a subsystem $S \subset \mathscr{S}$ we can write the second theorem (11.2.22) in the form

$$\frac{\partial}{\partial t} \iiint_{\mathscr{D}} \mathbf{r}^{(C)} \times \left(\mu \mathbf{v}^{(C)} \right) \mathrm{d} V = \iint_{\partial \mathscr{D}} \mathbf{r}^{(C)} \times \overset{n}{\mathbf{p}} \mathrm{d} S + \iiint_{\mathscr{D}} \mathbf{r}^{(C)} \times (\mu \mathbf{F}) \mathrm{d} V , \quad \forall D \subset \mathscr{D}, \tag{12.1.45}$$

and may enounce (a principle equivalent to the principle iii$_2$) or a theorem considered as a consequence of the latter principle)

iii'$_2$) **Principle of variation of moment of momentum with respect to the centre of mass.** *The derivative with respect to time, in a Koenig frame of reference, of the moment of momentum of any subsystem* $S \subset \mathscr{S}$, *in any configuration of it, with respect to the mass centre, is equal to the moment of the forces which act upon that subsystem, with respect to the same pole.*

For the continuous mechanical system $\mathscr{S}$ it results a formula of the form (11.2.24"), so that we can state (second theorem of Koenig for the moment of momentum)

Theorem 12.1.10 (*theorem of moment of momentum with respect to the centre of mass*). *The derivative with respect to time, in a Koenig frame of reference, of the moment of momentum of a continuous mechanical system subjected to constraints, with respect to the centre of mass, is equal to the moment of the given and constraint external forces, with respect to the same pole.*

The theorem of momentum in the form (11.2.15) corresponds to the motion of the centre of mass, while the theorem of moment of momentum in the form (11.2.24") describes the rotation of the continuous mechanical system about the centre of mass, the privileged rôle of which is thus put in evidence.

As in Sect. 11.2.2.1, we can prove a formula of the form (11.2.23'); the Theorem 11.2.9 of C. Iacob can be thus stated for a continuous mechanical system too. As well, the theorem of moment of momentum will have, in general, the same form (11.2.24). Obviously, in the particular cases $\boldsymbol{\omega} = \mathbf{0}$ or $\boldsymbol{\rho} = \mathbf{0}$ we obtain the same results.

We can introduce *the dynamic resultant* of the continuous mechanical system $\mathscr{S}$

$$\mathbf{A}' = \iiint_{\mathscr{D}} \mu(\mathbf{r}';t) \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}'(\mathbf{r}';t) \mathrm{d}V \tag{12.1.46}$$

and *the dynamic moment* of the same system

$$\mathbf{D}'_{O'} = \iiint_{\mathscr{D}} \mathbf{r}' \times \left[\mu(\mathbf{r}';t) \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}'(\mathbf{r}';t) \right] \mathrm{d}V, \tag{12.1.46'}$$

obtaining the results expressed by the formulae (11.2.11') and (11.2.24–11.2.26').

Conservation theorems (hence, first integrals) can be obtained only with respect to the inertial frame of reference $\mathscr{R}'$, in conditions analogous to those put in evidence in Sect. 11.2.2.1.

The kinetic energy is defined in the form

$$T' = \frac{1}{2} \iiint_{\mathscr{D}} \mu v'^2 \mathrm{d}V; \tag{12.1.47}$$

taking into account (12.1.40), (12.1.40'), we find again the formula (11.2.28) of the kinetic energy, where the velocity $\mathbf{v}'_C$ is given by (11.2.14), *the kinetic energy* $\overline{T}$ with respect to the frame $\overline{\mathscr{R}}$ being expressed in the form

$$\overline{T} = \frac{1}{2} \iiint_{\mathscr{D}} \mu (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r})^2 \, \mathrm{d}V = \frac{1}{2} \iiint_{\mathscr{D}} \mu \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)^2 \mathrm{d}V. \tag{12.1.48}$$

Introducing the moment of momentum (12.1.43), we can express the kinetic energy $\overline{T}$ in the form (11.2.28"), where the kinetic energy of the continuous mechanical system $\mathscr{S}$ with respect to the non-inertial frame $\mathscr{R}$ is given by

$$T = \frac{1}{2} \iiint_{\mathscr{D}} \mu v^2 \mathrm{d}V. \tag{12.1.47'}$$

The Theorem 11.2.10 of V. Vâlcovici can be thus stated also for a continuous mechanical system $\mathscr{S}$; as well, one can use also the formulae (11.2.29–11.2.30).

The elementary work and *the power* of the given and constraint, external and internal forces may be further expressed by the formulae (11.2.31), (11.2.31'), so that the Theorem 11.2.11 can be stated for a continuous mechanical system too. As a matter of fact, assuming catastatic internal constraints, we have $\mathrm{d}W'_{R\mathrm{int}} = \mathrm{d}W_{R\mathrm{int}} = P'_{R\mathrm{int}} = P_{R\mathrm{int}} = 0$, the elementary work of deformation being contained in $\mathrm{d}W'_{\mathrm{int}}$ and $\mathrm{d}W_{\mathrm{int}}$ (or in P'_{int} and P_{int}), respectively; indeed, we can assume that the mechanical system $\mathscr{S}$ is formed by several continuous subsystems, being thus subjected to internal constraints.

The theorem of kinetic energy is written in the form (11.2.34), (11.2.34') or in the form (11.2.34"). We notice that all the considerations made in Sect. 11.2.2.2, by particularization, hold. The same observations can be made for the formulae (11.2.41–11.2.42') and for the reciprocity relation (11.2.43) of C. Iacob.

A theorem of mechanical energy of a continuous mechanical system, free or subjected to constraints, as well as a conservation theorem of that energy (hence, a scalar first integral) may be obtained in conditions analogous to those of a discrete mechanical system.

12.2. One-dimensional Continuous Mechanical Systems

A study of the motion of threads and straight bars is presented in this paragraph; results concerning longitudinal and transverse vibrations are also included.

12.2.1 Motion of Threads

In what follows, one makes some general considerations on one-dimensional continuous mechanical systems; results concerning the motion and the vibrations of threads are then dealt with.

12.2.1.1 General Considerations

A one-dimensional continuous mechanical system $\mathscr{L}$ is modelled as a *material line* (a one-dimensional variety in the space E_3, to which we associate a mass depending on a single variable; see Chap. 1, Sect. 1.1.8 too). Let be a curve C, the geometric support of the mechanical system $\mathscr{L}$; its linear density is given by

$$\mu(s;t) = \frac{\mathrm{d}m}{\mathrm{d}s} > 0 , \quad m = m(s) , \tag{12.2.1}$$

s being the curvilinear co-ordinate along that curve (a possible dependence on time is also put in evidence). Hence, the mass of $\mathscr{L}$ is expressed in the form

$$M = \int_C \mu(s)\mathrm{d}s = \int_{P^0}^{P^1} \mu(s)\mathrm{d}s , \tag{12.2.1'}$$

P^0 and P^1 being the extremities of the material line; if that line is homogeneous, then we have

$$M = \mu l , \quad \mu = \text{const} , \tag{12.2.1"}$$

where l is its length. A material line can be a thread or, eventually, a bar (see Chap. 1, Sect. 1.1.10 too). If A is the area of the cross section (a finite area), then we can write $\mu(s;t) = \mu(\mathbf{r};t)A$, where $\mu(\mathbf{r};t) > 0$ is the volume density; it is necessary that $\mu(\mathbf{r};t) = \mathscr{O}(1/A)$, because $\mu(s;t)$ is a bounded function. The position of the mass centre is given by

$$\boldsymbol{\rho} = \frac{1}{M}\int_C \mu(s)\mathbf{r}\mathrm{d}s = \frac{1}{M}\int_{P^0}^{P^1} \mu(s)\mathbf{r}\mathrm{d}s\,, \tag{12.2.2}$$

with respect to an arbitrary pole O.

We assume that at the moment $t = t_0$ the material line occupies the initial position $C(t_0) = C_0$, while, at the moment t, the actual position is given by $C(t) = C$ (Fig. 12.4). Introducing a parameter $\lambda \in [\lambda_0, \lambda_1]$, we can express the equation of the curve C occupied by the material line at the moment t in the form

$$\mathbf{r} = \mathbf{r}(\lambda;t)\,; \tag{12.2.3}$$

for $t = t_0$ we obtain the curve C_0, while $\lambda = \lambda_0$ and $\lambda = \lambda_1$, respectively, specify the extremities of the material line at an arbitrary given moment. The parameter λ plays thus the rôle of a generalized co-ordinate. The velocity and the acceleration of a particle in motion are given by (λ is fixed)

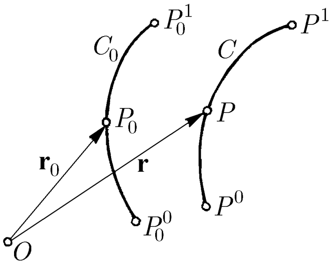

Fig. 12.4 Material line at the moment t_0 and at the moment t

$$\mathbf{v} = \frac{\partial \mathbf{r}(\lambda;t)}{\partial t}\,, \quad \mathbf{a} = \frac{\partial^2 \mathbf{r}(\lambda;t)}{\partial t^2}\,. \tag{12.2.4}$$

For t fixed, these formulae give the distribution of velocities and accelerations along the curve $C(t)$; in particular, for $t = t_0$ are obtained the velocities and the accelerations at the initial state (along the curve C_0)

$$\mathbf{r}(\lambda;t_0) = \mathbf{r}_0(\lambda)\,, \quad \mathbf{v}(\lambda;t_0) = \mathbf{v}_0(\lambda)\,, \quad \mathbf{a}(\lambda;t_0) = \mathbf{a}_0(\lambda)\,. \tag{12.2.4'}$$

For the curvilinear integral

$$I_{C(t)} = \int_C F(\mathbf{r};t)\mathrm{d}s \tag{12.2.5}$$

one obtains the formula (A.2.82'), which can be written also in the form

$$\frac{\mathrm{d}I_C}{\mathrm{d}t} = \int_C \frac{\mathrm{d}F}{\mathrm{d}t}\mathrm{d}s + \int_C F\frac{\dfrac{\partial\mathbf{r}}{\partial\lambda}\cdot\dfrac{\partial\mathbf{v}}{\partial\lambda}}{\left(\dfrac{\partial\mathbf{r}}{\partial\lambda}\right)^2}\mathrm{d}s\,. \tag{12.2.5'}$$

The principle of mass conservation is written in the form (the material line $P'P''$ is a subsystem of the continuous mechanical system P^0P^1, its extremities being specified by the values λ' and λ'' of the parameter λ)

$$\int_{\lambda'}^{\lambda''}\mu(s;t)\mathrm{d}s = \int_{\lambda'}^{\lambda''}\mu(s_0;t_0)\mathrm{d}s_0\,,$$

where the elements of arc are given by

$$\mathrm{d}s = \left|\frac{\partial\mathbf{r}}{\partial\lambda}\right|\mathrm{d}\lambda,\quad \mathrm{d}s_0 = \left|\frac{\partial\mathbf{r}_0}{\partial\lambda}\right|\mathrm{d}\lambda\,; \tag{12.2.3'}$$

the relation must take place for any λ' and λ'', so that one obtains *the continuity condition of d'Alembert* in the form

$$\mu(\lambda;t)\left|\frac{\partial\mathbf{r}}{\partial\lambda}\right| = \mu_0(\lambda)\left|\frac{\partial\mathbf{r}_0}{\partial\lambda}\right|,\quad \mu_0(\lambda) = \mu(\lambda;t_0) \tag{12.2.6}$$

or in the form

$$\mu(s;t)\mathrm{d}s = \mu(s_0)\mathrm{d}s_0\,,\quad \mu(s_0) = \mu(s_0;t_0)\,. \tag{12.2.6'}$$

One can write this principle in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\lambda'}^{\lambda''}\mu(s;t)\mathrm{d}s = 0$$

too; applying the formula (12.2.5') and noting that the limits λ' and λ'' are arbitrary, it results *the condition of continuity*

$$\left(\frac{\partial\mathbf{r}}{\partial\lambda}\right)^2\frac{\partial\mu(\lambda;t)}{\partial t} + \mu(\lambda;t)\frac{\partial\mathbf{r}}{\partial\lambda}\cdot\frac{\partial\mathbf{v}}{\partial\lambda} = 0\,, \tag{12.2.6''}$$

corresponding to the continuity condition of Euler (1.1.79). These conditions of continuity are, obviously, equivalent.

Taking $F(\mathbf{r};t) = \Phi(\lambda;t)\mu(\lambda;t)$, applying the formula (12.2.5') and taking into account (12.2.6''), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \Phi(\lambda;t)\mu(\lambda;t)\mathrm{d}s = \int_C \frac{\partial\Phi}{\partial t}\mu\mathrm{d}s\,; \tag{12.2.7}$$

analogously, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \Phi(\mathbf{r};t)\mu(\lambda;t)\mathrm{d}s = \int_C \frac{\partial \Phi}{\partial t}\mu \mathrm{d}s\,, \tag{12.2.7'}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \mathbf{\Psi}(\mathbf{r};t)\mu(\lambda;t)\mathrm{d}s = \int_C \frac{\partial \mathbf{\Psi}}{\partial t}\mu \mathrm{d}s\,, \tag{12.2.7''}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \mathbf{\Psi}(\lambda;t)\mu(\lambda;t)\mathrm{d}s = \int_C \frac{\partial \mathbf{\Psi}}{\partial t}\mu \mathrm{d}s\,. \tag{12.2.7'''}$$

Upon the material arc $P'P''$ in the actual state (at the moment t) there are exerted actions at distance (volume forces $\mathbf{p}(\lambda)\mathrm{d}s$ and volume moments $\mathbf{m}(\lambda)\mathrm{d}s$; in case of deformable solids it is convenient to report these quantities to the unity of volume) on all its length and actions of contact (the resultants $-\mathbf{R}(\lambda')$ and $\mathbf{R}(\lambda'')$ and the resultant moments $-\mathbf{M}(\lambda')$ and $\mathbf{M}(\lambda'')$) at its extremities (Fig. 12.5a).

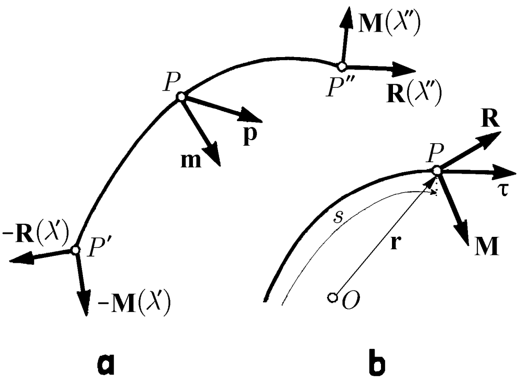

Fig. 12.5 Dynamic equilibrium of a material arc $P'P''$ (**a**).
Efforts on a cross section (**b**)

The momentum and the moment of momentum with respect to a fixed pole O are given by

$$\mathbf{H} = \int_{\lambda'}^{\lambda''} \mu\mathbf{v}(\lambda;t)\mathrm{d}s\,, \quad \mathbf{K}_O = \int_{\lambda'}^{\lambda''} \mathbf{r}\times[\mu\mathbf{v}(\lambda;t)]\mathrm{d}s\,.$$

The principle of variation of momentum and the principle of variation of moment of momentum allow to write

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \int_{\lambda'}^{\lambda''} \mathbf{p}(\lambda)\mathrm{d}s + \mathbf{R}(\lambda'') - \mathbf{R}(\lambda')\,,$$

$$\frac{\mathrm{d}\mathbf{K}_O}{\mathrm{d}t} = \int_{\lambda'}^{\lambda''} \mathbf{r}(\lambda)\times\mathbf{p}(\lambda)\mathrm{d}s + \mathbf{r}(\lambda'')\times\mathbf{R}(\lambda'') - \mathbf{r}(\lambda')\times\mathbf{R}(\lambda')$$
$$+\int_{\lambda'}^{\lambda''} \mathbf{m}(\lambda)\mathrm{d}s + \mathbf{M}(\lambda'') - \mathbf{M}(\lambda').$$

Eventual concentrated forces or moments acting along the material arc $P'P''$ can be contained in $\mathbf{p}(\lambda)\mathrm{d}s$ and $\mathbf{m}(\lambda)\mathrm{d}s$; but these quantities must be, in this case, represented by distributions. Taking into account (12.2.7''') and observing that

$$\mathbf{R}(\lambda'') - \mathbf{R}(\lambda') = \int_{\lambda'}^{\lambda''} \frac{\partial\mathbf{R}}{\partial\lambda}\mathrm{d}\lambda, \quad \mathbf{M}(\lambda'') - \mathbf{M}(\lambda') = \int_{\lambda'}^{\lambda''} \frac{\partial\mathbf{M}}{\partial\lambda}\mathrm{d}\lambda,$$
$$\mathbf{r}(\lambda'')\times\mathbf{R}(\lambda'') - \mathbf{r}(\lambda')\times\mathbf{R}(\lambda') = \int_{\lambda'}^{\lambda''} \frac{\partial}{\partial\lambda}(\mathbf{r}\times\mathbf{R})\mathrm{d}\lambda,$$

we obtain

$$\int_{\lambda'}^{\lambda''} \left(\mu\frac{\partial\mathbf{v}}{\partial t} - \mathbf{p} - \frac{\partial\mathbf{R}}{\partial\lambda}\frac{\partial\lambda}{\partial s}\right)\mathrm{d}s = 0,$$
$$\int_{\lambda'}^{\lambda''} \left[\mathbf{r}\times\left(\mu\frac{\partial\mathbf{v}}{\partial t} - \mathbf{p}\right) - \frac{\partial}{\partial\lambda}(\mathbf{r}\times\mathbf{R})\frac{\mathrm{d}\lambda}{\mathrm{d}s} - \mathbf{m}(\lambda) - \frac{\partial\mathbf{M}}{\partial\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}s}\right]\mathrm{d}s = 0.$$

Because the limits λ' and λ'' are arbitrary, we can write

$$\frac{\partial\mathbf{R}}{\partial s} + \mathbf{p} = \mu\frac{\partial\mathbf{v}}{\partial t} = \mu\mathbf{a}, \tag{12.2.8}$$

as well as

$$\mathbf{r}\times\left(\mu\frac{\partial\mathbf{v}}{\partial t} - \mathbf{p} - \frac{\partial\mathbf{R}}{\partial s}\right) - \frac{\partial\mathbf{r}}{\partial s}\times\mathbf{R} - \mathbf{m} - \frac{\partial\mathbf{M}}{\partial s} = \mathbf{0};$$

taking into account (12.2.8), we get the second equation of motion ($\boldsymbol{\tau} = \partial\mathbf{r}/\partial s$ is the unit vector of the tangent to the material curve $C(t)$ (Fig.12.5b))

$$\frac{\partial\mathbf{M}}{\partial s} + \boldsymbol{\tau}\times\mathbf{R} + \mathbf{m} = \mathbf{0}. \tag{12.2.8'}$$

In the static case, we find again the equations (4.2.36) corresponding to a curved bar (one-dimensional continuous mechanical system). We notice that, in the dynamic case, we use the partial derivative with respect to the curvilinear co-ordinate s, because the considered quantities depend on the temporal variable t too, e.g., $\mathbf{r} = \mathbf{r}(s;t)$. The condition of continuity (12.2.6'') must also be associated to these equations.

12.2.1.2 Equations of Motion of Threads

A thread will be modelled by its axis; thus, the points of application of the external forces as well as those of the efforts are situated on the same axis. If, in the equations of motion (12.2.8), (12.2.8'), we neglect the concentrated moments **M** and the external loads given by distributed moments **m** (the thread being *perfect flexible and torsionable*), then we can express these equations in the form

$$\frac{\partial \mathbf{R}(s;t)}{\partial s} + \mathbf{p}(s;t) = \mu \frac{\partial}{\partial t} \mathbf{v}(s;t), \quad \boldsymbol{\tau}(s;t) \times \mathbf{R}(s;t) = \mathbf{0}, \tag{12.2.9}$$

where $\mathbf{p}(s;t)$ is the external load (distributed on the unit length of the thread), applied at the point of curvilinear abscissa s at the moment t, while $\mathbf{R}(s;t)$ is the resultant of the efforts on the corresponding cross section. Taking into account the second equation (12.2.9) (corresponding to the chosen model for the thread), $\mathbf{R}(s;t)$ is reduced to the axial force $\mathbf{T}(s;t)$ of traction (*the tension* $\mathbf{T}(s;t)$ along the direction of the unit vector $\boldsymbol{\tau}$, tangent to the thread (see Fig. 4.47 too)) and we can state

Theorem 12.2.1 *At any point of a perfect flexible and torsionable thread, in any of its configurations, the tension is tangent to the curve occupied by the thread.*

The equation of motion is reduced to

$$\frac{\partial \mathbf{T}(s;t)}{\partial s} + \mathbf{p}(s;t) = \mu \frac{\partial}{\partial t} \mathbf{v}(s;t). \tag{12.2.10}$$

Noting that $\mathbf{T}(s;t) = T(s;t)\boldsymbol{\tau}(s;t)$, $T(s;t) \geq 0$, is the tension in the thread at the cross section of curvilinear co-ordinate s, it results

$$\frac{\partial}{\partial s}\left(T(s;t) \frac{\partial \mathbf{r}(s;t)}{\partial s} \right) + \mathbf{p}(s;t) = \mu \frac{\partial}{\partial t} \mathbf{v}(s;t); \tag{12.2.10'}$$

in Frenet's frame, the equations have the form

$$\frac{\partial T}{\partial s} + p_\tau = \mu \dot{v}, \quad \frac{T}{\rho} + p_\nu = \frac{\mu}{\rho} v^2, \quad p_\beta = 0. \tag{12.2.10''}$$

The third equation (12.2.10") shows that, to have dynamic equilibrium at any point of the thread, the external force **p** must be contained in the corresponding osculating plane (in other words, the osculating plane to the deformed configuration of the thread contains, at any moment, the support of the external force **p**).

Finally, the equations

$$\mu \frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{p} + \frac{1}{\left| \dfrac{\partial \mathbf{r}}{\partial \lambda} \right|} \frac{\partial}{\partial \lambda} \left(T \frac{\dfrac{\partial \mathbf{r}}{\partial \lambda}}{\left| \dfrac{\partial \mathbf{r}}{\partial \lambda} \right|} \right), \tag{12.2.11}$$

$$\frac{\partial \mu}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial \lambda}\right)^2 + \mu \frac{\partial \mathbf{r}}{\partial \lambda} \cdot \frac{\partial^2 \mathbf{r}}{\partial \lambda \partial t} = 0, \tag{12.2.11'}$$

constitute a system of four scalar equations with partial derivatives for five unknown functions: $T = T(\lambda;t)$, $\mu = \mu(\lambda;t)$ and $x_i = x_i(\lambda;t)$, $i = 1,2,3$. To solve the problem, it is necessary to associate a fifth equation, corresponding to the constitutive law of the material.

The curvilinear abscissa of the particle individualized by the parameter λ on the curve $C(t)$ at the moment t is given by

$$s(\lambda;t) = \int_{\lambda_0}^{\lambda} \left|\frac{\partial \mathbf{r}}{\partial \lambda}\right| \mathrm{d}\lambda ;$$

at the same moment t, we have

$$\mathrm{d}s = \left|\frac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda}\right| \mathrm{d}\lambda ,$$

while for the moment $t + \mathrm{d}t$ we can write

$$\mathrm{d}s' = \left|\frac{\partial \mathbf{r}(\lambda;t+\mathrm{d}t)}{\partial \lambda}\right| \mathrm{d}\lambda = \mathrm{d}s + \frac{\partial}{\partial t}\left|\frac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda}\right| \mathrm{d}\lambda \mathrm{d}t ,$$

neglecting the terms of higher order. The variation of the linear strain in the time interval $\mathrm{d}t$ is written in the form

$$\frac{\mathrm{d}s' - \mathrm{d}s}{\mathrm{d}s} = \frac{\partial}{\partial t}\left|\frac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda}\right| \frac{\mathrm{d}\lambda}{\mathrm{d}s} \mathrm{d}t = \frac{\dfrac{\partial}{\partial t}\left|\dfrac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda}\right|}{\left|\dfrac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda}\right|} \mathrm{d}t .$$

The velocity of strain is given by ($\varepsilon = \varepsilon(\lambda;t)$ is *the linear strain*; see the formula (12.1.30'))

$$\dot{\varepsilon} = \frac{\dfrac{\partial \mathbf{r}}{\partial \lambda} \cdot \dfrac{\partial^2 \mathbf{r}}{\partial \lambda \partial t}}{\left(\dfrac{\partial \mathbf{r}}{\partial \lambda}\right)^2} = \frac{\dfrac{\partial \mathbf{r}}{\partial \lambda} \cdot \dfrac{\partial \mathbf{v}}{\partial \lambda}}{\left(\dfrac{\partial \mathbf{r}}{\partial \lambda}\right)^2} ; \tag{12.2.12}$$

in this case, the condition of continuity (12.2.11') can be written in the form

$$\dot{\mu} + \mu \dot{\varepsilon} = 0 , \tag{12.2.12'}$$

so that

$$\varepsilon + \ln \mu = C\,, \quad \mu e^{\varepsilon} = \bar{C}\,, \tag{12.2.12''}$$

where $C, \bar{C}$ are constant with respect to time. An elastic constitutive law is of the form

$$T = T(\varepsilon)\,; \tag{12.2.13}$$

in particular, if

$$T = k\varepsilon\,, \quad k = \text{const}\,, \tag{12.2.13'}$$

then the constitutive law is linearly elastic.

In the case in which $\dot{\varepsilon} = 0$, hence if ε is constant in time (the tension T has the same property), then from (12.2.12') it results $\mu(\lambda;t) = \mu(\lambda;t_0) = \mu(\lambda)$ too, the thread being, in general, *non-homogeneous*; if the thread is *homogeneous*, then we have $\mu(\lambda) = \mu_0(\lambda) = \text{const}$. From (12.2.1) one obtains

$$\left| \frac{\partial \mathbf{r}(\lambda;t)}{\partial \lambda} \right| = \text{const}\,. \tag{12.2.14}$$

If in the motion of the thread we have $s(\lambda;t) = s(\lambda;t_0) = s(\lambda)$, then we can take as parameter on the curve $C(t)$ the curvilinear abscissa s (instead of λ). If, in particular, $\varepsilon = 0$, then the thread is *inextensible*.

The motion of a perfect flexible and torsionable thread is thus governed by the equation (taking into account (12.1.3'))

$$\mu(s) \frac{\partial^2 \mathbf{u}(s;t)}{\partial t^2} = \mathbf{p} + \frac{\partial \mathbf{T}}{\partial s} = \mathbf{p} + \frac{\partial}{\partial s}\left(T \frac{\partial \mathbf{r}}{\partial s} \right), \tag{12.2.15}$$

which, in projection on the three co-ordinate axes, leads to

$$\mu(s) \frac{\partial^2 u_i(s;t)}{\partial t^2} = p_i + \frac{\partial}{\partial s}\left(T \frac{\partial x_i(s;t)}{\partial s} \right), \quad i = 1,2,3\,. \tag{12.2.15'}$$

To this equation is associated the condition of continuity, which, for $\varepsilon = 0$, is of the form

$$\left| \frac{\partial \mathbf{r}(s;t)}{\partial s} \right| = 1\,. \tag{12.2.15''}$$

One can thus determine the four unknown functions $T = T(s;t)$ and $x_i = x_i(s;t)$, $i = 1,2,3$; the linear density is, in this case, a given function.

If the thread is subjected to constraints, the influence of the constraint forces is added too; if the thread is constrained to stay on a surface, then to the distributed force **p** is added a distributed constraint force **R**, which must be determined. In Chap. 4, Sect. 2.2.4 one has considered the static problem of a thread constrained to stay on a surface, in particular the case in which the thread is situated along the directrix of a circular cylinder; thus, one can change the direction of a force (Fig. 12.6). As well, a weight **P** can be raised by means of a force **Q**, so that, in the absence of friction, $Q = P$ (see also the static problem of the pulley in Chap. 4, Sect. 2.1.6); but if the phenomenon of friction appears too, it is possible to be necessary a force **Q** for which $Q \gg P$ (see Chap. 4, Sect. 2.2.4). In this case, one uses a *pulley*, hence a cylinder movable around a horizontal axle. Due to friction, the (perfect flexible) thread abuts on the cylinder along BC, moving as a rigid (Fig. 12.6). We assume that the forces **P** and **Q**, applied at A and D, respectively, are constant as direction, the angle $\widehat{BOC}$ being constant during the motion of the thread (the thread is inextensible and does not slide on the pulley). We neglect the rolling friction in the bearing of the axle through O, the normal constraint forces leading to a vanishing moment with respect to that point. The velocity of a point of the thread is $R\omega$, where ω is the angular velocity of rotation; if μ is the density of the thread, then its moment of momentum is

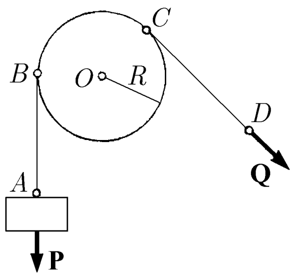

Fig. 12.6 Change of direction of a force with the aid of a pulley

$$\omega r^2 \left(\int_{\widehat{AB}} \mu \mathrm{d}s + \int_{\widehat{BC}} \mu \mathrm{d}s + \int_{\widehat{CD}} \mu \mathrm{d}s \right) = \omega r^2 m \,,$$

where m is the mass. Applying the theorem of moment of momentum to the system pulley-thread, we can write

$$\left(I_O + m r^2 \right) \dot{\omega} = (Q - P) r \,, \tag{12.2.16}$$

where I_O is the moment of inertia of the pulley with respect to the axis passing through O. Neglecting the mass of the thread and its moment of inertia with respect to I_O, it results, with a good approximation,

$$I_O \dot{\omega} = (Q - P) r \,. \tag{12.2.16'}$$

In case of equilibrium ($\omega = 0$) or in case of a uniform motion ($\dot{\omega} = 0$) we get $Q = P$; we obtain the same result if we assume that also the pulley has a negligible mass, hence that the moment of inertia I_O is very small. The pulley allows thus to change the direction of a force, modifying only a little its intensity. As an illustration of those results, we mention the Atwood engine, which was considered in Sect. 11.1.2.2.

For a complete formulation of the boundary value problem, one must give also the initial conditions and the conditions on the frontier. Thus, the initial conditions (at the moment $t = t_0$) are, in general, of the form

$$\mathbf{r}(s;t_0) = \mathbf{r}_0(s), \quad \left.\frac{\partial \mathbf{r}(s;t)}{\partial t}\right|_{t=t_0} = \mathbf{v}_0(s), \quad s \in [s_0, s_1], \tag{12.2.17}$$

specifying the position of the thread and the repartition of the velocities at that moment. In case of a thread of finite length, the conditions on the contour are bilocal conditions. For instance, one can give the position vectors

$$\mathbf{r}(s_0;t) = \mathbf{r}_0(t), \quad \mathbf{r}(s_1;t) = \mathbf{r}_1(t), \quad t \in [t_0, t_1], \tag{12.2.18}$$

or the velocities

$$\dot{\mathbf{r}}(s_0;t) = \dot{\mathbf{r}}_0(t), \quad \dot{\mathbf{r}}(s_1;t) = \dot{\mathbf{r}}_1(t), \quad t \in [t_0, t_1], \tag{12.2.18'}$$

at the extremities of the thread; eventually, one can put mixed bilocal conditions (position vector at one extremity and velocity at the other extremity) or one extremity or two extremities can be fixed. As well, one can give the external forces which act at those points

$$-\mathbf{T}(s_0;t) = \mathbf{F}_0(t), \quad \mathbf{T}(s_1;t) = \mathbf{F}_1(t), \quad t \in [t_0, t_1]. \tag{12.2.18''}$$

In the absence of the force **p**, assuming that $\left|\mu(s)\partial^2\mathbf{u}(s;t)/\partial t^2\right| \cong 0$, the equation (12.2.15) takes the form $\mathbf{T}(s;t) = \mathbf{C}(t)$, $T(s;t) = T(t)$, so that $\partial\mathbf{r}/\partial s = \mathbf{C}(t)/T(t)$, $T(t) \neq 0$; thus, it results

$$\mathbf{r}(s;t) = \frac{\mathbf{C}(t)}{T(t)} s + \mathbf{C}_0(t). \tag{12.2.19}$$

If $T(t) \equiv 0$, then the thread can take any form at the moment t.

12.2.1.3 Longitudinal and Transverse Vibrations of Threads

Let be a thread stretched between two points O and Q at a distance l, along the Ox_3-axis, by the action of a *static tension* $\mathbf{T}_0$; we assume that the thread is acted upon by a longitudinal force $\mathbf{p}_3$ too, of intensity $p_3 = p_3(x_3;t)$ (Fig. 12.7). Making $i = 3$ in the equation (12.2.15') and observing that $s = x_3$, we obtain

$$T_{,3} + p_3 = \mu\ddot{u}_3. \tag{12.2.20}$$

The tension in the thread is given by $T = T_0 + \sigma_{33} A$, where A is *the area of the cross section*, while the *normal stress* σ_{33} is linked to the linear strain ε_{33} by *Hooke's linear constitutive law* $\sigma_{33} = E\varepsilon_{33}$, where E is *the modulus of longitudinal elasticity*; we find thus a linear elastic constitutive law $T = T_0 + EA\varepsilon_{33}$, where EA is the *rigidity in traction* (longitudinal effort). Taking into account Cauchy's relation $\varepsilon_{33} = u_{3,3}$ (corresponding to (12.1.30')), *the equation* (12.2.20) *of longitudinal vibrations of threads* takes the form

$$EAu_{3,33} + p_3 = \mu \ddot{u}_3, \tag{12.2.20'}$$

where the unknown function $u_3 = u_3(x_3;t)$ is the displacement component along the Ox_3-axis. The boundary conditions are conditions on the contour and initial conditions. In our case, the conditions on the contour are bilocal conditions of the form (thread with fixed extremities)

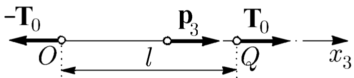

Fig. 12.7 Longitudinal vibrations of threads

$$u_3(0;t) = u_3(l;t) = 0, \quad t \geq t_0, \tag{12.2.21}$$

while the initial conditions are of Cauchy type

$$u_3(x_3;0) = u_3^0(x_3), \quad \dot{u}_3(x_3;0) = \dot{u}_3^0(x_3) \tag{12.2.21'}$$

for $x_3 \in [0,l]$. In case of a thread of infinite length, we put analogous conditions at infinity, the initial conditions being of the same form for $-\infty < x_3 < \infty$. Obviously, we can imagine also other bilocal conditions for a thread of finite length (e.g., imposed displacements or given tensions, depending on time).

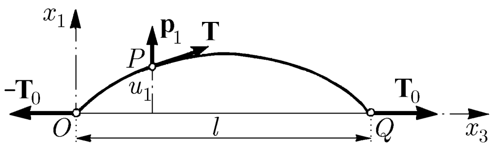

Fig. 12.8 Transverse vibrations of threads

Let us consider also a thread stretched between the points O and Q by the action of a static tension $\mathbf{T}_0$, acted upon by a transverse force $\mathbf{p}_1$, of intensity $p_1(x_3;t)$, in the plane Ox_1x_3 (Fig. 12.8); in this case, taking into account that the force $\mathbf{p}_1$ must be contained at each point of the thread in the osculating plane, it results that the actual configuration of the thread is contained in the same plane. The relation (12.1.3) allows to write $x_1 = x_1^0 + u_1$, where $u_1 = u_1(x_3;t)$ is the displacement component along the direction of the Ox_1-axis; because $x_1^0 = 0$, corresponding to a point of the non-deformed thread (along the Ox_3-axis), it results $x_1 = u_1$. The equation (12.2.15') is thus written in the form

$$\frac{\partial}{\partial s}\left(T\frac{\partial u_1}{\partial s}\right) + p_1 = \mu\ddot{u}_1 \tag{12.2.22}$$

for $i = 1$. Taking into account the relation $T\partial x_3/\partial s = T_3 = T_0$, as well as $\partial x_3/\partial s \cong 1$ in case of small deformations, we can assume that $T \cong T_0 = \text{const}$; on the other hand, we notice that $\partial/\partial s = (\partial/\partial x_3)(\partial x_3/\partial s) \cong \partial/\partial x_3$, according to the same linear approximation. *The equation* (12.2.22) *of the transverse vibrations of threads* will be of the form

$$T_0 u_{1,33} + p_1 = \mu\ddot{u}_1 . \tag{12.2.22'}$$

The contour (bilocal) conditions are, in general, of the form

$$u_1(0;t) = u_1(l;t) = 0, \quad t \geq t_0, \tag{12.2.23}$$

while the initial conditions (of Cauchy type) read

$$u_1(x_3;0) = u_1^0(x_3), \quad \dot{u}_1(x_3;0) = \dot{u}_1^0(x_3), \quad x_3 \in [0,l]. \tag{12.2.23'}$$

As well, we can write the transverse vibrations equation of threads in the plane Ox_2x_3 in the form

$$T_0 u_{2,33} + p_2 = \mu\ddot{u}_2, \tag{12.2.24}$$

and the boundary conditions are analogously put.

We notice that the equations (12.2.20') and (12.2.22') have the same form. Referring, for instance, only to the equation (12.2.22') we obtain, in absence of external forces, d'Alembert's equation of the vibrating string (the homogeneous equation)

$$u_{1,33} - \frac{1}{c^2}\ddot{u}_1 = 0, \quad c^2 = \frac{T_0}{\mu}. \tag{12.2.25}$$

We assume firstly that the thread is of infinite length, the initial conditions being of the form (12.2.23') for $-\infty < x_3 < \infty$. By means of Heaviside's function $\theta(t)$, we introduce the positive part of the function $u_1(x_3;t)$ in the form

$$\bar{u}_1(x_3;t) = u_1(x_3;t)\theta(t) = \begin{cases} 0 & \text{for } t < 0, \\ u_1(x_3;t) & \text{for } t \geq 0. \end{cases} \tag{12.2.26}$$

Passing from this piecewise continuous function to the corresponding regular distribution, the formula (1.1.50) allows to write

$$\frac{\partial \bar{u}_1}{\partial t} = \frac{\tilde{\partial} \bar{u}_1}{\partial t} + u_1^0 \delta(t)\,, \quad \frac{\partial^2 \bar{u}_1}{\partial t^2} = \frac{\tilde{\partial}^2 \bar{u}_1}{\partial t^2} + \dot{u}_1^0 \delta(t) + u_1^0 \dot{\delta}(t)\,, \quad \frac{\partial^2 \bar{u}_1}{\partial x_3^2} = \frac{\tilde{\partial}^2 \bar{u}_1}{\partial x_3^2}\,;$$

observing that for $t \geq 0$ we have

$$\frac{\tilde{\partial}^2 \bar{u}_1}{\partial t^2} = \ddot{u}_1\,, \quad \frac{\tilde{\partial}^2 \bar{u}_1}{\partial x_3^2} = u_{1,33}\,,$$

the equation (12.2.25) may be written with the aid of regular distributions in the form (assuming that $u_1^0,\ \dot{u}_1^0 \in K'(\mathbb{R})$, being thus distributions, we introduce the direct product)

$$\ddot{\bar{u}}_1 = c^2 \bar{u}_{1,33} + \dot{u}_1^0 \times \delta(t) + u_1^0 \times \dot{\delta}(t)\,, \tag{12.2.25'}$$

the initial conditions being included too. We assume that the equation maintains its form also if u_1 is a singular distribution; we apply the Fourier transform with respect to the space variable (the regularity conditions at infinity are thus ensured, see App., Sect. 3.2.1) and the Laplace transform with respect to the time variable (see App., Sect. 3.2.2). We get

$$(\alpha_3^2 c^2 + p^2)\mathrm{L}[\mathrm{F}[\bar{u}_1]] = \mathrm{F}[\dot{u}_1^0] + p\mathrm{F}[u_1^0]\,,$$

wherefrom

$$\mathrm{L}[\mathrm{F}[\bar{u}_1]] = \frac{1}{\alpha_3^2 c^2 + p^2}(\mathrm{F}[\dot{u}_1^0] + p\mathrm{F}[u_1^0])\,,$$

α_3 and p being the new variables in the spaces of Fourier and Laplace transforms, respectively. Taking into account the inverse Laplace transforms

$$\mathrm{L}^{-1}\left[\frac{1}{\alpha_3^2 c^2 + p^2}\right] = \frac{\sin \alpha_3 ct}{\alpha_3 c}\,, \quad \mathrm{L}^{-1}\left[\frac{p}{\alpha_3^2 c^2 + p^2}\right] = \cos \alpha_3 ct\,,$$

we can write

$$\mathrm{F}[\bar{u}_1] = \mathrm{F}[\dot{u}_1^0]\frac{\sin \alpha_3 ct}{\alpha_3 c} + \mathrm{F}[u_1^0]\cos \alpha_3 ct\,;$$

observing that

$$\mathrm{F}^{-1}\left[\frac{\sin \alpha_3 ct}{\alpha_3 c}\right] = \frac{1}{2c}\theta(t)\theta(c^2 t^2 - x_3^2)\,, \quad \mathrm{F}^{-1}[\cos \alpha_3 ct] = x_3 \delta(x_3^2 - c^2 t^2)\,,$$

where

$$\theta\left(c^2t^2 - x_3^2\right) = \theta(ct - |x_3|) = \theta(x_3 + ct) - \theta(x_3 - ct), \quad c > 0,$$

is a distribution corresponding to a characteristic function of interval (Fig. 12.9), while

$$\delta\left(x_3^2 - c^2t^2\right) = -\frac{1}{2x_3}[\delta(x_3 + ct) - \delta(x_3 - ct)], \quad c > 0,$$

and using the formula (A.3.15), we get

$$\mathrm{F}[\bar{u}_1] = \frac{1}{2c}\mathrm{F}\left[\dot{u}_1^0 * \theta(t)\theta(ct - |x_3|)\right]$$
$$+\frac{1}{2}\left(\mathrm{F}\left[u_1^0 * \delta(x_3 + ct)\right] + \mathrm{F}\left[u_1^0 * \delta(x_3 - ct)\right]\right).$$

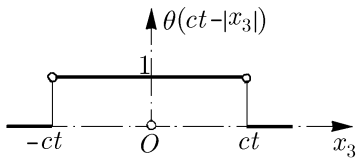

Fig. 12.9 Characteristic function of interval

By means of the formula (A.3.6'), it results, finally,

$$u_1(x_3;t) = \frac{1}{2}\left[u_1^0(x_3 + ct) + u_1^0(x_3 - ct)\right] + \frac{1}{2c}\dot{u}_1^0 * \theta(ct - |x_3|). \qquad (12.2.27)$$

If $u_1^0(x_3) \in C^1$, $\dot{u}_1^0(x_3) \in C^0$, then we write the solution of Cauchy's problem corresponding to the equation (12.2.25) in the form

$$u_1(x_3;t) = \frac{1}{2}\left[u_1^0(x_3 + ct) + u_1^0(x_3 - ct)\right] + \frac{1}{2c}\int_{x_3-ct}^{x_3+ct}\dot{u}_1^0(\xi)\mathrm{d}\xi, \qquad (12.2.27')$$

obtaining thus d'Alembert's formula. Denoting

$$s(z) = \int_0^z \dot{u}_1^0(\xi)\mathrm{d}\xi, \qquad (12.2.28)$$

we may also write

$$u_1(x_3;t) = \frac{1}{2}\left[u_1^0(x_3 + ct) + u_1^0(x_3 - ct)\right] + \frac{1}{2c}[s(x_3 + ct) - s(x_3 - ct)],$$
$$(12.2.28')$$

putting thus in evidence the symmetric and the antisymmetric parts of the deformed thread with respect to the origin $x_3 = 0$, respectively; an initial deformation of the thread (a "crest") given by $u_1^0(x_3)$ (Fig. 12.10a) will be symmetrically propagated with the propagation speed c, while a deformation $s(x_3)$ corresponding to the deformation velocity (Fig. 12.10b) will be antisymmetrically propagated, with the same propagation velocity c (the "wave" advances by c in a unity of time).

In case of a semi-infinite thread, we put the condition at finite distance $u_1^0(0;t) = 0$, being thus led to the solution

$$\begin{aligned} u_1(x_3;t) &= \frac{1}{2}\left[u_1^0(x_3+ct) + \theta(x_3-ct)u_1^0(x_3-ct) - \theta(ct-x_3)u_1^0(ct-x_3)\right] \\ &+ \frac{1}{2c}\left[s(x_3+ct) - s(x_3-ct)\right] = \frac{1}{2c}\left[s(x_3+ct) - s(x_3-ct)\right] \\ &+ \begin{cases} \dfrac{1}{2}\left[u_1^0(x_3+ct) - u_1^0(ct-x_3)\right] & \text{for } x_3 < ct, \\ \dfrac{1}{2}\left[u_1^0(x_3+ct) + u_1^0(x_3-ct)\right] & \text{for } x_3 > ct. \end{cases} \end{aligned} \tag{12.2.29}$$

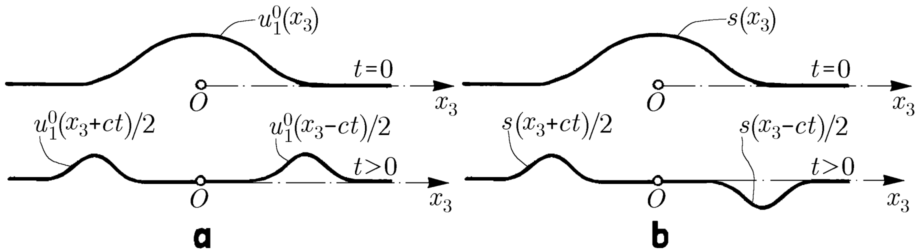

Fig. 12.10 Deformed configuration of the thread: symmetric propagation (**a**); antisymmetric propagation (**b**)

If a perturbing force $p_1 = p_1(x_3;t)$ intervenes too, the equation of motion becomes

$$u_{1,33} - \frac{1}{c^2}\ddot{u}_1 = f(x_3;t), \quad f(x_3;t) = -\frac{1}{T_0}p_1(x_3;t). \tag{12.2.30}$$

The solution $E = E(x_3;t)$ of the equation

$$E_{,33} - \frac{1}{c^2}\ddot{E} = \delta(x_3;t) = \delta(x_3)\times\delta(t) \tag{12.2.30'}$$

is the fundamental solution in the sense of the theory of distributions of the equation (12.2.30). Using the method of integral transforms, we get

$$E(x_3;t) = -\frac{c}{2}\theta(t)\theta(ct-|x_3|) = \begin{cases} -\dfrac{c}{2} & \text{for } |x_3| \leq ct, \\ 0 & \text{otherwise;} \end{cases} \tag{12.2.30''}$$

the fundamental solution is equal to $-c/2$ in the hatched zone and equal to zero in the exterior of it (Fig. 12.11). The corresponding solution of the equation (12.2.30) reads

$$u_1(x_3;t) = -\frac{c}{2}\theta(ct - |x_3|) * f(x_3;t); \qquad (12.2.31)$$

if $f \in C^0$, then we can write

$$u_1(x_3;t) = -\frac{c}{2}\int_0^t \int_{x_3 - c(t-\tau)}^{x_3 + c(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau\,. \qquad (12.2.31')$$

If the thread has a finite length, then we put boundary conditions of the form (12.2.23), (12.2.23'). In case of steady-state vibrations, we assume that (we can obtain separately solutions in $\cos\omega t$ or in $\sin\omega t$)

$$u_1(x_3;t) = u(x_3)\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathrm{e}^{\mathrm{i}\omega t} = \cos\omega t + \mathrm{i}\sin\omega t\,, \quad \omega > 0\,, \qquad (12.2.32)$$

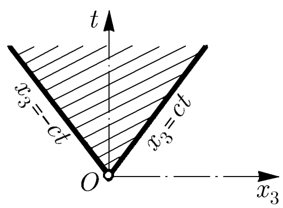

Fig. 12.11 Fundamental solution for the forced transverse vibrations of a thread

where i is the imaginary unity; replacing in the equation (12.2.31), we find the equation

$$u_{,33} + \alpha^2 u = 0\,, \quad \alpha^2 = \frac{\omega^2}{c^2} = \frac{\mu\omega^2}{T_0}\,,$$

whose general integral is $u(x_3) = A\cos\alpha x_3 + B\sin\alpha x_3$. Putting the bilocal conditions (12.2.23) (we cannot put initial conditions, because the vibrations are stationary), we get $A = 0$ and $B \neq 0$, with the characteristic equation $\sin\alpha l = 0$, which leads to *the eigenvalues*

$$\alpha_n = \frac{n\pi}{l}\,, \quad n \in \mathbb{N}\,, \qquad (12.2.33)$$

and then to the *proper pulsations* (*circular frequencies*)

$$\omega_n = \alpha_n c = \frac{n\pi}{l}\sqrt{\frac{T_0}{\mu}}\,; \qquad (12.2.33')$$

the proper form of the free vibrations is thus given by

$$u_{1n}(x_3;t) = B_n \mathrm{e}^{\mathrm{i}\omega_n t}\sin\alpha_n x_3\,, \quad n \in \mathbb{N}\,. \qquad (12.2.33'')$$

We see that $u_{1n}(x_3 + \lambda; t) = u_{1n}(x_3; t)$ for $\lambda = 2l$; λ is called *wave length*. As well, $u_{1n}(x_3; t + \tau_n) = u_{1n}(x_3; t)$ for *the period* $\tau_n = 2\pi / \omega_n = \lambda / nc$, where c is *the propagation velocity of the wave*. We notice that the transverse displacement $u_{1n}(x_3; t)$ vanishes for $x_3 = kl / n = k\lambda / 2n$, $k = 0,1,2,...,n$, the vibrations of the thread having thus n–1 *nodes* between the fixed extremities. *The frequency of the vibrations* (the number f_n of vibrations in a unity of time) is given by $f_n = 1/\tau_n = nc/\lambda$, defining thus *the height of the tone* emitted by the wave (important, e.g., in acoustics); for $n = 1$ one obtains *the fundamental tone*, while for $n = 2$ we have *the octave of the fundamental tone*.

In case of *forced vibrations*, due to *the perturbing force*

$$p_1(x_3; t) = p(x_3)\mathrm{e}^{\mathrm{i}\omega t}, \quad \omega > 0, \tag{12.2.32'}$$

the equation (12.2.30) reads

$$T_0 u_{,33} + \mu\omega^2 u + p = 0.$$

Applying the sinus Fourier transform and observing that

$$\begin{aligned}\int_0^l u_{,33} \sin\alpha_n x_3 \mathrm{d}x_3 &= u_{,3} \sin\alpha_n x_3 \Big|_0^l - \alpha_n u \cos\alpha_n x_3 \Big|_0^l - \alpha_n^2 \int_0^l u \sin\alpha_n x_3 \mathrm{d}x_3 \\ &= -\alpha_n^2 \int_0^l u \sin\alpha_n x_3 \mathrm{d}x_3 = -\alpha_n^2 \mathrm{F}_s[u],\end{aligned}$$

because $u(0) = u(l) = 0$ and $\sin\alpha_n 0 = \sin\alpha_n l = 0$, we get

$$-\mathrm{F}_s[u](\alpha_n^2 T_0 - \mu\omega^2) + \mathrm{F}_s[p] = 0,$$

so that (we use the proper pulsation (12.2.33'))

$$\mathrm{F}_s[u] = \frac{\mathrm{F}_s[p]}{T_0\alpha_n^2[1 - (\omega/\omega_n)^2]};$$

effecting the inverse Fourier transform, it results

$$u_1(x_3; t) = \frac{2}{T_0 l}\mathrm{e}^{\mathrm{i}\omega t}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3}{\alpha_n^2[1 - (\omega/\omega_n)^2]}\int_0^l p(\xi)\sin\alpha_n\xi\mathrm{d}\xi. \tag{12.2.34}$$

We notice that a *phenomenon of resonance* can take place if the pulsation ω of the forced vibrations is very close to one of the proper pulsations ω_n (the displacement u_1 tends to infinity and *the stability is lost by divergence*); it is true that, in this case, the hypothesis of small displacements with respect to the length of the thread does no more hold. In case of the action of a concentrated force $p_1(x_3; t) = \mathrm{e}^{\mathrm{i}\omega t}\delta(x_3 - \xi)$ at the point $x_3 = \xi$, we notice that, by abuse of notation, we can write

$$\int_0^l \delta(x_3 - \xi)\sin\alpha_n x_3 \mathrm{d}x_3 = \sin\alpha_n\xi ,$$

so that we obtain Green's function of the equation (12.2.30) in the form

$$u_1(x_3;t) = \frac{2}{T_0 l}\mathrm{e}^{\mathrm{i}\omega t}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3 \sin\alpha_n\xi}{\alpha_n^2\left[1-(\omega/\omega_n)^2\right]} = G(x_3,\xi;t), \tag{12.2.35}$$

corresponding to a line of influence for the thread. We notice the relation

$$G(x_3,\xi;t) = G(\xi,x_3;t), \tag{12.2.35'}$$

that is a *reciprocity theorem of Betti type* (the displacement at the point x_3, due to a unitary concentrated force applied at ξ, is equal to the displacement at ξ, due to a unitary concentrated force applied at x_3). In this case, we obtain

$$u_1(x_3;t) = \int_0^l p(\xi)G(x_3,\xi;t)\mathrm{d}\xi \tag{12.2.35''}$$

for an arbitrary load (12.2.32'). We can express Green's function also in a finite form by the relation (expanding this relation into a Fourier series, we get (12.2.35))

$$G(x_3,\xi;t) = \frac{\mathrm{e}^{\mathrm{i}\omega t}}{T_0\alpha\sin\alpha l}\begin{cases}\sin\alpha x_3\sin\alpha(l-\xi) & \text{for}\quad 0 < x_3 < \xi,\\ \sin\alpha\xi\sin\alpha(l-x_3) & \text{for}\quad \xi < x_3 < l.\end{cases} \tag{12.2.35'''}$$

If the vibrations are arbitrary (non-stationary), we impose the initial conditions (12.2.23') too. Effecting a Laplace transform and a finite sinus Fourier transform on the equation (12.2.25), brought to the form (12.2.25') in distributions, we get

$$\mathrm{F}_s\left[\mathrm{L}[\bar{u}_1]\right] = \frac{1}{p^2+\alpha^2c^2}\left(\mathrm{F}_s\left[\dot{u}_1^0\right] + p\mathrm{F}_s\left[u_1^0\right]\right);$$

using the same method of computation, it results, finally,

$$u_1(x_3;t) = \frac{2}{l}\sum_{n=1}^{\infty}\left[\cos\omega_n t\int_0^l u_1^0(\xi)\sin\alpha_n\xi\mathrm{d}\xi + \frac{\sin\omega_n t}{\omega_n}\int_0^l \dot{u}_1^0(\xi)\sin\alpha_n\xi\mathrm{d}\xi\right]\sin\alpha_n x_3 , \tag{12.2.36}$$

with the notations (12.2.33), (12.2.33').

Taking into account the action of a perturbing force $p_1 = p_1(x_3;t)$, we apply, analogously, the Laplace transform and the finite sinus Fourier transform to the equation (12.2.30); neglecting the influence of the initial conditions (homogeneous initial conditions), we obtain

$$\mathrm{F}_s\left[\mathrm{L}[\bar{u}_1]\right] = \frac{1}{\mu(p^2+\alpha^2c^2)}\mathrm{F}_s\left[\mathrm{L}[\bar{p}_1]\right],$$

wherefrom, using the convolution theorem, we get

$$u_1(x_3;t) = \frac{2c}{T_0}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3}{\alpha_n l}\int_0^l \sin\alpha_n\xi \mathrm{d}\xi\int_0^t p_1(\xi;\tau)\sin\omega_n(t-\tau)\mathrm{d}\tau\,. \tag{12.2.36'}$$

If the thread is acted upon by a concentrated unitary force at the initial moment (a shock), at the point $x_3 = \xi$, we can write $p_1(x_3;t) = \delta(x_3 - \xi)\times\delta(t)$, obtaining the corresponding Green function

$$G(x_3,\xi;t) = \frac{2c}{T_0}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3}{\sin\alpha_n l}\sin\alpha_n\xi\sin\omega_n(t) = G(\xi,x_3;t)\,, \tag{12.2.37}$$

so that, for an arbitrary load $p_1(x_3;t)\in C^0$, it results

$$u_1(x_3;t) = \int_0^l \mathrm{d}\xi\int_0^t G(x_3,\xi;t-\tau)p_1(\xi;\tau)\mathrm{d}\tau\,. \tag{12.2.37'}$$

12.2.2 Motion of a Straight Bar

The straight bar is a rectilinear one-dimensional continuous mechanical system for which, in the computation of the momentum **H** and of the moment of momentum $\mathbf{K}_O$, one must take into account the cross section of area A. In what follows, we present some results concerning the motion and the vibrations of a straight bar.

12.2.2.1 Equations of Motion of a Straight Bar

Assuming that the linear unit mass μ is constant on the cross section of area A (it does not depend on the position vector $\mathbf{r}$ situated in the plane of that section and having the origin on the bar axis), we can write ($\mathrm{d}s = \mathrm{d}x_3$)

$$\mathbf{H} = \frac{1}{A}\int_{\lambda'}^{\lambda''}\mu\left[\int_A \mathbf{v}(\lambda,\bar{\mathbf{r}};t)\mathrm{d}A\right]\mathrm{d}x_3\,,$$

$$\mathbf{K}_O = \frac{1}{A}\int_{\lambda'}^{\lambda''}(\mathbf{r}+\bar{\mathbf{r}})\times\left[\mu\int_A \mathbf{v}(\lambda,\bar{\mathbf{r}};t)\mathrm{d}A\right]\mathrm{d}x_3\,.$$

Proceeding as in Sect. 12.2.1.1 and observing that ($\partial(\mathbf{r}+\bar{\mathbf{r}})/\partial t = \mathbf{v}$)

$$\frac{\partial}{\partial t}\int_A(\mathbf{r}+\bar{\mathbf{r}})\times\mathbf{v}\mathrm{d}A - \mathbf{r}\times\frac{\partial}{\partial t}\int_A \mathbf{v}\mathrm{d}A = \int_A \bar{\mathbf{r}}\times\mathbf{a}\mathrm{d}A\,,$$

we obtain the equations

$$\mathbf{R}_{,3} + \mathbf{p} = \frac{\mu}{A}\frac{\partial}{\partial t}\int_A \mathbf{v}\mathrm{d}A = \frac{\mu}{A}\int_A \mathbf{a}\mathrm{d}A = \frac{\mu}{A}\int_A \ddot{\mathbf{u}}\mathrm{d}A\,, \tag{12.2.38}$$

$$\mathbf{M}_{,3} + \mathbf{i}_3\times\mathbf{r} + \mathbf{m} = \frac{\mu}{A}\int_A \bar{\mathbf{r}}\times\mathbf{a}\mathrm{d}A = \frac{\mu}{A}\int_A \bar{\mathbf{r}}\times\ddot{\mathbf{u}}\mathrm{d}A\,, \tag{12.2.38'}$$

where we have used an orthonormed right-handed fixed frame $Ox_1x_2x_3$, the Ox_3-axis being along the bar axis, and where we took into account the displacement acceleration (12.1.3'). In the frame of the notation convention used in technical mechanics of deformable solids, the components of the torsor of internal forces (the efforts on the cross section) are given by (4.2.41) and the external load is expressed in the form (4.2.41') (Fig. 12.12ab); there results the relations (the rectilinear continuous mechanical system is flexible, torsionable and extensible)

$$-T_{2,3} + p_1 = \frac{\mu}{A}\int_A \ddot{u}_1 \mathrm{d}A\,, \quad T_{1,3} + p_2 = \frac{\mu}{A}\int_A \ddot{u}_2 \mathrm{d}A\,, \qquad N_{1,3} + p_3 = \frac{\mu}{A}\int_A \ddot{u}_3 \mathrm{d}A\,, \tag{12.2.39}$$

$$M_{1,3} - T_1 + m_1 = \frac{\mu}{A}\int_A x_2\ddot{u}_3 \mathrm{d}A\,, \quad M_{2,3} - T_2 + m_2 = -\frac{\mu}{A}\int_A x_1\ddot{u}_3 \mathrm{d}A\,, \qquad M_{t,3} + m_3 = \frac{\mu}{A}\int_A (x_1\ddot{u}_2 - x_2\ddot{u}_1)\mathrm{d}A\,. \tag{12.2.39'}$$

Eliminating the shearing forces, we get

$$M_{1,33} + m_{1,3} + p_2 = \frac{\mu}{A}\int_A (\ddot{u}_2 + x_2\ddot{u}_{3,3})\mathrm{d}A\,, \qquad M_{2,33} + m_{2,3} - p_1 = -\frac{\mu}{A}\int_A (\ddot{u}_1 + x_1\ddot{u}_{3,3})\mathrm{d}A\,. \tag{12.2.40}$$

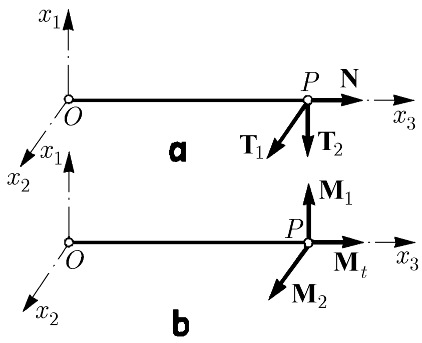

Fig. 12.12 Efforts on a cross section of a straight bar: forces (**a**), moments (**b**)

In case of external loads contained in the plane Ox_1x_3 we have $p_2 = 0$, $m_1 = m_3 = 0$; it results

$$T_{1,3} = \frac{\mu}{A}\int_A \ddot{u}_2 \mathrm{d}A\,, \quad M_{1,3} = T_1 + \frac{\mu}{A}\int_A x_2\ddot{u}_3 \mathrm{d}A\,, \qquad M_{t,3} = \frac{\mu}{A}\int_A (x_1\ddot{u}_2 - x_2\ddot{u}_1)\mathrm{d}A\,, \tag{12.2.41}$$

wherefrom, integrating with respect to x_3, we obtain, successively, the efforts T_1, M_1 and M_t. If we have $m_2 = 0$ too, then the other relations can be written in the form

$$T_{2,3} = p_1 - \frac{\mu}{A}\int_A \ddot{u}_1 \mathrm{d}A\,, \quad N_{,3} = -p_3 + \frac{\mu}{A}\int_A \ddot{u}_3 \mathrm{d}A\,,$$
$$M_{2,3} = T_2 - \frac{\mu}{A}\int_A x_1 \ddot{u}_3 \mathrm{d}A\,, \tag{12.2.41'}$$

wherefrom

$$M_{2,33} = p_1 - \frac{\mu}{A}\int_A (\ddot{u}_1 + x_1 \ddot{u}_{3,3})\mathrm{d}A\,. \tag{12.2.41''}$$

2.2.2 Longitudinal and Torsional Vibrations of a Straight Bar

If the efforts on the cross section are reduced to the axial force *N*, we are in the particular case in which $p_1 = p_2 = 0$, $m_1 = m_2 = m_3 = 0$; the components of the displacement vector are $u_1 = u_2 = 0$, $u_3 = u_3(x_1, x_2; t)$. The equations (12.2.39), (12.2.39') are reduced to the second equation (12.2.41'), the other equations being identically verified; observing that the axial force is given by $N = A\sigma_{33} = AE\varepsilon_{33} = EAu_{3,3}$, where EA is *the rigidity by axial efforts*, this equation takes the form (12.2.10') or the form

$$u_{3,33} - \frac{1}{c^2}u_3 = 0\,, \quad c^2 = \frac{EA}{\mu}\,, \tag{12.2.42}$$

corresponding to the equation (12.2.15), in the absence of the perturbing force. We can thus state that *the longitudinal vibrations of the straight bar* are governed by the same partial derivative equations as the longitudinal vibrations of threads.

The solutions of the boundary value problem depend on the conditions on the end cross sections of the bar (bilocal conditions) and on the initial conditions. Thus, in case of a bar of infinite length with initial conditions (12.2.21') for $x_3 \in (-\infty, \infty)$, we find

$$u_3(x_3; t) = \frac{1}{2}\left[u_3^0(x_3 + ct) + u_3^0(x_3 - ct)\right] + \frac{1}{2c}\int_{x_3 - ct}^{x_3 + ct} \dot{u}_3^0(\xi)\mathrm{d}\xi\,. \tag{12.2.42'}$$

In case of free longitudinal vibrations of a simply supported bar of finite length *l*, hence for which the bilocal conditions (12.2.21) are put, we impose the same initial conditions (12.2.21'); we obtain thus

$$u_3(x_3; t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin\alpha_n x_3 \left[\cos\omega_n t\int_0^l u_3^0(\xi)\sin\alpha_n\xi\mathrm{d}\xi + \frac{\sin\omega_n t}{\omega_n}\int_0^l \dot{u}_3^0(\xi)\sin\alpha_n\xi\mathrm{d}\xi\right], \tag{12.2.43}$$

with the proper pulsations

$$\omega_n = \alpha_n c = \frac{n\pi}{l}\sqrt{\frac{EA}{\mu}}\,, \quad \alpha_n = \frac{n\pi}{l}\,, \tag{12.2.43'}$$

corresponding to the formula (12.2.36). For the forced vibrations of the bar, due to the perturbing force $p_3 = p_3(x_3;t)$, the formula (12.2.36') allows to write

$$u_3(x_3;t) = \frac{2c}{EA}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3}{\alpha_n l}\int_0^l \sin\alpha_n\xi \mathrm{d}\xi \int_0^t p_3(\xi;\tau)\sin\omega_n(t-\tau)\mathrm{d}\tau\,; \tag{12.2.43''}$$

in particular, in case of the perturbing force $p_3(x_3;t) = p(x_3)\mathrm{e}^{\mathrm{i}\omega t}$, $\omega > 0$, it results

$$u_3(x_3;t) = \frac{2}{EAl}\mathrm{e}^{\mathrm{i}\omega t}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x_3}{\alpha_n^2\left[1-(\omega/\omega_n)^2\right]}\int_0^l p(\xi)\sin\alpha_n\xi\mathrm{d}\xi\,, \tag{12.2.43'''}$$

corresponding to the formula (12.2.34). In case of other types of supports, the corresponding boundary conditions are modified; e.g., if the right end cross section ($x_3 = l$) is free, then the normal stress σ_{33} must vanish, i.e.

$$u_{3,3}(l;t) = 0\,,\quad t \geq t_0\,, \tag{12.2.44}$$

taking into account Hooke’s law.

In case of a *phenomenon of torsion*, the moment of torsion is given by $M_t = GI_t\theta$, where GI_t is *the torsional rigidity* (G is *the modulus of transverse elasticity* and I_t is *the moment of inertia by torsion*, which is reduced to the polar moment of inertia I_O in case of a circular or annular cross section), while $\theta = \vartheta_{,3}$ is *the angle of unit rotation* (ϑ is *the rotation angle*); one can show that $u_1 = -x_2\vartheta$, $u_3 = x_1\vartheta$, so that from the third equation (12.2.39') we obtain *the equation of free torsional vibrations* of the straight bar in the form

$$GI_t\vartheta_{,33} + m_3 = \frac{\mu}{A}I_O\ddot{\vartheta}\,, \tag{12.2.45}$$

where I_O is the polar moment of inertia of the cross section. This equation is of the same form as the equation of longitudinal vibrations of the bar. In the absence of the perturbing moment m_3, the homogeneous equation (12.2.45) can be written in the form

$$\vartheta_{,33} - \frac{1}{c^2}\ddot{\vartheta} = 0\,,\quad c^2 = \frac{GA}{\mu}\frac{I_t}{I_O}\,, \tag{12.2.45'}$$

where GA is *the shearing rigidity*. If the left end cross section of the bar is built-in, the condition

$$\vartheta(0;t) = 0 \tag{12.2.46}$$

is imposed, while if it is free one must have

$$\vartheta_{,3}(0;t) = 0\,; \tag{12.2.46'}$$

the initial conditions are put in the form

$$\vartheta(x_3;0) = \vartheta_0(x_3), \quad \dot{\vartheta}(x_3;0) = \dot{\vartheta}_0(x_3), \quad x_3 \in [0,l]. \tag{12.2.46''}$$

Thus, the study of the boundary value problems is made as in the preceding cases.

2.2.3 Transverse Vibrations of the Straight Bar

In case of a straight bar subjected to *bending and shearing* in the plane Ox_1x_3, due to an external load $p_1 \neq 0$, it results $N = T_1 = M_1 = M_t = 0$ and we have $M_2 = EI_2u_{1,33}$, corresponding to *the Bernoulli-Euler equation*, $u_3 = -x_1u_{1,3}$ and $u_{1,1} = u_{2,2} = 0$, corresponding to *the hypothesis of plane cross sections* of Jacob Bernoulli (a plane cross section, normal to the bar axis before deformation remains plane and normal to the deformed axis after application of the external loads; a segment of a line normal to this axis is not subjected to linear strains); here, EI_2 is *the bending rigidity* of the bar, I_2 being the moment of inertia with respect to the Ox_2-axis (*the neutral axis* of the bar). The equation (12.2.41") becomes

$$EI_2u_{1,3333} + \mu\ddot{u}_1 - \frac{\mu}{A}I_2\ddot{u}_{1,33} = p_1. \tag{12.2.47}$$

The influence of the term $(1/A)I_2\ddot{u}_{1,33} = i_2^2\ddot{u}_{1,33}$, where i_2 is *the gyration radius* with respect to the Ox_2-axis, given by a formula of the form (3.1.30'), on the acceleration $\ddot{u}_1$ is small; the researches of Rayleigh showed that this term is important only for high frequencies of the vibrations. We will use thus *the* simplified *equation of the transverse vibrations of the straight bar*

$$c^2u_{1,3333} + \ddot{u}_1 = \frac{p_1}{\mu}, \quad c^2 = \frac{EI_2}{\mu}, \tag{12.2.47'}$$

to which we associate the initial conditions (12.2.23'), together with two other conditions at the end cross sections of the bar (bilocal conditions). The efforts on an arbitrary cross section of the bar are given by (the third relation (12.2.41') leads to $T_2 = EI_2u_{1,333} - \mu i_2^2u_{1,3}$, so that we make the same approximation as above)

$$M_2(x_3;t) = EI_2u_{1,33}, \quad T_2(x_3;t) = EI_2u_{1,333}, \tag{12.2.47''}$$

where $u_1 = u_1(x_3;t)$ is the solution of the equation (12.2.47'). If the end cross section $x_3 = 0$ is hinged, then it results

$$u_1(0;t) = 0, \quad M_2(0;t) = EI_2u_{1,33}(0;t) = 0, \tag{12.2.48}$$

if this section is built-in, then we have

$$u_1(0;t) = 0, \quad u_{1,3}(0;t) = 0 \tag{12.2.48'}$$

and, finally, if the respective cross section is free, then we put the conditions

$$M_2(0;t) = EI_2 u_{1,33}(0;t) = 0\,, \quad T_2(0;t) = EI_2 u_{1,333}(0;t) = 0\,. \tag{12.2.48''}$$

In case of free vibrations of a bar of infinite length one puts only initial conditions of the form

$$\lim_{t\to 0+0} u_1(x_3;t) = u_1^0(x_3)\,, \quad \lim_{t\to 0+0} \dot{u}_1(x_3;t) = cU_{1,33}^0(x_3)\,, \tag{12.2.49}$$

where c is a constant, while $u_1^0(x_3)$ and $U_1^0(x_3)$ are given distributions, the differentiation being effected in the sense of the theory of distributions; the equation of motion will be

$$c^2 u_{1,3333} + \ddot{u}_1 = 0\,, \quad t > 0\,. \tag{12.2.49'}$$

To can replace the differential equation (12.2.49') by a differential equation in distributions, as it was shown by W. Kecs and P.P. Teodorescu, we introduce *the generalized displacement* ($\theta(t)$ is Heaviside’s distribution)

$$\overline{u}_1(x_3;t) = \theta(t)u_1(x_3;t)\,, \tag{12.2.50}$$

by a prolongation at left with null values; at the initial moment $t = 0$ appears a discontinuity of the first kind. Using the formula (1.1.51), we can write

$$\frac{\partial^4}{\partial x_3^4}\overline{u}_1(x_3;t) = \frac{\tilde{\partial}^4}{\partial x_3^4}\overline{u}_1(x_3;t)\,, \quad \frac{\partial}{\partial t}\overline{u}_1(x_3;t) = \frac{\tilde{\partial}}{\partial t}\overline{u}_1(x_3;t) + u_1^0(x_3)\delta(t)\,,$$
$$\frac{\partial^2}{\partial t^2}\overline{u}_1(x_3;t) = \frac{\tilde{\partial}^2}{\partial t^2}\overline{u}_1(x_3;t) + cU_{1,33}^0(x_3)\delta(t) + u_1^0(x_3)\dot{\delta}(t)\,,$$

so that the equation (12.2.49') becomes, in a modified form (we have $\ddot{u}_1 = \tilde{\partial}^2\overline{u}_1/\partial t^2$ and $u_{1,3333} = \tilde{\partial}^4\overline{u}_1/\partial x_3^4$ for $t > 0$),

$$c^2\overline{u}_{1,3333}(x_3;t) + \ddot{\overline{u}}_1(x_3;t) = cU_{1,33}^0(x_3)\delta(t) + u_1^0(x_3)\dot{\delta}(t)\,, \tag{12.2.49''}$$

which contains the initial conditions too, being written in distributions. To find the solution of the equation (12.2.49''), we apply the Laplace transform, with respect to the time variable, and the Fourier transform, with respect to the space variable. We obtain thus

$$c^2(-\mathrm{i}\alpha_3)^4\,\mathrm{F}[\mathrm{L}[\overline{u}_1(x_3;t)]] + p^2\mathrm{F}[\mathrm{L}[\overline{u}_1(x_3;t)]]$$
$$= (-\mathrm{i}\alpha_3)^2\, c\mathrm{F}\left[U_1^0(x_3)\right] + p\mathrm{F}\left[u_1^0(x_3)\right],$$

wherefrom

$$\mathrm{F}[\mathrm{L}[\overline{u}_1(x_3;t)]] = \frac{p}{p^2+\alpha_3^4c^2}\mathrm{F}\left[u_1^0(x_3)\right] - \frac{\alpha_3^2c}{p^2+\alpha_3^4c^2}\mathrm{F}\left[U_1^0(x_3)\right].$$

Applying the inverse Laplace transform and taking into account formulae of the form

$$\mathrm{L}[\theta(x)\cos\omega x] = \frac{p}{p^2+\omega^2}, \quad \mathrm{L}[\theta(x)\sin\omega x] = \frac{\omega}{p^2+\omega^2}, \quad \mathrm{Re}\, p > 0, \tag{12.2.51}$$

we can write

$$\mathrm{F}[\overline{u}_1(x_3;t)] = \mathrm{F}\left[u_1^0(x_3)\right]\cos\left(\alpha_3^2ct\right) - \mathrm{F}\left[U_1^0(x_3)\right]\sin\left(\alpha_3^2ct\right).$$

Applying the inverse Fourier transform to the latter relation, we get

$$\overline{u}_1(x_3;t) = \mathrm{F}^{-1}\left[\mathrm{F}\left[u_1^0(x_3)\right]\cos\left(\alpha_3^2ct\right)\right] - \mathrm{F}^{-1}\left[\mathrm{F}\left[U_1^0(x_3)\right]\sin\left(\alpha_3^2ct\right)\right].$$

Observing that

$$\mathrm{F}\left[\cos\frac{x_3^2}{4ct} + \sin\frac{x_3^2}{4ct}\right] = 2\sqrt{2\pi ct}\cos\left(\alpha_3^2ct\right),$$
$$\mathrm{F}\left[\cos\frac{x_3^2}{4ct} - \sin\frac{x_3^2}{4ct}\right] = 2\sqrt{2\pi ct}\sin\left(\alpha_3^2ct\right)$$

and taking into account the formula (A.3.15) concerning the Fourier transform of a convolution product, in the case in which one of the factors is a temperate distribution, while the other factor is a distribution with bounded support, we can write the solution of the problem in the form

$$\begin{aligned}\overline{u}_1(x_3;t) = \frac{1}{2\sqrt{2\pi ct}}\Bigg[&u_1^0(x_3) * \left(\cos\frac{x_3^2}{4ct} + \sin\frac{x_3^2}{4ct}\right)\\ &-U_1^0(x_3) * \left(\cos\frac{x_3^2}{4ct} - \sin\frac{x_3^2}{4ct}\right)\Bigg].\end{aligned} \tag{12.2.52}$$

If $u_1^0(x_3)$ and $U_1^0(x_3)$ are locally integrable functions, then we have

$$\begin{aligned}u_1(x_3;t) = \frac{1}{2\sqrt{2\pi ct}}\int_{-\infty}^{\infty}\Bigg[&u_1^0(x_3-\xi)\left(\cos\frac{\xi^2}{4ct} + \sin\frac{\xi^2}{4ct}\right)\\ &-U_1^0(x_3-\xi)\left(\cos\frac{\xi^2}{4ct} - \sin\frac{\xi^2}{4ct}\right)\Bigg]\mathrm{d}\xi, \quad t \geq 0;\end{aligned} \tag{12.2.52'}$$

introducing a new variable λ by the relation $\lambda^2 = \xi^2/4ct$, we find again the solution given by Boussinesq

$$u_1(x_3;t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[u_1^0 \left(x_3 - 2\lambda\sqrt{ct} \right) \left(\cos\lambda^2 + \sin\lambda^2 \right) \right.$$
$$\left. - U_1^0 \left(x_3 - 2\lambda\sqrt{ct} \right) \left(\cos\lambda^2 - \sin\lambda^2 \right) \right] \mathrm{d}\lambda, \quad t \geq 0. \tag{12.2.52''}$$

In case of a semi-infinite bar ($x_3 \in [0,\infty)$) with homogeneous initial conditions ($u_1(x_3;0) = 0$, $\dot{u}_1(x_3;0) = 0$) and for which the displacement $u_1(0;t) = u(t)$, $t > 0$, has been imposed, the end cross section being free of stresses ($u_{1,33}(0;t) = 0$, $t > 0$), we find an analogous solution, given also by Boussinesq ($u(t)$ is a locally integrable function)

$$u_1(x_3;t) = \frac{1}{\sqrt{\pi}} \int_{x_3/\sqrt{2ct}}^{\infty} u\left(t - \frac{x_3^2}{2c\lambda^2} \right) \left(\sin\frac{\lambda^2}{2} + \cos\frac{\lambda^2}{2} \right) \mathrm{d}\lambda, \quad t \geq 0. \tag{12.2.53}$$

To determine the free vibrations of a bar of finite length l, we assume that

$$u_1(x_3;t) = U(x_3)\mathrm{e}^{\mathrm{i}\omega t}, \quad x_3 \in [0,l] \tag{12.2.54}$$

the homogeneous equation (12.2.47') leading to the differential equation

$$U_{,3333} - \lambda^4 U = 0, \quad \lambda^4 = \frac{\omega^2}{c^2}; \tag{12.2.54'}$$

on the same way, using the formula (1.1.51), we can write this equation in distributions in the form ($\bar{U}(x_3) = \theta(x_3)U(x_3)$)

$$\bar{U}_{,3333}(x_3) - \lambda^4 \bar{U}(x_3) = U_{,333}(0)\delta(x_3) + U_{,33}(0)\delta_{,3}(x_3)$$
$$+ U_{,3}(0)\delta_{,33}(x_3) + U(0)\delta_{,333}(x_3). \tag{12.2.54''}$$

Applying the Laplace transform and taking into account the formula (A.3.21), we obtain

$$\mathrm{L}[\bar{U}(x_3)] = \frac{1}{p^4 - \lambda^4} \left[p^3 U(0) + p^2 U_{,3}(0) + pU_{,33}(0) + U_{,333}(0) \right],$$

which leads to

$$U(x_3) = U(0)f_1(\lambda x_3) + \frac{1}{\lambda} U_{,3}(0) f_2(\lambda x_3)$$
$$+ \frac{1}{\lambda^2} U_{,33}(0) f_3(\lambda x_3) + \frac{1}{\lambda^3} U_{,333}(0) f_4(\lambda x_3), \quad x_3 \geq 0, \tag{12.2.55}$$

where the functions

$$
\begin{aligned}
f_1(\lambda x_3) &= \frac{1}{2}(\cosh \lambda x_3 + \cos \lambda x_3), \quad f_2(\lambda x_3) = \frac{1}{2}(\sinh \lambda x_3 + \sin \lambda x_3), \\
f_3(\lambda x_3) &= \frac{1}{2}(\cosh \lambda x_3 - \cos \lambda x_3), \quad f_4(\lambda x_3) = \frac{1}{2}(\sinh \lambda x_3 - \sin \lambda x_3),
\end{aligned}
\tag{12.2.56}
$$

verify the relations

$$
\begin{aligned}
f_{1,3}(\lambda x_3) &= \lambda f_4(\lambda x_3), \quad f_{2,3}(\lambda x_3) = \lambda f_1(\lambda x_3), \\
f_{3,3}(\lambda x_3) &= \lambda f_2(\lambda x_3), \quad f_{4,3}(\lambda x_3) = \lambda f_3(\lambda x_3).
\end{aligned}
\tag{12.2.56'}
$$

Taking into account (12.1.47"), we notice that $U(0)$, $U_{,3}(0)$, $U_{,33}(0)$ and $U_{,333}(0)$, hence the quantities which appear in the Cauchy type conditions, are quantities in direct proportion to the displacement, the rotation, the bending moment and the shearing force, respectively, at the left end cross section of the bar. As a matter of fact, these quantities can be easily calculated in any cross section of the bar ($\forall x_3 \in [0,l]$) in the form

$$
\begin{aligned}
U_{,3}(x_3) &= \lambda U(0) f_4(\lambda x_3) + U_{,3}(0) f_1(\lambda x_3) \\
&\quad + \frac{1}{\lambda} U_{,33}(0) f_2(\lambda x_3) + \frac{1}{\lambda^2} U_{,333}(0) f_3(\lambda x_3), \\
U_{,33}(x_3) &= \lambda^2 U(0) f_3(\lambda x_3) + \lambda U_{,3}(0) f_4(\lambda x_3) \\
&\quad + U_{,33}(0) f_1(\lambda x_3) + \frac{1}{\lambda} U_{,333}(0) f_2(\lambda x_3), \\
U_{,333}(x_3) &= \lambda^3 U(0) f_2(\lambda x_3) + \lambda^2 U_{,3}(0) f_3(\lambda x_3) \\
&\quad + \lambda U_{,33}(0) f_4(\lambda x_3) + U_{,333}(0) f_1(\lambda x_3).
\end{aligned}
\tag{12.2.55'}
$$

Using bilocal conditions of the form (12.2.48 – 12.2.48"), two of the quantities $U(0)$, $U_{,3}(0)$, $U_{,33}(0)$, $U_{,333}(0)$ are equated to zero, while the relations (12.2.55), (12.2.55') allow to calculate the other two quantities. One can thus obtain the frequency of the proper vibrations and their corresponding modes. For instance, in case of a simply supported bar we impose the conditions $U(0) = U_{,33}(0) = U(l) = U_{,33}(l) = 0$, wherefrom

$$
U_{,3}(0) f_2(\lambda l) + \frac{1}{\lambda^2} U_{,333}(0) f_4(\lambda l) = 0, \quad U_{,3}(0) f_4(\lambda l) + \frac{1}{\lambda^2} U_{,333}(0) f_2(\lambda l) = 0;
$$

this homogeneous system has non-trivial solutions if $f_2^2(\lambda l) - f_4^2(\lambda l) = 0$ leading to $\sinh \lambda l \sin \lambda l = 0$ or to $\lambda_n l = n\pi$, $n \in \mathbb{N}$. With the notations (12.2.47'), (12.2.54') we obtain the proper pulsations

$$
\omega_n = \frac{n^2 \pi^2}{l^2} \sqrt{\frac{EI_2}{\mu}}, \quad n = 1,2,\ldots. \tag{12.2.57}
$$

In this case $U_{,333}(0)/\lambda^2 = -[f_2(\lambda l)/f_4(\lambda l)]U_{,3}(0) = -U_{,3}(0)$, so that the modes of the proper vibrations are sinusoidal

$$U_n(x_3) = \frac{1}{\lambda_n} U_{,3}(0) \sin \lambda_n x_3 = C_n \sin \lambda_n x_3 , \tag{12.2.57'}$$

where $C_n = \text{const}$ for a certain mode of vibration. If the bar is built-in at the left end cross section and simply supported at the right one (complex problem, the bar being statically indeterminate), we put the conditions $U(0) = U_{,3}(0) = 0$, $U(l) = U_{,33}(l) = 0$; we are thus led to the system of homogeneous equations

$$\frac{1}{\lambda^2} U_{,33}(0) f_3(\lambda l) + \frac{1}{\lambda^3} U_{,333}(0) f_4(\lambda l) = 0 ,$$
$$U_{,33}(0) f_1(\lambda l) + \frac{1}{\lambda} U_{,333}(0) f_2(\lambda l) = 0 ,$$

which admits non-trivial solutions if

$$f_3(\lambda l) f_2(\lambda l) - f_4(\lambda l) f_1(\lambda l) = 0 .$$

We obtain thus the characteristic equation $\tanh \lambda l = \tan \lambda l$ with the roots $\lambda_1 = 3.927/l$, $\lambda_2 = 7.069/l$, $\lambda_3 = 10.210/l$, $\lambda_4 = 13.352/l$, $\lambda_5 = 16.493/l$; for $n > 5$ we can use the asymptotic formula $\lambda_n = (4n+1)\pi/4l$. The proper pulsations are thus given by

$$\omega_n = \lambda_n^2 c = \lambda_n^2 \sqrt{\frac{EI_2}{\mu}}, \quad n = 1, 2, \ldots . \tag{12.2.58}$$

The modes of vibrations are

$$U_n(x_3) = C\left[f_3(\lambda_n x_3) - \frac{f_3(\lambda_n l)}{f_4(\lambda_n l)} f_4(\lambda_n x_3) \right], \tag{12.2.58'}$$

where $C = U_{,33}(0)/\lambda_n^2$ for a certain mode of vibration. Other cases of support may be analogously studied.

Considering also the particular solution of the equation (12.2.47') for $p_1 \neq 0$, we obtain results which put in evidence the forced transverse vibrations of the straight bar.

Chapter 13

Other Considerations on Dynamics of Mechanical Systems

In this chapter, we consider firstly the case of motions with discontinuity, putting in evidence the phenomenon of collision in case of a discrete mechanical system. We deal then with some problems concerning mechanical systems of variable mass.

13.1 Motions with Discontinuity

Using the results obtained in Chap. 5, Sect. 1.2.6 concerning the acceleration of a particle, its motion with discontinuity has been considered in Chap. 10, Sect. 1; the results thus obtained have been then applied to the study of the corresponding phenomenon of collision. The relations of jump which have been put in evidence with this occasion, as well as the relations of jump presented in Sect. 11.1.2, concerning the discrete mechanical systems, are also useful in the study of the phenomenon of collision. Starting from the case of only one particle, we pass to a finite system of particles, modelling mathematically the phenomenon of collision and presenting the corresponding general theorems. We make then a general study of elastic collisions, while for the plastic collisions we introduce a space of plastic collisions.

13.1.1 Phenomenon of Collision in Case of a Discrete Mechanical System

After some general considerations, we present various aspects of the phenomenon of collision in case of a single particle; passing to the case of a discrete mechanical system, a particular attention is given to the general theorems, including the Carnot and Kelvin ones, as well as the principle of virtual work.

13.1.1.1 General Considerations

In general, during the motion of a mechanical system, the velocity (hence, the momentum) of each particle has a continuous variation. If in the evolution of these quantities appear discontinuities, then we say that the mechanical system is subjected to a *shock*; the corresponding mechanical phenomenon is called *collision*. We mention thus various cases of simple technique: nail beating, modelling by beating with a hammer etc. or forging, pressing, boring, riveting, beating a pile with a drop hammer etc., as well as the collision of a car with a wall (the velocity decreases from a finite value to zero in a very short time). The phenomenon of collision appears also by applying suddenly a rigid link upon a mechanical system in motion (the velocities and the momenta have a sudden variation); but a sudden loss of a link does not lead to

P.P. Teodorescu, *Mechanical Systems, Classical Models*,

collision, because the velocities and the momenta have a continuous variation. As well, the sudden rigid linking of two mechanical systems, one of them being in motion and the other at rest (*the clutch* of two mechanisms) is a collision; but the sudden separation of those mechanical systems (*the clutch release* of two mechanisms) is a continuous phenomenon from the point of view of velocities and momenta.

In the frame of the collision phenomenon, in a very short but finite interval of time $[t', t'']$, called *interval of collision*, the velocities of the particles of the (discrete or continuous) mechanical system have finite variations (in magnitude and direction), their positions remaining practically unchanged. As a matter of fact, a deformation of the bodies in collision takes place in case of continuous mechanical systems; this deformation may be permanent (a plastic one) or not (an elastic one). On the other hand, during the mechanical phenomenon, a growth of heat takes place, but it will be neglected in what follows. *The fundamental problem* consists in the determination of the velocities of the particles of a mechanical system after collision, assuming that the corresponding velocities before the phenomenon are known.

In the interval of collision, between the bodies (in general, mechanical systems) in contact are developed forces, the intensity of which increases quickly, reaching very great values, and then decreases; under the action of these forces, the bodies are deformed and we distinguish a *compression phase* and a *relaxation phase*, so that the model of rigid solid is no more sufficient. It is extremely difficult to establish theoretically or to determine experimentally the variation law of these forces in time. We are thus led to the notion of *percussion*, as it was defined in Chap. 10, Sect. 1.2.3, starting from the notions of generalized force and impulse of the generalized force.

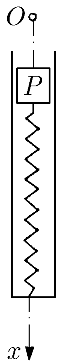

Fig. 13.1 Collision of a heavy body with an elastic spring (model of a collision phenomenon)

We illustrate the above affirmations, together with R. Voinaroski, considering the falling along the vertical of a heavy body (modelled as a particle P of mass m), which comes in contact with a spring vertically guided; the spring is deformed till a position for which the velocity vanishes and, due to its relaxation, the body is launched up, and then is separated from the spring. The equation of motion of the particle P in contact with the spring is $m\ddot{x} = mg - kx$, where $k > 0$ is the elastic constant of the spring, the Ox-axis being directed along the descendent vertical (Fig. 13.1). At the initial moment $t = 0$ we impose the conditions $x(0) = 0$, $\dot{x}(0) = v_0 = \sqrt{2gh}$, h being the height of free falling of the particle P. Denoting $\omega^2 = k/m$, $\omega > 0$, we obtain

$$x(t) = a\cos(\omega t - \varphi) + \frac{g}{\omega^2}, \quad v(t) = -a\sin(\omega t - \varphi), \tag{13.1.1}$$

where

$$a = \frac{1}{\omega^2}\sqrt{g^2 + v^2\omega^2}, \quad \varphi = \operatorname{arc\,tan}\left(-\frac{v_0\omega}{g}\right); \tag{13.1.1'}$$

the spring is compressed in the time interval beginning with $t_0 = \arctan(-v_0\omega/g)/\omega$ till $\dot{x}(t_0) = 0$, and then it is relaxed till the moment $\Delta t = 2t_0$ for which $v(\Delta t) = -v_0$, the jump of the velocity being $\Delta v = v(0) - v(\Delta t) = 2v_0$. The maximal variation of the position of the contact point between the body and the spring is given by $\Delta x = x(t_0) = \left(g + \sqrt{g^2 + v^2\omega^2}\right)/\omega^2$. Let be a helical cylindrical spring of elastic constant $k = 3\cdot 10^4\ \mathrm{daN/m}$ and a particle of mass $m = 10\ \mathrm{kg}$, which falls from $1\ \mathrm{m}$ height; it results $\omega^2 = 3\cdot 10^4\ \mathrm{s}^{-2}$, $v_0 \cong 4.429\ \mathrm{m/s}$ (we have taken $g = 9.81\ \mathrm{m/s^2}$), $\Delta v \cong 8.858\ \mathrm{m/s}$, $t_0 \cong 9.142\cdot 10^{-3}\,\mathrm{s}$, $\Delta t \cong 1.829\cdot 10^{-2}\,\mathrm{s}$, $\Delta x \cong 0.026\ \mathrm{m}$, $\Delta H = m\Delta v = 88.58\ \mathrm{kg\cdot m/s} = 8.858\ \mathrm{daN\cdot s}$. The maximal intensity of the force by which the spring acts upon the particle P is given by $F_{\max} = k\Delta x \cong 780\ \mathrm{daN}$, hence much greater than the weight $mg \cong 9.81\ \mathrm{daN}$ of it. The percussions given by each of these forces will be

$$\begin{aligned} 2\int_0^{t_0} kx(t)\mathrm{d}t &= 2mv_0 + 2mgt_0 = m\Delta v + mg\Delta t \\ &\cong (88.58 + 1.79)\ \mathrm{kg\cdot m/s} = 90.37\ \mathrm{kg\cdot m/s} = 9.037\ \mathrm{daN\cdot s}, \\ 2\int_0^{t_0} mg\mathrm{d}t &= 2mgt_0 = mg\Delta t = 1.79\ \mathrm{kg\cdot m/s} = 0.179\ \mathrm{daN\cdot s}. \end{aligned}$$

The above case can be considered to be a first modelling of a collision, in which some aspects of this phenomenon (e.g., the mass of the spring, the variation of the unknown intensity of the percussive force, modelled as an elastic force, the deformation of the body modelled as a particle etc.) have been neglected; we mention also that the moment of detachment of the particle from the spring (in calculation has been considered a relaxation till the moment Δt) has not been specified. As well, replacing the spring by a damper or by a model formed by a spring and a damper (in parallel or in series), we can obtain a model closer to the reality for the considered mechanical phenomenon. But we can obtain thus useful conclusions to set up a mathematical model with a general character for this phenomenon. We notice thus that $0.179/9.037 \cong 0.020$, hence the percussion due to usual forces (which act permanently) may be neglected with respect to the percussion due to the forces which appear only in the interval of percussion (in the considered case, the usual forces have a contribution of only 2% in what concerns the percussion). We notice also that the interval of percussion Δt is small with respect to the time in which the mechanical phenomenon takes place and that the displacement Δx can be neglected with respect to the dimensions of the mechanical systems which intervene; indeed, if we assume, e.g., that the body modelled by the particle P is of steel, then its volume would have approximately $0.001282\ \mathrm{m}^3$, so that, for any of its form, it would have dimensions much greater that Δx (moreover, the particle modelling implies the neglecting of the dimensions of the body, so much the more the neglecting of the displacement Δx).

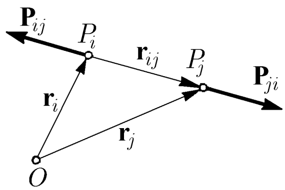

Fig. 13.2 Internal percussions of two particles

The interval of collision is called also *interval of percussion*. The forces which appear in the interval of percussion and which have a great intensity (provoking great variations of the momentum), in a very short interval of time (the percussion interval), are called *percussive forces*, the other forces (the weight of bodies, the elastic forces, the resistance of the air etc.) being *usual* (*non-percussive*) *forces*. Therefore, the percussive forces as well as the percussion interval are thus difficult to estimate; we can put better in evidence the mechanical phenomenon with the aid of *the percussion* defined by the formula (10.1.40). Thus, the two quantities (a vector and a scalar one), of very different order of magnitude, are replaced by a vector quantity of a mean order of magnitude. Because the percussions are obtained starting from the percussive forces, these ones can be classified analogously. We distinguish thus between *given percussive forces* and *constraint percussive forces*, as well as between *external percussive forces* and *internal percussive forces*. Obviously, in case of a mechanical system, starting from the internal forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, which represent the actions of the particles P_i and P_j, respectively, one upon the other, having as support the straight line P_iP_j and being linked by the relation (1.1.81), we can define *the internal percussions* $\mathbf{P}_{ij}$ of support P_iP_j ($\mathbf{P}_{ij} = \lambda \overrightarrow{P_iP_j}$, λ scalar) and the internal percussion $\mathbf{P}_{ji}$, respectively, both percussions being linked by the relation (Fig. 13.2)

$$\mathbf{P}_{ij} + \mathbf{P}_{ji} = \mathbf{0}; \tag{13.1.2}$$

corresponding to the relation (2.2.50), the above properties of the internal percussions (either given or constraint ones) can be expressed in a concise form (the pole O is arbitrary)

$$\tau_O\{\mathbf{P}_{ij}, \mathbf{P}_{ji}\} = \mathbf{0}. \tag{13.1.2'}$$

Before passing to limit in the formula (10.1.40), we can write the relation corresponding to a *mean percussion* in the form

$$\mathbf{P}_m = \int_{t'}^{t''} \mathbf{F}(t)\mathrm{d}t; \tag{13.1.3}$$

if $\mathbf{F}_m$ is the *mean percussive force* in the percussion interval, then we have

$$\mathbf{P}_m = \mathbf{F}_m(t'' - t') = \mathbf{F}_m \Delta t. \tag{13.1.3'}$$

Assuming that, till the moment $t_0 \in [t', t'']$ takes place a *compression phase* (in the interval $[t', t_0]$), a *relaxation phase* being then developed in the interval $[t_0, t'']$, we can define the corresponding mean percussions

$$\mathbf{P}_{cm} = \int_{t'}^{t_0} \mathbf{F}(t)\mathrm{d}t, \quad \mathbf{P}_{rm} = \int_{t_0}^{t''} \mathbf{F}(t)\mathrm{d}t, \tag{13.1.4}$$

with the obvious relation

$$\mathbf{P}_m = \mathbf{P}_{cm} + \mathbf{P}_{rm}. \tag{13.1.4'}$$

Passing to limit in the sense of the theory of distributions (**F** is a generalized force in the sense defined in Chap. 10, Sect. 1.1.2), we obtain

$$\mathbf{P}_c = \lim_{t_0 - t' \to 0+0} \int_{t'}^{t_0} \mathbf{F}(t)\mathrm{d}t, \quad \mathbf{P}_r = \lim_{t'' - t_0 \to 0+0} \int_{t_0}^{t''} \mathbf{F}(t)\mathrm{d}t, \tag{13.1.5}$$

with the relation

$$\mathbf{P} = \mathbf{P}_c + \mathbf{P}_r. \tag{13.1.5'}$$

Together with the generalized force $\mathbf{F}(t)$ appears Dirac's distribution too, so that we can write

$$\mathbf{F}(t) = \mathbf{P}\delta(t - t_0), \tag{13.1.6}$$

justifying thus the denomination of shock given to the respective phenomenon; the percussive forces $\mathbf{F}(t)$ are thus *temporal distributions*.

In case of ideal constraints, *the constraint percussions* $\mathbf{P}_{Rn}$ are normal to the surface elements of contact, while, in case of constraints with friction, the percussions have not only a normal component $\mathbf{P}_{Rn}$ but also a tangential one $\mathbf{P}_{Rt}$, collinear with the sliding velocity and opposite to the latter one; we can assume the Coulombian law $|\mathbf{P}_{Rt}| \leq f|\mathbf{P}_{Rn}|$, f being a coefficient of sliding friction. The dimensional equation of the percussion is $[P] = \mathrm{FT} = \mathrm{LMT}^{-1}$.

We notice that the phenomenon of collision is characterized by the relative velocity of two bodies and not by the absolute velocity of each one. If the support of the relative velocity with which a body strikes another body (*the collision line*) is normal to the surface of the latter one, then the collision is *normal*, while, otherwise, it is *oblique* (in fact, the component of the relative velocity along the collision line intervenes); as well, if the support of this velocity passes through the mass centre, then the collision is *central*. For instance, the normal collision of two homogeneous spherical balls is a central one.

For a mathematical modelling of the collision phenomenon, we make some hypotheses which, taking into account the above considerations and the simplified model studied, correspond sufficiently well to the physical reality. We assume thus that:

i) The principles of mechanics are applied in the conditions considered in Chap. 10, Sect. 1.1.2; especially, the second principle of mechanics is applied in the form (1.1.89), the differentiation being in the sense of the theory of distributions, using generalized forces of the form (10.1.5), (10.1.5').

ii) The usual (non-percussive) forces are neglected with respect to the percussive ones (as it was shown in Chap. 10, Sect. 1.2.3).

iii) It is assumed that, in the interval of percussion, the bodies have not rigid motions (translation or rotation), but only deformations; the position vectors of the points of contact are constant in this interval.

It is assumed that, for two given materials, the ratio between the magnitudes of the normal components of relaxation and compression percussions, respectively, is constant

$$\frac{P_{nr}}{P_{nc}} = k\,, \tag{13.1.7}$$

the constant k being a *restitution coefficient* (*coefficient of elasticity* by collision). Experimentally, it is seen that $0 < k < 1$, the magnitude of the normal collision in the relaxation phase being smaller than the magnitude corresponding to the compression phase; the respective collision is called *elastic-plastic* (*natural*) collision too. In the ideal case $k = 1$ we have $P_{nr} = P_{nc}$, the collision being *elastic* (e.g., for steel, ivory etc.), while for $k = 0$ it results $P_{nr} = 0$ and we have to do with a *plastic* collision (the bodies, e.g., wax, plasticine, clay etc., remain in contact also after the phenomenon of collision).

13.1.1.2 Collision Phenomenon in Case of a Single Particle

In case of a particle subjected to collision, the corresponding mathematical model is based on the hypotheses in the preceding subsection and the fundamental equation which replaces Newton's one is the jump relation (10.1.41); the Theorem 10.1.11 of the momentum may be thus considered as a basic principle. Unlike Newton's equation, which is a differential equation, the relation (10.1.41) is an *algebraic* (*finite*) *relation*, which implies *the jump* of the momentum (in fact, of the velocity) at the theoretic moment t_0 of collision and the percussion at that moment. Assuming that the particle is subjected to constraints too, we can write this relation (or the relation (10.1.43)) in the form

$$(\Delta \mathbf{H})_0 = m\mathbf{v}_0 = m(\mathbf{v}'' - \mathbf{v}') = \mathbf{P} + \mathbf{P}_R, \tag{13.1.8}$$

where $\mathbf{v}'$ is the velocity of the particle before collision, $\mathbf{v}''$ is the velocity of the same particle after collision, while $\mathbf{P}$ and $\mathbf{P}_R$ are the resultants of the percussions corresponding to the given and constraint forces, respectively, at the moment t_0.

Starting from the equation $\mathrm{d}\mathbf{r} = \mathbf{v}\mathrm{d}t$, taking into account the formula (13.1.8) and integrating between the limits of the interval of percussion (we use a mean value formula), we obtain (by $\mathbf{P}_c$ and $\mathbf{P}_{Rc}$ and by $\mathbf{P}_r$ and $\mathbf{P}_{Rr}$ we mean the percussions in the compression and relaxation phase, respectively, at a moment t)

$$\begin{aligned}\mathbf{r}'' - \mathbf{r}' &= \int_{t'}^{t''} \mathbf{v}\mathrm{d}t = \int_{t'}^{t_0} \mathbf{v}\mathrm{d}t + \int_{t_0}^{t''} \mathbf{v}\mathrm{d}t = \int_{t'}^{t_0} \left(\mathbf{v}' + \frac{\mathbf{P}_c + \mathbf{P}_{Rc}}{m} \right) \mathrm{d}t \\ &+ \int_{t_0}^{t''} \left(\mathbf{v}'' - \frac{\mathbf{P}_r + \mathbf{P}_{Rr}}{m} \right) \mathrm{d}t = \left[\mathbf{v}' + \frac{1}{m} (\mathbf{P}_{cm} + \mathbf{P}_{Rcm}) \right] (t_0 - t') \\ &+ \left[\mathbf{v}'' - \frac{1}{m} (\mathbf{P}_{rm} + \mathbf{P}_{Rrm}) \right] (t'' - t_0);\end{aligned}$$

we thus see that the displacement at the contact zone (the difference $|\mathbf{r}'' - \mathbf{r}'|$) is of the order of magnitude of the collision time. Passing to limit ($t_0 - t' \to 0 + 0$ and $t'' - t_0 \to 0 + 0$) and noting that the velocities and the percussions are finite magnitudes, it results $\mathbf{r}'' - \mathbf{r}' \to \mathbf{0}$; the hypothesis iii) is thus theoretically justified at the limit (at the theoretically collision moment).

Analogously, the relation (10.1.42) leads to the relation

$$(\Delta \mathbf{K}_O)_0 = \mathbf{r}_0 \times (\Delta \mathbf{H})_0 = \mathbf{r}_0 \times (\mathbf{P} + \mathbf{P}_R), \tag{13.1.8'}$$

corresponding to the theorem of the moment of momentum.

Starting from the relations (10.1.45) and (10.1.45'), we obtain the theorem of kinetic energy in the form ($T_0 = mv_0^2/2$, $(\Delta T)_0 = T'' - T' = mv''^2/2 - mv'^2/2$)

$$(\Delta T)_0 + T_0 = (\mathbf{P} + \mathbf{P}_R) \cdot \mathbf{v}'', \tag{13.1.9}$$

an analogue of the theorem of kinetic energy being given by

$$(\Delta T)_0 - T_0 = (\mathbf{P} + \mathbf{P}_R) \cdot \mathbf{v}'. \tag{13.1.9'}$$

In particular, if $(\mathbf{P} + \mathbf{P}_R) \cdot \mathbf{v}'' = 0$, then we obtain *Carnot's theorem* in the same form (10.1.47), while if $(\mathbf{P} + \mathbf{P}_R) \cdot \mathbf{v}' = 0$, then an *analogue* of this theorem is given by (10.1.47'). As a matter of fact, Carnot's theorem takes place at the moment of a sudden apparition of a rigid constraint, corresponding to a plastic collision. As well, starting from (10.1.48) to (10.1.48'), we can establish *Kelvin's theorem* or an *analogue* of this theorem, respectively, in the form

$$(\Delta T)_0 = \frac{1}{2}(\mathbf{P} + \mathbf{P}_R) \cdot (\mathbf{v}' + \mathbf{v}''), \tag{13.1.10}$$

$$T_0 = \frac{1}{2}(\mathbf{P} + \mathbf{P}_R) \cdot \mathbf{v}_0 . \tag{13.1.10'}$$

We notice that all the relations obtained above are algebraic (finite) relations, which– obviously – influence (and simplify, in a great measure) the mathematical character of the considered problems. Thus, in case of a free particle P, the velocity $\mathbf{v}'$ before collision and the percussion $\mathbf{P}$ which appears in the interval of collision are considered as known; the velocity of the particle after collision will be given, in this case, by the relation (13.1.8) with $\mathbf{P}_R = \mathbf{0}$, in the form

$$\mathbf{v}'' = \mathbf{v}' + \frac{1}{m}\mathbf{P} . \tag{13.1.11}$$

Taking into account (13.1.11), the relations (13.1.10), (13.1.10') have the remarkable form

$$(\Delta T)_0 = \mathbf{P} \cdot \mathbf{v}' + \frac{1}{2m}\mathbf{P}^2 = \mathbf{P} \cdot \mathbf{v}'' - \frac{1}{2m}\mathbf{P}^2 , \tag{13.1.12}$$

$$T_0 = \frac{1}{2m}\mathbf{P}^2 . \tag{13.1.12'}$$

Let us suppose now that the particle P is subjected to a *unilateral holonomic* (finite), *rheonomous constraint*, of the form

$$f(\mathbf{r};t) \equiv f(x_1, x_2, x_3; t) \geq 0 ; \tag{13.1.13}$$

hence, it can be on a surface S or aside it, so as it was shown in Chap. 3, Sect. 2.2.5. We consider, at the beginning, that the constraint is weak (it is a strict constraint), the particle having a free motion given by the equation $\mathbf{r} = \mathbf{r}(t)$, $t \in [0, \infty)$; let t' be the smallest positive root of the equation $\mathbf{r} = \mathbf{r}(x_1(t), x_2(t), x_3(t); t) = \mathbf{0}$, for which the particle P reaches the surface S with the velocity $\mathbf{v}'$. If the particle moves on the unilateral constraint or leaves it at the moment t, then its velocity must verify the condition $\mathrm{d}f / \mathrm{d}t \geq 0$ or

$$\operatorname{grad} f \cdot \mathbf{v} + \dot{f} \geq 0 , \tag{13.1.14}$$

corresponding to the results in Chap. 3, Sect. 2.2.5; the inequality appears when the particle leaves the constraint. The condition verified by the velocity $\mathbf{v}'$ when the particle reaches the surface S is $\mathrm{d}f / \mathrm{d}t|_{t=t'} = 0$. Let us suppose now that $\mathbf{v}_0$ is thus that $\mathrm{d}f / \mathrm{d}t|_{t=t'} < 0$. To put this relation in concordance with the condition (13.1.14), we must assume that, in the interval of percussion, appears a constraint percussive force, which leads to a constraint percussion $\mathbf{P}_R$; thus, arises a jump of the particle velocity, which – at the end of the percussion interval – will be $\mathbf{v}''$, verifying the condition (13.1.14). Hence, in the frame of the mathematical model previously set up, we must have

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'} < 0\,, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} \geq 0\,.$$

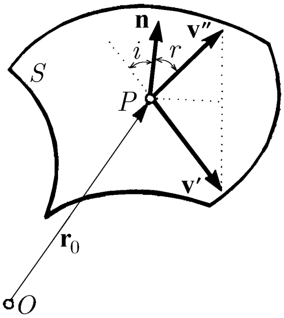

Fig. 13.3 Particle subjected to a unilateral constraint – phenomenon of collision

We can suppose that, at a moment $t_0 \in [t', t'']$, the particle reaches the surface S, where it reaches a velocity $\mathbf{v}^0$, so that to have $\mathrm{d}f / \mathrm{d}t|_{t=t_0} = 0$. The velocity $\mathbf{v}'$ by which the particle reaches the surface is called *incidental velocity*, while the velocity $\mathbf{v}''$ by which that one leaves it *reflected velocity*; the angles formed by these velocities with the external normal to the surface, of unit vector $\mathbf{n}$, are *the incidental angle* i and *the reflected angle* r, respectively (Fig. 13.3). In case of ideal constraints, the constraint percussion $\mathbf{P}_R$ is directed along the unit vector $\mathbf{n}$ (along grad f), so that $\mathbf{P}_{Rn} = P_{Rn}\mathbf{n}$. The fundamental equation (13.1.8) allows to write $m\mathbf{v}_0 = P_{Rn}\mathbf{n}$, the variation $\mathbf{v}_0$ of the velocity taking place along the unit vector $\mathbf{n}$, in its positive sense; it results that both velocities $\mathbf{v}'$ and $\mathbf{v}''$ are contained in a plane normal to the surface S, their projections on the tangent plane being equal

$$v'' \sin r = v' \sin i\,. \tag{13.1.15}$$

The fundamental problem which is put consists in the determination of the reflected velocity $\mathbf{v}''$ and of the constraint percussion $\mathbf{P}_{Rn}$ if the incidental velocity $\mathbf{v}'$ and the position of the particle in the interval of collision (given by the position vector $\mathbf{r}_0$) are known. If $\mathbf{P}'_{Rn}$ and $\mathbf{P}''_{Rn}$ are the constraint percussions corresponding to the first phase and to the second phase of the collision phenomenon, respectively (in case of a deformable body, they correspond to the compression phase and to the relaxation phase, respectively), then we have

$$m(\mathbf{v}^0 - \mathbf{v}') = \mathbf{P}'_{Rn}\,, \quad m(\mathbf{v}'' - \mathbf{v}^0) = \mathbf{P}''_{Rn}\,, \tag{13.1.16}$$

as well as the obvious relation

$$m(\mathbf{v}'' - \mathbf{v}') = \mathbf{P}_{Rn} = \mathbf{P}'_{Rn} + \mathbf{P}''_{Rn}. \tag{13.1.16'}$$

The condition $\mathrm{d}f / \mathrm{d}t|_{t=t_0} = \operatorname{grad} f|_{t=t_0} \cdot \mathbf{v}^0 + \dot{f}\big|_{t=t_0} = 0$ can be transcribed also in one of the forms (we take into account the hypothesis iii)

$$\operatorname{grad} f|_{t=t'} \cdot \mathbf{v}^0 + \dot{f}\big|_{t=t'} = 0, \tag{13.1.17}$$

$$\operatorname{grad} f|_{t=t''} \cdot \mathbf{v}^0 + \dot{f}\big|_{t=t''} = 0; \tag{13.1.17'}$$

the unknown quantities are thus given by the equations (13.1.16) or (13.1.16') and (13.1.17) or (13.1.17').

In the particular case of a *plastic collision* (the first phase of the collision phenomenon), the problem to determine the velocity $\mathbf{v}^0$ and the constraint percussion $\mathbf{P}'_{Rn}$ with the aid of the first equation (13.1.16) and of the equation (13.1.17) is put; by eliminating the velocity $\mathbf{v}^0$ between these equations, one obtains

$$-m \operatorname{grad} f|_{t=t'} \cdot \mathbf{v}^0 - m\dot{f}\big|_{t=t'} = -m\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'} = \operatorname{grad} f|_{t=t'} \cdot \mathbf{P}'_{Rn} = |\operatorname{grad} f|_{t=t'} P'_{Rn},$$

so that

$$\mathbf{P}'_{Rn} = -\frac{m\dfrac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}}{|\operatorname{grad} f|_{t=t'}|^2}\operatorname{grad} f|_{t=t'}, \quad \mathbf{v}^0 = \mathbf{v}' - \frac{\dfrac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}}{|\operatorname{grad} f|_{t=t'}|^2}\operatorname{grad} f|_{t=t'}. \tag{13.1.18}$$

In general, if *the second phase* of the collision phenomenon takes place, then one must determine the quantities $\mathbf{v}''$ and $\mathbf{P}''_{Rn}$ too, but we dispose only on the second equation (13.1.16); we must add a supplementary relation of experimental nature to complete the mathematical model of the collision phenomenon. Newton assumed that the ratio P''_{Rn} / P'_{Rn} does not depend on the incidental velocity but only on the physical properties of the bodies in collision, so that one may write (corresponding to the relation (13.1.7))

$$P''_{Rn} = kP'_{Rn}, \tag{13.1.19}$$

where k is *the restitution* (*damping*) *coefficient*; in this case, taking into account (13.1.16'), we obtain

$$\mathbf{P}_{Rn} = (1+k)\mathbf{P}'_{Rn} = -(1+k)\frac{m\dfrac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}}{|\operatorname{grad} f|_{t=t'}|^2}\operatorname{grad} f|_{t=t'}, \tag{13.1.20}$$

$$\mathbf{v}'' = \mathbf{v}' - (1+k)\frac{\left.\dfrac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'}}{\left|\operatorname{grad} f|_{t=t'}\right|^2}\operatorname{grad} f|_{t=t'} . \tag{13.1.20'}$$

Starting from (13.1.16') to (13.1.17'), we get

$$\mathbf{P}''_{Rn} = \frac{m\left.\dfrac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''}}{\left|\operatorname{grad} f|_{t=t'}\right|^2}\operatorname{grad} f|_{t=t'} , \tag{13.1.20''}$$

too, so that, together with (13.1.18), (13.1.19), we are led to the remarkable relation

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} = -k\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'} . \tag{13.1.21}$$

If, in particular, the constraint is scleronomic (the surface S is fixed), then we have $\dot{f} = 0$ and the relation (13.1.21) becomes

$$\operatorname{grad} f|_{t=t''} \cdot \mathbf{v}'' = -k \operatorname{grad} f|_{t=t'} \cdot \mathbf{v}'$$

or, taking into account Fig. 13.3, has the form (we notice that $\cos(\operatorname{grad} f|_{t=t'} \cdot \mathbf{v}') = -\cos i$)

$$v''\cos r = kv'\cos i ; \tag{13.1.21'}$$

if we use also the relation (13.1.15), it results, finally,

$$k = \frac{\cot r}{\cot i} = \frac{\tan i}{\tan r} . \tag{13.1.21''}$$

With the aid of the first formula (13.1.10), where we make $\mathbf{P} = \mathbf{0}$, we may write (for the two phases of the motion)

$$T^0 - T' = \frac{1}{2}\mathbf{P}'_{Rn}\cdot(\mathbf{v}' + \mathbf{v}^0), \quad T'' - T^0 = \frac{1}{2}\mathbf{P}''_{Rn}\cdot(\mathbf{v}^0 + \mathbf{v}''),$$

where T^0 is the kinetic energy at the moment t_0; taking into account (13.1.17), (13.1.17'), we can write

$$\begin{aligned} T^0 - T' &= \frac{P'_{Rn}}{2\left|\operatorname{grad} f|_{t=t'}\right|}\operatorname{grad} f|_{t=t'}\cdot(\mathbf{v}' + \mathbf{v}^0) = \frac{P'_{Rn}}{2\left|\operatorname{grad} f|_{t=t'}\right|}\left(\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'} - 2\dot{f}|_{t=t'}\right), \\ T'' - T^0 &= \frac{P''_{Rn}}{2\left|\operatorname{grad} f|_{t=t'}\right|}\operatorname{grad} f|_{t=t'}\cdot(\mathbf{v}^0 + \mathbf{v}'') = \frac{P''_{Rn}}{2\left|\operatorname{grad} f|_{t=t'}\right|}\left(\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} - 2\dot{f}|_{t=t''}\right) \end{aligned} \tag{13.1.22}$$

too. If, in the case of scleronomic constraints, we take into account (13.1.18), (13.1.20"), then we obtain

$$T^0 - T' = -\frac{P_{Rn}'^2}{2m}, \quad T'' - T^0 = \frac{P_{Rn}''^2}{2m}; \tag{13.1.22'}$$

using also the relations (13.1.19), (13.1.20), it results

$$(\Delta T)_0 = T'' - T' = -\frac{1}{2m}(1 - k^2) P_{Rn}'^2 = -\frac{1}{2m}\frac{1-k}{1+k} P_{Rn}''^2 . \tag{13.1.22''}$$

A relation of the form (13.1.12'), written for a constraint percussion, allows to write

$$(\Delta T)_0 + \frac{1-k}{1+k} T_0 = 0 , \tag{13.1.23}$$

so that we can state

Theorem 13.1.1 (*Carnot's generalized theorem*). *In the motion of a particle subjected to collision, due to a holonomic and scleronomic unilateral constraint, the sum of the variation of the kinetic energy of that particle at the moment of discontinuity and the kinetic energy of the lost velocity at the same moment, multiplied by the number* $(1-k)/(1+k)$, *where k is the restitution coefficient, vanishes.*

For $k = 1$ (elastic collision) we obtain $(\Delta T)_0 = 0$, hence $T'' = T'$, so that a loss of kinetic energy cannot take place, for $0 < k < 1$ (elastic-plastic collision) we have $(\Delta T)_0 < 0$ (because $T_0 > 0$), hence the variation of the kinetic energy is negative. As a matter of fact, we can replace the notion of variation of the kinetic energy ($(\Delta T)_0 = T'' - T'$) by *the loss of kinetic energy* ($(\Delta T)^0 = -(\Delta T)_0 = T' - T''$), so that $(\Delta T)^0 > 0$ (the loss of kinetic energy is positive, hence the kinetic energy diminishes) and we may write

$$(\Delta T)^0 = \frac{1-k}{1+k} T_0 , \tag{13.1.23'}$$

corresponding to the generalized theorem of Carnot. Finally, in the limit case of a plastic collision ($k = 0$), we find again the Theorem 10.1.14 in the form (*the loss of kinetic energy is equal to the kinetic energy of the lost velocities*)

$$(\Delta T)^0 = T_0 . \tag{13.1.23''}$$

13.1.1.3 Collisions in Case of a Discrete Mechanical System

We consider a discrete mechanical system $\mathcal{S}$ of particles P_i, $i = 1,2,...,n$, in an inertial frame of reference $\mathcal{R}$, subjected to the action of percussive and non-percussive, given and constraint, external and internal forces. Corresponding to the hypotheses i) and ii) of Sect. 11.1.1.1, we will use Newton's law and will neglect the non-percussive forces with respect to the percussive ones. Assuming that the time interval $[t', t'']$ contains only one moment of discontinuity t_0, so that $|t'' - t'| < \varepsilon$, $\varepsilon > 0$ arbitrary, and passing to limit in the sense of the theory of distributions, as in Chap. 10, Sect. 1.2.3, the relation (11.1.54) allows to write

$$(\Delta \mathbf{H})_0 = \lim_{t''-t' \to 0+0} \sum_{i=1}^{n} \left[\int_{t'}^{t''} \mathbf{F}_i(t) \mathrm{d}t + \int_{t'}^{t''} \mathbf{R}_i(t) \mathrm{d}t \right],$$

wherefrom

$$(\Delta \mathbf{H})_0 = \sum_{i=1}^{n} (\mathbf{P}_i + \mathbf{P}_{Ri}) = \mathbf{R} + \bar{\mathbf{R}}, \tag{13.1.24}$$

so that we can state

Theorem 13.1.2 (*theorem of momentum*). *The jump of a momentum of a discrete mechanical system subjected to constraints, at a moment of discontinuity, is equal to the resultant of the given and constraint external percussions which act upon that system at the same moment.*

Taking into account (11.1.19), we obtain

$$(\Delta \mathbf{H})_0 = M(\Delta \mathbf{v}_C)_0, \tag{13.1.24'}$$

so that we can write the relation (13.1.24) also in the form

$$M \Delta \mathbf{v}_C = M \Delta \dot{\boldsymbol{\rho}} = M(\dot{\boldsymbol{\rho}}'' - \dot{\boldsymbol{\rho}}') = \mathbf{R} + \bar{\mathbf{R}}, \tag{13.1.24''}$$

putting thus in evidence the jump of the velocity of the mass centre and being led to the

Theorem 13.1.2' (*theorem of motion of the mass centre*). *The product of the mass of a discrete mechanical system subjected to constraints by the jump of the velocity of the mass centre, at a moment of discontinuity, is equal to the resultant of the given and constraint external percussions which act upon that system at the same moment.*

Hence, the centre of mass of a discrete mechanical system subjected to constraints, at a moment of discontinuity, moves as a particle at which would be concentrated the whole mass of the system and which would be acted upon, at that moment, by the resultant of the given and constraint percussions which act upon that system.

Taking into account the hypothesis iii) in Sect. 11.1.1.1, in conformity to which $\mathbf{r}_i = \overrightarrow{\text{const}}$ in the interval of percussion, and passing to limit, as in the preceding case, the relation (11.1.54') allows to write

$$(\Delta \mathbf{K}_O)_0 = \lim_{t''-t' \to 0+0} \sum_{i=1}^{n} \left[\int_{t'}^{t''} \mathbf{r}_i \times \mathbf{F}_i(t) \mathrm{d}t + \int_{t'}^{t''} \mathbf{r}_i \times \mathbf{R}_i(t) \mathrm{d}t \right]$$
$$= \lim_{t''-t' \to 0+0} \sum_{i=1}^{n} \mathbf{r}_i \times \left[\int_{t'}^{t''} \mathbf{F}_i(t) \mathrm{d}t + \int_{t'}^{t''} \mathbf{R}_i(t) \mathrm{d}t \right],$$

wherefrom

$$(\Delta\mathbf{K}_O)_0 = \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{P}_i + \mathbf{P}_{Ri}) = \mathbf{M}_O + \bar{\mathbf{M}}_O, \tag{13.1.25}$$

so that we can state

Theorem 13.1.3 (*theorem of moment of momentum*). *The jump of the moment of momentum of a discrete mechanical system subjected to constraints, with respect to a fixed pole, at a moment of discontinuity, is equal to the resultant moment of the given and constraint external percussions which act upon that system, with respect to the same pole, at that moment.*

As well, the formula (11.1.54") leads to

$$(\Delta\tau_O\{\mathbf{H}_i\})_0 = \tau_O\{\mathbf{P}_i\} + \tau_O\{\mathbf{P}_{Ri}\}, \tag{13.1.26}$$

and we can state

Theorem 13.1.4 (*theorem of torsor*). *The jump of the torsor of a discrete mechanical system subjected to constraints, with respect to a fixed pole, at a moment of discontinuity, is equal to the torsor of the given and constraint external percussions which act upon that system, with respect to the same pole, at that moment.*

We notice that the theorem of moment of momentum and the theorem of torsor, which depend on the fixed pole O, maintain their form also with respect to another pole Q, fixed with respect to the frame of reference $\mathscr{R}$. If the pole Q is movable, the calculation being made with respect to the same frame $\mathscr{R}$, then we start from the formula (11.1.23); applying a mean value theorem and observing that under the integral we have finite quantities, we can write

$$\lim_{t''-t'\to 0+0} \int_{t'}^{t''} \mathbf{v}_Q(t) \times \mathbf{H}(t)\mathrm{d}t = \mathbf{0},$$

so that

$$(\Delta\mathbf{K}_Q)_0 = \mathbf{M}_Q + \bar{\mathbf{M}}_Q, \tag{13.1.25'}$$

the theorems of moment of momentum and of torsor maintaining their form with respect to the movable pole Q too.

Analogously, starting from the relations (11.1.66"), (11.1.67") and (11.1.68'), we can express the theorems of the dynamic resultant, of the dynamic moment and of the dynamic torsor in the form

$$(\Delta\mathbf{A})_0 = \mathbf{R} + \bar{\mathbf{R}}, \tag{13.1.27}$$

$$(\Delta\mathbf{D}_O)_0 = \mathbf{M}_O + \bar{\mathbf{M}}_O, \tag{13.1.27'}$$

$$(\Delta\tau_O\{\mathbf{A}_i\})_0 = \tau_O\{\mathbf{P}_i\} + \tau_O\{\mathbf{P}_{Ri}\}, \tag{13.1.27''}$$

respectively.

Let us suppose now that the frame of reference $\mathscr{R}$ of pole O is a non-inertial one, having a continuous motion (hence, the components $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ of the finite rototranslation are continuous functions) with respect to an inertial frame $\mathscr{R}'$ of pole O'; to find the form of the general theorems with respect to such a frame, we use the results in Sect. 11.2.2. Thus, starting from the formulae (11.2.12), (11.2.12') and observing, by applying a mean value theorem on intervals of continuity, that

$$\lim_{t''-t'\to 0+0}\left[\int_{t'}^{t''}\mathbf{F}_t^{(C)}(t)\mathrm{d}t+\int_{t'}^{t''}\mathbf{F}_C^{(C)}(t)\mathrm{d}t\right]=\mathbf{0},$$

because the complementary forces (the transportation and Coriolis forces), corresponding to the centre of mass, vary continuously or have finite jumps (see the formula (13.1.24') too), we can state that the Theorem 13.1.2 of the momentum (formula (13.1.24)) and the Theorem 13.1.2' of motion of the mass centre (formula (13.1.24")) take place also with respect to a non-inertial frame of reference. Applying the relation (11.2.10') at the moments t' and t'', subtracting the relations thus obtained one of the other, multiplying by the mass m_i, summing for all the particles of the discrete mechanical system, passing to limit in the sense of the theory of distributions ($t''-t'\to 0+0$) and taking into account the hypothesis iii) of the considered mathematical model and that the rototranslation $\{\mathbf{v}'_O,\boldsymbol{\omega}\}$ is continuous, we obtain

$$(\Delta\mathbf{H}')_0=(\Delta\mathbf{H})_0\,. \tag{13.1.28}$$

As a matter of fact, applying the operator Δ to the relation (11.2.11) we obtain the same result, justifying thus the preceding affirmations (the resultants $\mathbf{R}$ and $\bar{\mathbf{R}}$ are invariant to a change of pole).

Starting from the formula (11.2.18), corresponding to the theorem of moment of momentum with respect to an inertial frame of reference, and observing that

$$\lim_{t''-t'\to 0+0}\int_{t'}^{t''}\boldsymbol{\rho}(t)\times[M\mathbf{a}'_O(t)]\mathrm{d}t=\mathbf{0},$$

we get $(\Delta\mathbf{K}^O)_0=\mathbf{M}_O+\bar{\mathbf{M}}_O$; hence, the jump (with respect to an inertial frame) of the pseudomoment of momentum of a discrete mechanical system subjected to constraints, with respect to an arbitrary pole, at a moment of discontinuity, is equal to the resultant moment of the given and constraint external percussions which act upon the system, with respect to the same pole, at that moment. Taking into account the relation (11.2.17') and observing that $(\Delta(\mathbf{I}_O\boldsymbol{\omega}))_0=\mathbf{0}$, it results that $(\Delta\mathbf{K}^O)_0=(\Delta\mathbf{K}_O)_0$; we find thus again the formula (13.1.25'), written in the form (13.1.25). In general, starting from (11.2.12') to (11.2.18"') and noting that

$$\lim_{t''-t'\to 0+0}\sum_{i=1}^{n}\left\{\int_{t'}^{t''}\mathbf{r}_i\times\left[\mathbf{F}_t^{(i)}(t)+\mathbf{F}_C^{(i)}(t)\right]\mathrm{d}t\right\}=\mathbf{0},$$

we can state that the Theorem 13.1.3 of the moment of momentum (the formula (13.1.25)) takes place with respect to a non-inertial frame of reference too. As a matter of fact, starting from the relation (11.2.10') or applying the operator Δ to the relations

(11.2.16), (11.2.17'), taking into account that $\boldsymbol{\rho} = \overrightarrow{\text{const}}$ in the interval of percussion and using the relation (11.2.24'), we may write

$$(\Delta \mathbf{K}'_{O'})_0 = (\Delta \mathbf{K}_O)_0 + \mathbf{r}'_O \times (\Delta \mathbf{H})_0 \,; \tag{13.1.28'}$$

we find thus again the above affirmation, because $\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{r}'_O \times \mathbf{R}$ and $\bar{\mathbf{M}}_{O'} = \bar{\mathbf{M}}_O + \mathbf{r}'_O \times \bar{\mathbf{R}}$.

Moreover, from (11.2.16) one observes that this affirmation holds also in the case in which the velocity of the pole O has a jump for $O \equiv C$, with the condition $\Delta \boldsymbol{\omega} = \mathbf{0}$.

Finally, we can affirm that the Theorem 13.1.4 of torsor may be stated also with respect to a non-inertial frame in continuous motion with respect to an inertial frame; *the fundamental equations of the mathematical model of the collision phenomenon do not need a privileged frame of reference.* Hence, *in the collision phenomenon of a discrete mechanical system, the jump relations are invariant to a change of frame of reference or of pole.*

If, in particular, we have $\mathbf{R} + \bar{\mathbf{R}} = \mathbf{0}$, it results

$$(\Delta \mathbf{H})_0 = \mathbf{0} \,, \quad \Delta \mathbf{v}_C = \mathbf{0} \,, \tag{13.1.29}$$

or

$$\mathbf{H}'' = \mathbf{H}' \,, \quad \mathbf{v}''_C = \mathbf{v}'_C \,, \tag{13.1.29'}$$

so that the momentum of the mechanical system and the velocity of its mass centre remain constant in the percussion interval. We find thus again the relation (1.1.26) obtained by the collision of two homogeneous spheres of negligible dimensions, for which the inertial character of the mass has been put in evidence (see Chap. 1, Sect. 1.1.6).

As well, if $\mathbf{M}_O + \bar{\mathbf{M}}_O = \mathbf{0}$, then we can write

$$(\Delta \mathbf{K}_O)_0 = \mathbf{0} \,, \tag{13.1.30}$$

wherefrom

$$\mathbf{K}''_O = \mathbf{K}'_O \,, \tag{13.1.30'}$$

the moment of momentum of the mechanical system remaining constant in the interval of percussion.

13.1.1.4 Case of a Discrete Mechanical System Subjected to Sudden Constraints

We introduce the notations

$$\xi_{3(i-1)+j} = \sqrt{m_i}\, x_j^{(i)} \,, \qquad \phi_{3(i-1)+j} = \frac{1}{\sqrt{m_i}} F_j^{(i)} \,, \qquad i = 1,2,\ldots,n \,, \quad j = 1,2,3 \,, \tag{13.1.31}$$

without summation with respect to i, where $x_j^{(i)}$ and $F_j^{(i)}$ are the components of the position vectors $\mathbf{r}_i$ and of the given forces $\mathbf{F}_i$, respectively; as in Chap. 3, Sect. 2.2.2, one can pass from the geometric support Ω of the discrete mechanical system $\mathscr{S}$ in the space E_3 (formed by the geometric points P_i) to a representative point P (of co-ordinates ξ_k, $k = 1,2,\ldots,3n$) in a representative space $\mathbb{R}^{3n}$ (of dimensional equation $M^{1/2}$). We assume that the mechanical system $\mathscr{S}$ is subjected to p bilateral geometric constraints of the form (3.2.8"), expressed by

$$f_l(\xi_k;t) \equiv f_l(\xi_1,\xi_2,\ldots,\xi_{3n};t) = 0, \quad l = 1,2,\ldots,p, \tag{13.1.32}$$

and by a unilateral geometric constraint

$$f(\xi_k;t) \geq 0. \tag{13.1.32'}$$

Lagrange's equations of motion (11.1.64), written for the mechanical system $\mathscr{S}$ in the space E_3, take in the space $\mathbb{R}^{3n}$ the form

$$\ddot{\xi}_k = \phi_k + \sum_{l=1}^{p} \lambda_l \frac{\partial f_l}{\partial \xi_k} + \lambda \frac{\partial f}{\partial \xi_k} = 0, \quad k = 1,2,\ldots,3n, \tag{13.1.33}$$

where λ_l, $l = 1,2,\ldots,p$, and λ are the Lagrange multipliers. If we assume that the constraint (13.1.32') is weak ($f(\xi_k;t) > 0$), the motion of the mechanical system $\mathscr{S}$ takes place as the respective constraint would not exist, after a law of the form $\xi_k = \xi_k(t)$, $k = 1,2,\ldots,3n$. Let t' be the smallest positive root of the equation $f(\xi_k(t);t) = 0$; this is the moment at which the mechanical system reaches the constraint, the corresponding co-ordinates and velocities being $\xi_k' = \xi_k(t')$ and $\dot{\xi}_k' = \dot{\xi}_k(t')$, respectively. In this case, the velocities thus found satisfy the conditions

$$\left.\frac{\mathrm{d}f_l}{\mathrm{d}t}\right|_{t=t'} = \sum_{k=1}^{3n} \left.\frac{\partial f_l}{\partial \xi_k}\right|_{t=t'} \dot{\xi}_k' + \dot{f}_l\Big|_{t=t'} = 0, \quad l = 1,2,\ldots,p, \tag{13.1.34}$$

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'} = \sum_{k=1}^{3n} \left.\frac{\partial f}{\partial \xi_k}\right|_{t=t'} \dot{\xi}_k' + \dot{f}\Big|_{t=t'} \geq 0. \tag{13.1.34'}$$

If the velocities $\dot{\xi}_k'$ satisfy the relation (13.1.34'), then the mechanical system $\mathscr{S}$ leaves of new the constraint (strict inequality, sufficient condition) or remains on the constraint (equality, necessary condition). But if the relation $\mathrm{d}f/\mathrm{d}t|_{t=t'} < 0$, takes place, to be in concordance with the condition (13.1.34') we must assume that, in the interval of collision, arise constraint percussive forces, which lead to constraint percussions $P_{Rl}^{(i)}$ and $P_R^{(i)}$, $i = 1,2,\ldots,n$, respectively; there arise jumps of the velocities, which become $\dot{\xi}_k''$, and a collision of the mechanical system $\mathscr{S}$ with the constraint (13.1.32') takes thus place. If at the moment t' the mechanical system leaves the constraint, then one must see what happens for the following positive root of the equation $f(\xi_k(t);t) = 0$, till the exhaustion of all those roots or till one has a root for which the system remains on the constraint, intervening the phenomenon of collision. We assume that the three hypotheses in Sect. 13.1.1.1 which are mathematically modelling this phenomenon hold further, and add a supplementary hypothesis:

iv) The collision phenomenon does not destroy the bilateral geometric constraints (the constraints (13.1.32)) of the mechanical system.

At the end of the percussion interval (the moment t'') we have

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} = \sum_{k=1}^{3n} \left.\frac{\partial f}{\partial \xi_k}\right|_{t=t'} \dot{\xi}_k'' + \left.\dot{f}\right|_{t=t'} > 0 . \tag{13.1.35}$$

We suppose that, at a moment $t_0 \in [t', t'']$, the velocities $\dot{\xi}_k^0$ are so that

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t_0} = \sum_{k=1}^{3n} \left.\frac{\partial f}{\partial \xi_k}\right|_{t=t'} \dot{\xi}_k^0 + \left.\dot{f}\right|_{t=t'} = 0 . \tag{13.1.35'}$$

In the last two relations, we took into account the hypothesis iii). If the phenomenon of collision is reduced to the first phase (the interval $[t', t_0]$), then the collision is a plastic one; if the second phase (the interval $[t_0, t'']$) takes also place, then the collision has an elastic-plastic character. We notice that the constraint percussions are given by

$$\mathbf{P}_{Rl}'^{(i)} = \lambda_l' \operatorname{grad}_i f_l \big|_{t=t'}, \quad \mathbf{P}_R'^{(i)} = \lambda' \operatorname{grad}_i f \big|_{t=t'},$$
$$\mathbf{P}_{Rl}''^{(i)} = \lambda_l'' \operatorname{grad}_i f_l \big|_{t=t'}, \quad \mathbf{P}_R''^{(i)} = \lambda'' \operatorname{grad}_i f \big|_{t=t'},$$
$$\mathbf{P}_{Rl}^{(i)} = \mathbf{P}_{Rl}'^{(i)} + \mathbf{P}_{Rl}''^{(i)} = \lambda_l \operatorname{grad}_i f_l \big|_{t=t'}, \quad \mathbf{P}_R^{(i)} = \mathbf{P}_R'^{(i)} + \mathbf{P}_R''^{(i)} = \lambda \operatorname{grad}_i f \big|_{t=t'},$$

where the first and the second phase of the collision are specified by "prime" and by "second", respectively, the corresponding Lagrange multipliers being given by

$$\lambda_l' = \int_{t'}^{t_0} \lambda_l(t)\mathrm{d}t , \quad \lambda_l'' = \int_{t_0}^{t''} \lambda_l(t)\mathrm{d}t ,$$
$$\lambda' = \int_{t'}^{t_0} \lambda(t)\mathrm{d}t , \quad \lambda'' = \int_{t_0}^{t''} \lambda(t)\mathrm{d}t ,$$
$$\lambda_l = \lambda_l' + \lambda_l'' , \quad \lambda = \lambda' + \lambda'' .$$

Lagrange's equations (13.1.33) lead thus to (we take into consideration the hypothesis ii)

$$\dot{\xi}_k^0 - \dot{\xi}_k' = \sum_{l=1}^{p} \lambda_l' \left.\frac{\partial f_l}{\partial \xi_k}\right|_{t=t'} + \lambda' \left.\frac{\partial f}{\partial \xi_k}\right|_{t=t'} , \tag{13.1.36}$$

$$\dot{\xi}_k'' - \dot{\xi}_k^0 = \sum_{l=1}^{p} \lambda_l'' \frac{\partial f_l}{\partial \xi_k}\bigg|_{t=t'} + \lambda'' \frac{\partial f}{\partial \xi_k}\bigg|_{t=t'}, \tag{13.1.36'}$$

$$\dot{\xi}_k'' - \dot{\xi}_k' = \sum_{l=1}^{p} \lambda_l \frac{\partial f_l}{\partial \xi_k}\bigg|_{t=t'} + \lambda \frac{\partial f}{\partial \xi_k}\bigg|_{t=t'}, \tag{13.1.36''}$$

for $k = 1,2,\ldots,3n$. Assuming that the collision is plastic, from the $3n + 1$ equations (13.1.35'), (13.1.36), to which we associate the p equations

$$\frac{\mathrm{d}f_l}{\mathrm{d}t}\bigg|_{t=t_0} = \sum_{k=1}^{3n} \frac{\partial f_l}{\partial \xi_k}\bigg|_{t=t'} \dot{\xi}_k^0 + \dot{f}_l\big|_{t=t'} = 0\,, \quad l = 1,2,\ldots,p\,, \tag{13.1.37}$$

corresponding to hypothesis iv), we obtain the $3n + p + 1$ unknowns $\dot{\xi}_k^0$, λ_l and λ. To this goal, we multiply the equation (13.1.36) by $\partial f / \partial \xi_k \big|_{t=t'}$ and by $\partial f_l / \partial \xi_k \big|_{t=t'}$, respectively, and sum with respect to k; taking into account (13.1.34), (13.1.35') and (13.1.37), we obtain

$$\begin{gathered} \sum_{l=1}^{p} \lambda_l' [f_l, f] + \lambda' [f, f] = -\frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}, \\ \sum_{l=1}^{p} \lambda_l' [f_l, f_q] + \lambda' [f, f_q] = 0\,, \quad q = 1,2,\ldots,p\,, \end{gathered} \tag{13.1.38}$$

with the notation

$$[\varphi, \psi] = [\psi, \varphi] = \sum_{k=1}^{3n} \frac{\partial \varphi}{\partial \xi_k}\bigg|_{t=t'} \frac{\partial \psi}{\partial \xi_k}\bigg|_{t=t'}. \tag{13.1.38'}$$

It results

$$\lambda' = -\frac{\Delta_{00}}{\Delta} \frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}, \quad \lambda_l' = -\frac{\Delta_{l0}}{\Delta} \frac{\mathrm{d}f}{\mathrm{d}t}\bigg|_{t=t'}, \quad l = 1,2,\ldots,p\,, \tag{13.1.39}$$

where Δ is the determinant of the system (13.1.38), (13.1.38') of $p + 1$ equations, while Δ_{00} and Δ_{l0} are the normalized minors of the elements $[f, f]$ and $[f_l, f]$ of the first line, respectively; replacing in the system (13.1.36), we find the unknown velocities $\dot{\xi}_k^0$.

In the general case in which takes place the second phase of the collision phenomenon, we use the equations (13.1.36), (13.1.36') or the equations (13.1.36''), which represent, in fact, a consequence of the first system of equations; in this last case, we dispose of $3n$ equations, to which we associate p equations

$$\frac{\mathrm{d}f_l}{\mathrm{d}t}\bigg|_{t=t''} = \sum_{k=1}^{3n} \frac{\partial f_l}{\partial \xi_k}\bigg|_{t=t'} \dot{\xi}_k'' + \dot{f}_l\big|_{t=t'} = 0\,, \quad l = 1,2,\ldots,p\,, \tag{13.1.40}$$

corresponding to the hypothesis iv), for the $3n + p + 1$ unknowns $\dot{\xi}_k''$, $k = 1,2,\ldots,3n$, λ_l, $l = 1,2,\ldots,p$, and λ. Using the method of calculation presented above, we obtain, analogously,

$$\lambda = \frac{\Delta_{00}}{\Delta}\left(\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} - \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'}\right), \quad \lambda_l = \frac{\Delta_{l0}}{\Delta}\left(\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} - \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'}\right), \quad l = 1,2,\ldots,p; \tag{13.1.41}$$

we notice that we do not know the total derivative $\mathrm{d}f/\mathrm{d}t|_{t=t''}$, which contains the unknown velocities $\dot{\xi}_k''$, missing thus a last necessary equation to solve the problem. As well, from (13.1.39), (13.1.41) it results

$$\lambda'' = \frac{\Delta_{00}}{\Delta}\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''}, \quad \lambda_l'' = \frac{\Delta_{l0}}{\Delta}\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''}, \quad l = 1,2,\ldots,p. \tag{13.1.41'}$$

To solve the problem in the frame of the mathematical model considered above, we assume that Lagrange's multipliers corresponding to the unilateral constraint verify the condition

$$\lambda'' = k\lambda', \tag{13.1.42}$$

where k is the coefficient of restitution introduced in Sect. 1.1.1. The first relations (13.1.39) and (13.1.41') lead to

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t''} + k\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t=t'} = 0; \tag{13.1.42'}$$

it results

$$\lambda_l'' = k\lambda_l', \quad l = 1,2,\ldots,p, \tag{13.1.42''}$$

too for the bilateral constraints. We obtain, finally,

$$\dot{\xi}_k'' - \dot{\xi}_k' = (1+k)\left(\sum_{l=1}^{p}\lambda_l\left.\frac{\partial f_l}{\partial \xi_k}\right|_{t=t'} + \lambda\left.\frac{\partial f}{\partial \xi_k}\right|_{t=t'}\right), \tag{13.1.43}$$

the problem being solved as in the preceding case.

13.1.1.5 Carnot and Kelvin Theorems. Principle of Virtual Work. Conservation Theorems

In what follows, we make a study of the variation of the kinetic energy

$$T = \frac{1}{2}\sum_{k=1}^{3n}\dot{\xi}_k^2, \tag{13.1.44}$$

corresponding to the notations (13.1.31). Multiplying the equations (13.1.36) by $\dot{\xi}_k^0$ and $\dot{\xi}_k'$, respectively, and summing, we obtain

$$2\left(T^0 - \text{ح}'\right) = -\sum_{l=1}^{p} \lambda_l' \dot{f}_l \Big|_{t=t_0} - \lambda' \dot{f}\Big|_{t=t_0},$$

$$2\left(\text{ح}' - T'\right) = -\sum_{l=1}^{p} \lambda_l' \dot{f}_l \Big|_{t=t'} - \lambda' \dot{f}\Big|_{t=t'} + \lambda' \frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{t=t'},$$

with

$$\text{ح}' = \frac{1}{2}\sum_{k=1}^{3n} \dot{\xi}_k^0 \dot{\xi}_k', \tag{13.1.45}$$

where we used the notations in Sect. 13.1.1.2, introducing the kinetic energy at the beginning of the collision phenomenon and at the end of it and where we took into account (13.1.34), (13.1.35') and (13.1.37). We assume that we have to do with scleronomic constraints (in case of rheonomous constraints, the variation of the kinetic energy depends explicitly on the variation of the constraints), so that $\dot{f} = 0$, $\dot{f}_l = 0$, $l = 1,2,\ldots,p$; it results

$$T^0 = \text{ح}', \quad \text{ح}' - T' = \frac{1}{2}\lambda' \frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{t=t'} = -\frac{1}{2}\lambda'^2 \frac{\Delta}{\Delta_{00}},$$

corresponding to the relations (13.1.39). From the above relations and by means of the notations which have been introduced, we obtain, analogously,

$$T^0 + T' - 2\text{ح}' = T_0',$$

where we have introduced also *the kinetic energy of the lost velocities in the first phase of the collision phenomenon*

$$T_0' = \frac{1}{2}\sum_{k=1}^{3n}\left(\dot{\xi}_k^0 - \dot{\xi}_k'\right)^2 = \frac{1}{2}\lambda'^2 K^2, \quad K^2 = \frac{\Delta}{\Delta_{00}}; \tag{13.1.46}$$

it is thus shown that the ratio Δ / Δ_{00} is positive. Eliminating $\text{ح}'$, it results

$$(\Delta T)_0' + T_0' = 0, \quad (\Delta T)_0' = T^0 - T' \tag{13.1.47}$$

and we can state

Theorem 13.1.5 (*Carnot*, I). *In the motion of a discrete mechanical system subjected to holonomic and scleronomic constraints, the sum of the variation of the kinetic energy in the first phase of the collision phenomenon, due to a sudden unilateral constraint, and the kinetic energy of the lost velocities in the same interval of time, vanishes.*

One can see that, in the first phase of the collision phenomenon, the variation of the kinetic energy is negative, corresponding thus to a loss of kinetic energy $(\Delta T)'^{0} = -(\Delta T)'_0$.

Starting from the equations (13.1.36'), multiplying by $\dot{\xi}''_k$ and $\dot{\xi}^0_k$, respectively, summing and taking into account (13.1.35'), (13.1.37) and (13.1.40), we can write, analogously,

$$2(\mathcal{T}'' - T^0) = -\sum_{l=1}^{p} \lambda''_l \dot{f}_l \Big|_{t=t_0} - \lambda'' \dot{f} \Big|_{t=t_0},$$

$$2(T'' - \mathcal{T}'') = -\sum_{l=1}^{p} \lambda''_l \dot{f}_l \Big|_{t=t''} - \lambda'' \dot{f} \Big|_{t=t''} + \lambda'' \frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{t=t''},$$

with

$$\mathcal{T}'' = \frac{1}{2} \sum_{k=1}^{3n} \dot{\xi}^0_k \dot{\xi}''_k, \tag{13.1.45'}$$

where, using the notation in Sect. 13.1.1.2, we have introduced the kinetic energy at the end of the second phase of the collision phenomenon. Assuming, further, that we remain in the case of the scleronomic constraints, it results, taking into account the relation (13.1.41'),

$$\mathcal{T}'' = T^0, \quad T'' - \mathcal{T}'' = \frac{1}{2} \lambda'' \frac{\mathrm{d}f}{\mathrm{d}t}\Big|_{t=t''} = \frac{1}{2} \lambda''^2 K^2.$$

We see thus that, in the frame of the hypotheses made above, we have

$$\mathcal{T}' = \mathcal{T}'', \tag{13.1.45''}$$

so that we may write the relation

$$\sum_{k=1}^{3n} \dot{\xi}^0_k (\dot{\xi}''_k - \dot{\xi}'_k) = 0 \tag{13.1.45'''}$$

too. Analogously, we have

$$T^0 + T'' - 2\mathcal{T}'' = T''_0,$$

where we have introduced *the kinetic energy of the lost velocities in the second phase of the collision phenomenon*

$$T''_0 = \frac{1}{2} \sum_{k=1}^{3n} (\dot{\xi}''_k - \dot{\xi}^0_k)^2 = \frac{1}{2} \lambda''^2 K^2; \tag{13.1.48}$$

eliminating ζ'', it results

$$(\Delta T)_0'' = T_0'', \quad (\Delta T)_0'' = T'' - T_0, \tag{13.1.49}$$

and we can state

Theorem 13.1.6 (*Carnot*, II). *In the motion of a discrete mechanical system subjected to holonomic and scleronomic constraints, due to a sudden unilateral constraint, the variation of the kinetic energy in the second phase of the collision phenomenon is equal to the kinetic energy of the lost velocities in the same interval of time.*

Hence, in the second phase of the collision phenomenon, the variation of the kinetic energy is positive, corresponding thus to an *increase of kinetic energy* $(\Delta T)_0''$.

From (13.1.46 to 13.1.49) it results

$$T'' - T' = \frac{1}{2} K^2 \left(\lambda''^2 - \lambda'^2 \right)$$

too; taking into account (13.1.42), we can write

$$(\Delta T)_0 = -\frac{1}{2}\left(1 - k^2\right) \lambda'^2 K^2 = -\frac{1}{2}\left(\frac{1}{k^2} - 1\right) \lambda''^2 K^2,$$

wherefrom

$$(\Delta T)_0 + \left(1 - k^2\right) T_0' = 0 \tag{13.1.50}$$

or

$$(\Delta T)_0 + \left(\frac{1}{k^2} - 1\right) T_0'' = 0, \tag{13.1.50'}$$

with $(\Delta T)_0 = T'' - T'$. We thus state:

Theorem 13.1.7 (*Carnot*, III). *In the motion of a discrete mechanical system subjected to holonomic and scleronomic constraints, the sum of the variation of the kinetic energy in the collision interval, due to a sudden unilateral constraint, and the kinetic energy of the lost velocities in the first phase of the collision phenomenon, multiplied by the number* $1 - k^2$, *where* k *is the coefficient of restitution by collision, vanishes.*

Theorem 13.1.7' (*Carnot*, III'). *In the motion of a discrete mechanical system subjected to holonomic and scleronomic constraints, the sum of the variation of the kinetic energy in the collision interval, due to a sudden unilateral constraint, and the kinetic energy of the lost velocities in the second phase of the collision phenomenon, multiplied by the number* $(1/k)^2 - 1$, *where* k *is the coefficient of restitution by collision, vanishes.*

We can state thus that, in the collision interval, the variation of the kinetic energy is negative, being put in evidence a loss of kinetic energy $(\Delta T)^0 = -(\Delta T)_0$.

Adding the relations (13.1.47) and (13.1.49) and observing that $(\Delta T)_0' + (\Delta T)_0'' = -(\Delta T)_0$, we can write

$$(\Delta T)_0 = T_0'' - T_0', \quad (\Delta T)^0 = T_0' - T_0'', \tag{13.1.51}$$

hence $T_0' > T_0''$; from (13.1.36 to 13.1.36") it results, easily,

$$\dot{\xi}_r'' - \dot{\xi}_r' = (1+k)(\dot{\xi}_r^0 - \dot{\xi}_r') = \left(\frac{1}{k} + 1\right)(\dot{\xi}_r'' - \dot{\xi}_r^0), \quad r = 1,2,...,3n, \tag{13.1.52}$$

so that the relations (13.1.51) lead to the same formula (13.1.23) as in the case of a single particle; thus, the generalized Theorem 13.1.1 of Carnot is stated analogously for a discrete mechanical system subjected to holonomic and scleronomic constraints. All the considerations made in Sect. 13.1.1.2 hold further.

We notice that the relations (13.1.52) justify the relations (13.1.45"), (13.1.45"') too.

As in Sect. 1.1.3, we consider the relations (11.1.24), (11.1.25'), written for the time interval $[t', t'']$, $|t'' - t'| < \varepsilon$, $\varepsilon > 0$ arbitrary, which contains only one moment of discontinuity t_0; passing to limit in the sense of the theory of distributions, we can write (the sign "prime" at the sum indicates $k \neq i$)

$$(\Delta T)_0 + T_0 = \sum_{i=1}^n \mathbf{v}_i'' \cdot \lim_{t''-t'\to 0+0} \int_{t'}^{t''} \left\{ \mathbf{F}_i(t) + \mathbf{R}_i(t) + \sum_{k=1}^{n}{}' [\mathbf{F}_{ik}(t) + \mathbf{R}_{ik}(t)] \right\} \mathrm{d}t,$$

$$(\Delta T)_0 - T_0 = \sum_{i=1}^n \mathbf{v}_i' \cdot \lim_{t''-t'\to 0+0} \int_{t'}^{t''} \left\{ \mathbf{F}_i(t) + \mathbf{R}_i(t) + \sum_{k=1}^{n}{}' [\mathbf{F}_{ik}(t) + \mathbf{R}_{ik}(t)] \right\} \mathrm{d}t,$$

where, for the sake of generality, we have introduced also the influence of the constraint forces. We obtain thus

$$(\Delta T)_0 + T_0 = \sum_{i=1}^n \mathbf{v}_i'' \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right], \tag{13.1.53}$$

$$(\Delta T)_0 - T_0 = \sum_{i=1}^n \mathbf{v}_i' \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right], \tag{13.1.53'}$$

so that we can state:

Theorem 13.1.8 (*theorem of kinetic energy*). *The sum of the variation of the kinetic energy of a discrete mechanical system subjected to constraints, at a moment of discontinuity, and the kinetic energy of the lost velocities at the same moment is equal to the sum of the scalar products of the percussions which act upon the particles by their velocities after that moment of discontinuity.*

Theorem 13.1.8' (*analogue of the theorem of kinetic energy*). *The difference between the variation of the kinetic energy of a discrete mechanical system subjected to constraints, at a moment of discontinuity, and the kinetic energy of the lost velocities at the same moment is equal to the sum of the scalar products of the percussions which act upon the particles by their velocities before that moment of discontinuity.*

If

$$\sum_{i=1}^n \mathbf{v}_i'' \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right] = \mathbf{0}, \tag{13.1.54}$$

then we find again Carnot's generalized theorem, written in the form (10.1.47), analogue to that in case of plastic collisions; as well, if

$$\sum_{i=1}^{n} \mathbf{v}_i' \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right] = \mathbf{0}, \tag{13.1.54'}$$

then we can write an analogue of the generalized theorem of Carnot, in the form (10.1.47').

Summing the relations (13.1.53) and (13.1.53'), it results

$$(\Delta T)_0 = \frac{1}{2} \sum_{i=1}^{n} (\mathbf{v}_i' + \mathbf{v}_i'') \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right], \tag{13.1.55}$$

so that we can state

Theorem 13.1.9 (*Kelvin*). *The variation of the kinetic energy of a discrete mechanical system subjected to constraints, at a moment of discontinuity, is equal to the sum of the scalar products of the percussions which act upon the particles by the semisum of their velocities before and after the phenomenon of discontinuity.*

If we subtract the relation (13.1.53') from the relation (13.1.53), then we obtain

$$T_0 = \frac{1}{2} \sum_{i=1}^{n} \mathbf{v}_0^{(i)} \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right] \tag{13.1.55'}$$

and we are led to

Theorem 13.1.9' (*analogue of Kelvin's theorem*). *The kinetic energy of the lost velocities of a discrete mechanical system subjected to constraints, at a moment of discontinuity, is equal to the semisum of the scalar products of the percussions which act upon the particles by the jumps of their velocities at that moment of discontinuity.*

Starting from the formula (13.1.8), we can write the relation

$$\mathbf{P}_i + \mathbf{P}_{Ri} - m_i \Delta \mathbf{v}_i = \mathbf{0}, \quad i = 1, 2, \ldots, n, \tag{13.1.56}$$

for a particle P_i; effecting a scalar product by the virtual displacements $\delta \mathbf{r}_i$ at the beginning of the interval of percussion and summing for all the particles of the mechanical system $\mathcal{S}$, one obtains a necessary condition to describe the phenomenon of collision

$$\sum_{i=1}^{n} (\mathbf{P}_i - m_i \Delta \mathbf{v}_i) \cdot \delta \mathbf{r}_i = 0, \tag{13.1.57}$$

where we have considered that for the ideal constraints we have

$$\sum_{i=1}^{n} \mathbf{P}_{Ri} \cdot \delta \mathbf{r}_i = 0, \tag{13.1.58}$$

corresponding to their relation of definition (3.2.36). Assuming that the condition (13.1.57) is fulfilled and that p holonomic constraints of the form (3.2.21'') and

m non-holonomic constraints of the form (3.2.15) take place, we use the method of Lagrange's multipliers; we can thus write

$$\sum_{i=1}^{n}\left(\mathbf{P}_i - m_i\Delta\mathbf{v}_i + \sum_{l=1}^{p}\lambda_l\nabla_i f_l\Big|_{t=t'} + \sum_{k=1}^{m}\mu_k\boldsymbol{\alpha}_{ki}\Big|_{t=t'}\right)\cdot\delta\mathbf{r}_i = 0,$$

where the Lagrange's multipliers λ_l, $l = 1,2,\ldots,p$, and

$$\mu_k = \mu_k' + \mu_k'', \quad \mu_k' = \int_{t'}^{t_0}\mu_k(t)\mathrm{d}t, \quad \mu_k'' = \int_{t_0}^{t''}\mu_k(t)\mathrm{d}t, \quad k = 1,2,\ldots,m,$$

are non-determinate scalars and where we took into account that in a finite double sum one can invert the order of summation. As in Sect. 11.1.2.10, we find again the relations (13.1.56), the constraint percussions being given by

$$\mathbf{P}_{Ri} = \sum_{l=1}^{p}\lambda_l\nabla_i f_l\Big|_{t=t'} + \sum_{k=1}^{m}\mu_k\boldsymbol{\alpha}_{ki}\Big|_{t=t'}, \quad i = 1,2,\ldots,n. \tag{13.1.59}$$

The relation (13.1.57) becomes a sufficient condition too, and we can state

Theorem 13.1.10 (*theorem of virtual work*). *The motion of a discrete mechanical system subjected to ideal constraints, in the collision interval, takes place so that the virtual work of the lost percussions which act upon that system vanishes for any system of virtual displacements of the respective system.*

We have used, in this theorem, the denomination of *lost percussion* for the difference $\mathbf{P}_i - m_i\Delta\mathbf{v}_i$ which equilibrates the constraint percussion $\mathbf{P}_{Ri}$ (by analogy to the denomination of lost force of d'Alembert). As in the case of continuous mechanical phenomena, this theorem can be considered as a principle (*the principle of virtual work* or *the principle of virtual displacements*), because, starting from it, one can solve the basic problems of the phenomenon of collision.

The equation (13.1.56) with (13.1.59) are *Lagrange's equations of the first kind for the collision phenomenon*.

Introducing the virtual velocities (3.2.1'), we can write the condition (13.1.57) in the form

$$\sum_{i=1}^{n}(\mathbf{P}_i - m_i\Delta\mathbf{v}_i)\cdot\mathbf{v}_i^* = 0, \tag{13.1.57'}$$

the considered principle being thus called *the principle of virtual velocities* too.

In case of a discrete mechanical system subjected to unilateral constraints, we can express the principle of virtual work in the form

$$\sum_{i=1}^{n}(\mathbf{P}_i - m_i\Delta\mathbf{v}_i)\cdot\delta\mathbf{r}_i \leq 0. \tag{13.1.57''}$$

It is interesting to notice that the relation (13.1.57) allows to find again the theorems of Carnot and Kelvin.

Starting from the form taken by the general theorems of mechanics in case of the collision phenomenon, we can state some conservation theorems, particularly useful. Thus, if $\mathbf{R} + \bar{\mathbf{R}} = \mathbf{0}$, then it results $(\Delta\mathbf{H})_0 = \mathbf{0}$, obtaining

Theorem 13.1.11 (*conservation theorem of momentum*). *The momentum of a discrete mechanical system subjected to constraints is conserved in a collision interval if and only if the resultant of the given and constraint external percussions which act upon the system vanishes in that interval.*

Analogously, if $\mathbf{M}_O + \bar{\mathbf{M}}_O = \mathbf{0}$, then it results $(\Delta\mathbf{K}_O)_0 = \mathbf{0}$, and we can state

Theorem 13.1.12 (*conservation theorem of moment of momentum*). *The moment of momentum of a discrete mechanical system subjected to constraints, with respect to a given pole, is conserved in a collision interval if and only if the resultant moment of the given and constraint external percussions which act upon the system, with respect to the same pole, vanishes in that interval.*

Hence, if the given and constraint percussions which act upon a discrete mechanical system subjected to constraints are equilibrated in their totality in the collision interval (the mechanical system is considered as non-deformable), then the kinetic torsor of the system is conserved in that interval.

Assuming that the relation

$$\sum_{i=1}^{n} (\mathbf{v}_i' + \mathbf{v}_i'') \cdot \left[\mathbf{P}_i + \mathbf{P}_{Ri} + \sum_{k=1}^{n}{}' (\mathbf{P}_{ik} + \mathbf{P}_{Rik}) \right] = 0 \tag{13.1.60}$$

takes place, the theorem of Kelvin allows to state

Theorem 13.1.13 (*conservation theorem of kinetic energy*). *The kinetic energy of a discrete mechanical system subjected to constraints is conserved in a collision interval if and only if the sum of the scalar products of the given and constraint, external and internal percussions which act upon the particles by the sum of their velocities before and after a moment of discontinuity vanishes.*

From (13.1.23), it results that the relation $(\Delta T)_0 = 0$ takes place in case of an elastic collision ($k = 1$). As well, from (13.1.60) we can state that the conservation Theorem 13.1.13 can be obtained if $\mathbf{v}_i'' = -\mathbf{v}_i'$, $i = 1,2,\ldots,n$, or if the given and constraint, external and internal percussions are equilibrated for each particle of the mechanical system (considered as non-deformable).

13.1.2 Elastic and Plastic Collisions of Discrete Mechanical Systems

In the following, we consider the general case of elastic and plastic collisions of the particles, including the problems at the atomic level. The case of plastic collisions is studied by introducing the space of plastic collisions.

13.1.2.1 Elastic Collisions of Particles. Disintegration and Diffusion of Particles. Rutherford's Formula

Let be two particles P_1 and P_2 of masses m_1 and m_2, respectively, having the velocities $\mathbf{v}_1'$, $\mathbf{v}_2'$ and $\mathbf{v}_1''$, $\mathbf{v}_2''$ before and after the interaction, respectively (it is assumed that the particles come from infinite and tend to infinite). In the absence of percussions, we can write a conservation theorem of momentum (corresponding to the formulae (1.1.26)) and a conservation theorem of kinetic energy in the form

$$m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2' = m_1 \mathbf{v}_1'' + m_2 \mathbf{v}_2'' , \quad m_1 v_1'^2 + m_2 v_2'^2 = m_1 v_1''^2 + m_2 v_2''^2 . \tag{13.1.61}$$

We notice that these relations can be written also in the form

$$m_1 (\mathbf{v}_1'' - \mathbf{v}_1') + m_2 (\mathbf{v}_2'' - \mathbf{v}_2') = \mathbf{0} ,$$
$$m_1 (\mathbf{v}_1'' - \mathbf{v}_1') \cdot (\mathbf{v}_1'' + \mathbf{v}_1') + m_2 (\mathbf{v}_2'' - \mathbf{v}_2') \cdot (\mathbf{v}_2'' + \mathbf{v}_2') = 0 .$$

Denoting $\lambda \mathbf{u} = m_1 (\mathbf{v}_1'' - \mathbf{v}_1') = -m_2 (\mathbf{v}_2'' - \mathbf{v}_2')$, $|\mathbf{u}| = 1$, λ scalar, and introducing the relative velocities

$$\overline{\mathbf{v}}' = \mathbf{v}_2' - \mathbf{v}_1' , \quad \overline{\mathbf{v}}'' = \mathbf{v}_2'' - \mathbf{v}_1'' , \tag{13.1.62}$$

we find $\mathbf{u} \cdot \overline{\mathbf{v}}' + \mathbf{u} \cdot \overline{\mathbf{v}}'' = 0$ and $\mathbf{v}'' = \mathbf{v}' - (\lambda / m)\mathbf{u}$, where we introduce the reduced mass m given by (8.1.14); a scalar product of the latter relation by $\mathbf{u}$ leads to $\lambda = 2m\mathbf{u} \cdot \overline{\mathbf{v}}'$ and $\overline{\mathbf{v}}'' = \overline{\mathbf{v}}' - 2(\mathbf{u} \cdot \overline{\mathbf{v}}')\mathbf{u}$. The velocities after interaction can be thus expressed as functions of the velocities before interaction in the form

$$\mathbf{v}_1'' = \mathbf{v}_1' + 2\frac{m}{m_1}(\mathbf{u} \cdot \overline{\mathbf{v}}')\mathbf{u} , \quad \mathbf{v}_2'' = \mathbf{v}_2' - 2\frac{m}{m_2}(\mathbf{u} \cdot \overline{\mathbf{v}}')\mathbf{u} ; \tag{13.1.63}$$

as well, we can write

$$\mathbf{v}_1' = \mathbf{v}_1'' + 2\frac{m}{m_1}(\mathbf{u} \cdot \overline{\mathbf{v}}'')\mathbf{u} , \quad \mathbf{v}_2' = \mathbf{v}_2'' - 2\frac{m}{m_2}(\mathbf{u} \cdot \overline{\mathbf{v}}'')\mathbf{u} . \tag{13.1.63'}$$

In case of the phenomenon of diffraction considered in Chap. 8, Sect. 1.2.1, $\mathbf{u} = \pm \operatorname{vers} \overrightarrow{OQ}$; assuming that $\mathbf{u} \cdot \overline{\mathbf{v}}' > 0$, we have $\mathbf{u} \cdot \overline{\mathbf{v}}' = \overline{v}' \sin(\varkappa / 2)$, where $\varkappa$ is the diffraction angle.

The problem considered above is also called *the problem of biparticle collision*. One assumes that the two particles are, at the initial moment, at a great distance one of the other, so that each one of these particles can be considered as being free; if during the motion the particles become nearer, then interaction forces arise, becoming in *real* (collision or capture) or *fictitious* (diffusion or diffraction) *contact*. Using the velocity $\mathbf{v}_C$ of the mass centre (invariant in the collision interval, in conformity to the conservation theorem of momentum – first relation (13.1.61))

$$\mathbf{v}_C = \frac{1}{M}(m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2') = \frac{1}{M}(\mathbf{H}_1' + \mathbf{H}_2') , \quad M = m_1 + m_2 , \tag{13.1.64}$$

and taking into account (13.1.62), we find

$$\mathbf{v}_1' = \mathbf{v}_C - \frac{m_2}{M}\overline{\mathbf{v}}' , \quad \mathbf{v}_2' = \mathbf{v}_C + \frac{m_1}{M}\overline{\mathbf{v}}' , \tag{13.1.65}$$

$$\mathbf{v}_1'' = \mathbf{v}_C - \frac{m_2}{M}\overline{\mathbf{v}}'', \quad \mathbf{v}_2'' = \mathbf{v}_C + \frac{m_1}{M}\overline{\mathbf{v}}''. \tag{13.1.65'}$$

The collision being elastic, the interaction forces are conservative, the potential energy tending to zero if the distance between the particles tends to infinite (practically, it is very great), and we can write a conservation theorem of kinetic energy; taking into account (13.1.65), (13.1.65'), the second relation (13.1.61) takes the form $Mv_C^2 + m\overline{v}'^2 = Mv_C^2 + m\overline{v}''^2$, where m is given by (8.1.14). Hence, the relative velocity is constant in modulus ($\overline{v}' = \overline{v}''$).

Observing that the considered mechanical system is closed, the centre of mass C has a rectilinear and uniform motion with the velocity $\mathbf{v}_C$ given by (13.1.64) with respect to an inertial frame of reference; we refer now the motion to an inertial frame $\mathscr{R}_C$ with respect to which the centre C is at rest (*the frame* $\mathscr{R}_C$ *of the mass centre*), using the index C (the origin of the frame can be taken even at C). Because $\mathbf{v}_1' = \mathbf{v}_C + \mathbf{v}_{1C}'$, $\mathbf{v}_2' = \mathbf{v}_C + \mathbf{v}_{2C}'$, we can write $\mathbf{H}_{1C}' = -m\overline{\mathbf{v}}'$, $\mathbf{H}_{2C}' = m\overline{\mathbf{v}}'$, where we took into account (13.1.65'); hence, $\mathbf{H}_{1C}' = -\mathbf{H}_{2C}'$ and, analogously, $\mathbf{H}_{1C}'' = -\mathbf{H}_{2C}''$. Finally, $H_{1C}' = H_{1C}'' = H_{2C}' = H_{2C}''$ (we have $\overline{v}' = \overline{v}''$), the four momenta having the same modulus; moreover, we have $v_{1C}' = v_{1C}''$, $v_{2C}' = v_{2C}''$. The angle θ_C between the momenta $\mathbf{H}_{1C}'$, $\mathbf{H}_{1C}''$ of the particle P_1 before and after collision, respectively (the angle between the directions of motion in $\mathscr{R}_C$) is called *diffusion angle* (Fig. 13.4).

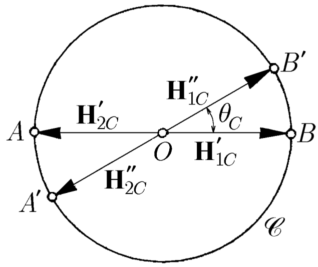

Fig. 13.4 Diffusion angle

In various experiments, one of the particles (e.g., the particle P_2) represents the "target", being at rest with respect to the measuring devices (with respect to the laboratory); the frame in which we have $\mathbf{v}_{2L}' = \mathbf{0}$ will be called *the laboratory frame* (denoted by $\mathscr{R}_L$), and the quantities in this frame will be indexed by L. In this case, from (13.1.64) it results $\mathbf{v}_C = (m_1 / M)\mathbf{v}_{1L}'$; observing that $\mathbf{v}_{2C}' = \mathbf{v}_{2L}' - \mathbf{v}_C = -\mathbf{v}_C$, we have $\mathbf{v}_C = -\mathbf{v}_{2C}' = -\mathbf{H}_{2C}' / m_2 = \mathbf{H}_{1C}' / m_2$, so that

$$\mathbf{H}_{1L}' = \frac{M}{m_2}\mathbf{H}_{1C}' = \left(1 + \frac{m_1}{m_2}\right)\mathbf{H}_{1C}'. \tag{13.1.66}$$

After collision, we obtain

$$\mathbf{H}''_{1L} = m_1(\mathbf{v}''_{1C} + \mathbf{v}_C) = \mathbf{H}''_{1C} + \frac{m_1}{m_2}\mathbf{H}'_{1C}, \tag{13.1.66'}$$

the recoil momentum of the target being (the conservation theorem of momentum in the frame $\mathscr{R}_L$)

$$\mathbf{H}''_{2L} = \mathbf{H}'_{1L} - \mathbf{H}''_{1L} = \mathbf{H}'_{1C} - \mathbf{H}''_{1C}, \tag{13.1.66''}$$

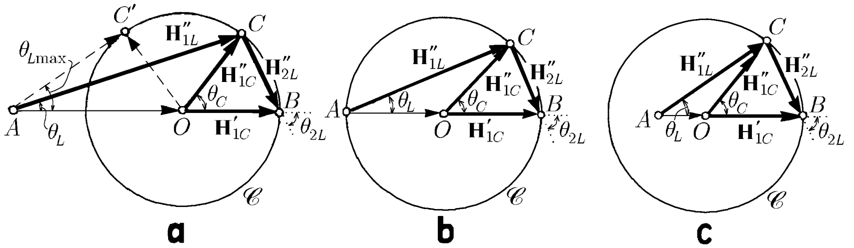

Fig. 13.5 Diffusion angles for: $m_1 > m_2$ (**a**), $m_1 = m_2$ (**b**), $m_1 < m_2$ (**c**)

The relations (13.1.66–13.1.66") will be geometrically represented in Fig. 13.5a,b,c for $m_1/m_2 > 1$, $m_1 = m_2$ and $m_1/m_2 < 1$, respectively. We denote $\overrightarrow{OB} = \mathbf{H}'_{1C}$, $\overrightarrow{OC} = \mathbf{H}''_{1C}$, $\overrightarrow{AO} = (m_1/m_2)\mathbf{H}'_{1C}$ and obtain $\overrightarrow{AB} = \mathbf{H}'_{1L}$, $\overrightarrow{AC} = \mathbf{H}''_{1L}$, $\overrightarrow{CB} = \mathbf{H}''_{2L}$; the diffusion angles are θ_C and θ_L in $\mathscr{R}_C$ and $\mathscr{R}_L$, respectively, while the diffusion angle of the particle "target" is given by $\theta_{2L} = (\pi - \theta_C)/2$. Because we can write the relation $\tan\theta_L = \left|\overrightarrow{OC}\right|\sin\theta_C / \left(\left|\overrightarrow{AO}\right| + \left|\overrightarrow{OC}\right|\cos\theta_C\right)$, we obtain

$$\tan\theta_L = \frac{m_2\sin\theta_C}{m_1 + m_2\cos\theta_C} = \frac{\tan\theta_C}{1 + (m_1/m_2)\sec\theta_C}. \tag{13.1.67}$$

We notice that $\theta_C \in [0,\pi]$. If $m_1 > m_2$, then the point A is outside the circle $\mathscr{C}$ so that $\theta_L \in [0,\theta_{L\max}]$, where $\sin\theta_{L\max} = \left|\overrightarrow{OC'}\right| / \left|\overrightarrow{AO}\right| = m_2/m_1$, $\theta_{L\max} < \pi/2$ (Fig. 13.5a), while if $m_1 < m_2$ (Fig. 13.5c), then the point A is inside the circle $\mathscr{C}$, the angle of diffusion θ_L having the same interval of variation as θ_C; if the particles have equal masses ($m_1 = m_2$), then the point A will be on the circle $\mathscr{C}$, resulting $\theta_L = \theta_C/2$ (Fig. 13.5b), hence $\theta_{L\max} = \pi/2$, $\theta_L \in [0,\pi/2]$ and $\theta_L + \theta_{2L} = \pi/2$. In the latter case, we can write the remarkable relations ($v'_{1L} = 2v'_{1C} = \overline{v}'$)

$$v''_{1L} = \overline{v}'\cos\frac{\theta_C}{2}, \quad v''_{2L} = \overline{v}'\sin\frac{\theta_C}{2}, \tag{13.1.68}$$

and after collision the particles move away under a right angle.

To can determine the diffusion angles θ_C and θ_L one must know the law of motion during the collision of the particles as well as their reciprocal position. We have seen that the angle θ_L has an upper limit if $m_1 > m_2$, while if $m_1 \gg m_2$ we obtain $\theta_{L\max} \cong 0$, hence $\theta_L \cong 0$; in this case, after collision with the target, the incidental particle is practically moving along the same initial direction. If $m_1 < m_2$, then the angle θ_L can be anyone, depending essentially on the interaction law and on the initial conditions, while if $m_1 \ll m_2$ we have $\theta_L \cong \theta_C$, the "target" particle remaining practically at rest in $\mathscr{R}_L$. In case of a *back diffusion* ($\theta_L = \theta_C - \pi$), it results $\mathbf{H}''_{1C} = -\mathbf{H}'_{1C}$, while the relations (13.1.66') and (13.1.66") lead us to the momenta $\mathbf{H}''_{1L} = (m_1 / m_2 - 1)\mathbf{H}'_{1C}$, $\mathbf{H}''_{2L} = 2\mathbf{H}'_{1C}$.

After collision, the kinetic energy of the "target" particle, in the frame $\mathscr{R}_L$, will be (Fig. 13.5)

$$T''_{2L} = \frac{1}{2m_2}|\mathbf{H}''_{2L}|^2 = \frac{2}{m_2}|\mathbf{H}'_{1C}|^2 \sin^2\frac{\theta_C}{2}, \tag{13.1.69}$$

while if $\theta_C = \pi$ we may write (we use the formula (13.1.66))

$$T''_{2L\max} = \frac{2}{m_2}|\mathbf{H}'_{1C}|^2 = \frac{2m_2}{M^2}|\mathbf{H}'_{1L}|^2 = \frac{4m}{M}T'_{1L}; \tag{13.1.69'}$$

hence, in case of a back diffusion, *the recoil energy* (the energy of the "target" particle after collision) is smaller than (or at the most equal to) the initial kinetic energy of the incidental particle (which coincides with the initial mechanical energy); indeed, $4m / M = \sqrt{m_1 m_2} / [(m_1 + m_2)/2] \le 1$. As a matter of fact, this is the maximal kinetic energy of the "target" particle after collision.

We emphasize that the above results have a general character, non-depending on the specific interaction law of the two particles. We can thus include in this study also the case of the "spontaneous" disintegration of a particle in two "fragments" (two particles which, after disintegration, move independently one of the other). If $\mathbf{v}'_L$ is the velocity of the particle before disintegration and $\mathbf{v}''_L$ and $\mathbf{v}''_C$ are the velocities of one of the particles resulting from this phenomenon, respectively, then we will have $\mathbf{v}''_L = \mathbf{v}'_L + \mathbf{v}''_C$, so that $v''^2_L + v'^2_L - 2v'_L v''_L \cos\theta_L = v''^2_C$, where θ_L is the angle under which the particle is deviated from the direction of the velocity $\mathbf{v}'_L$. If $v'_L > v''_C$, then $\theta_L \in [0, \theta_{L\max}]$, with $\sin\theta_{L\max} = v''_C / v'_L$. In general,

$$\tan\theta_L = \frac{v''_C \sin\theta_C}{v''_C \cos\theta_C + v'_L}, \tag{13.1.70}$$

wherefrom

$$\cos\theta_C = -\frac{v'_L}{v''_C}\sin^2\theta_L \pm \cos\theta_L\sqrt{1 - \frac{v'^2_L}{v''^2_C}\sin^2\theta_L}; \tag{13.1.70'}$$

if $v''_C > v'_L$ (Fig. 13.6a), then one takes the sign + before the radical (the relation is univocal), if $v''_C < v'_L$ (Fig. 13.6b), then one can take both signs, two solutions being possible, while in the limit case $v''_C = v'_L$ it results $\theta''_C = 2\theta_L$.

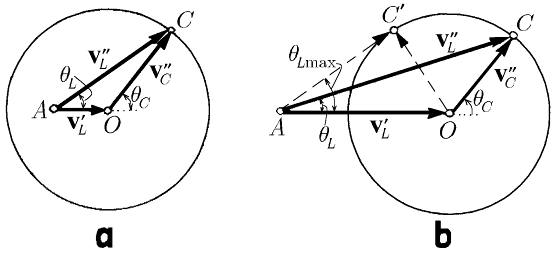

Fig. 13.6 Diffraction angle

In Chap. 8, Sect. 1.2.1, we have considered the problem of deviation of a particle of mass m in a field $U(\mathbf{r})$ by a fixed centre of force (in the case considered above, it is situated at the mass centre C); the trajectory of the particle is contained between two asymptotes the angle of which is *the diffraction angle* $\varkappa = \mp(\pi - 2\overline{\theta})$ (Fig. 8.6), given by (8.1.17). The parameters which intervene are the velocity $\overline{v} = v_\infty$ of the incidental particle at infinity and *the collision parameter* b (the distance from the centre of force to the incidental asymptote of the particle trajectory), the formula (8.1.15) specifying the constants to be determined. These results complete the problems considered above, by introducing an interaction law of the two particles (the field $U(r)$).

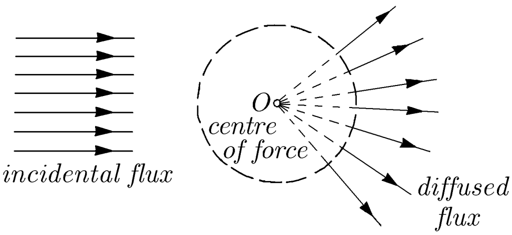

Fig. 13.7 Phenomenon of diffusion

In general, besides the problem of diffraction (deviation) of a single particle by the "target", the problem of diffusion of a great number of particles (flux of particles) by a *centre of force* (a field of central forces) is put (Fig. 13.7). Obviously, if the mechanical phenomenon takes place at an atomic level, then the results which can be obtained in the frame of a classical model may represent only an approximation (not always the best one), the quantum effects having an essential importance in this case. Nevertheless, some results are true, with a good approximation; moreover, the methods used to describe *the diffusion phenomenon* are the same in classical as in quantum mechanics. We will consider an *incidental flux* of particles which move independently on parallel rectilinear trajectories, having the same mass m and the same velocity $\mathbf{v}_\infty$ (a homogeneous incidental beam of particles, which can be electrons, particles or celestial bodies). Passing in the vicinity of the centre of force O, the particles are influenced by that one; being diffused with velocities in various directions, but of the same magnitude (*elastic diffusion*) and becoming again independent. To this goal, the centre of force must have a finite radius of action; hence, one must have $U(r) \neq 0$ for $r < r_0$ and $U(r) = 0$ for $r > r_0$. Practically, it is sufficient to have $\lim\limits_{r \to \infty} U(r) = 0$; in this case, $E = T - U > 0$ because $E = T > 0$ at infinity. The trajectories of the particles are curves symmetric with respect to the straight line which passes through O and through the pericentre P (see Fig. 8.6). The diffusion angle $\varkappa$ is the angle made by the two asymptotes of the trajectory and is the same for all the particles which have the same collision parameter b. To can evaluate the way in which the particles having a collision parameter contained in a certain interval are deviated in directions contained, as well, in a given interval, one defines *the efficacious* (differential) *section of diffusion* $\mathrm{d}\sigma$ in the form

$$\mathrm{d}\sigma = \frac{\mathrm{d}N}{n}, \tag{13.1.71}$$

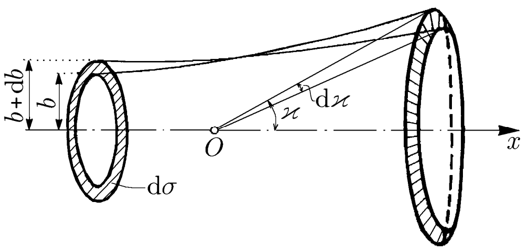

Fig. 13.8 Efficacious section of diffusion

where $\mathrm{d}N$ is the number of particles diffused in the solid angle $\mathrm{d}\omega$ in a unity of time, while n is the number of incident particles which cross the unit area in the same time; this ratio constitutes the most important characteristic of the process of diffusion. Assuming that between $\varkappa$ and b there exists a one-to-one correspondence, hence that the function $\varkappa = \varkappa(b)$ is a monotone decreasing function, the only particles diffused in the interval $[\varkappa, \varkappa + \mathrm{d}\varkappa]$ are those the collision parameter of which is contained between $b(\varkappa)$ and $b(\varkappa) + \mathrm{d}b(\varkappa)$; the number of these particles will be, obviously, $\mathrm{d}N = 2\pi b \mathrm{d}b n$, resulting the efficacious section (an annular section, the phenomenon being with axial symmetry with respect to the Ox-axis (Fig. 13.8)

$$\mathrm{d}\sigma = 2\pi b \mathrm{d}b. \tag{13.1.71'}$$

The relation between the efficacious section of diffusion and the diffusion angle (plane angle) will result in the form

$$\mathrm{d}\sigma = 2\pi b(\varkappa)\left|\frac{\mathrm{d}b(\varkappa)}{\mathrm{d}\varkappa}\right|\mathrm{d}\varkappa\,, \tag{13.1.71''}$$

where the derivative is taken in absolute value because, in general, it is negative. With reference to the element of solid angle $\mathrm{d}\omega = 2\pi\sin\varkappa\mathrm{d}\varkappa$, we obtain

$$\mathrm{d}\sigma = \frac{b(\varkappa)}{\sin\varkappa}\left|\frac{\mathrm{d}b(\varkappa)}{\mathrm{d}\varkappa}\right|\mathrm{d}\omega = \frac{1}{2\sin\varkappa}\left|\frac{\mathrm{d}b^2(\varkappa)}{\mathrm{d}\varkappa}\right|\mathrm{d}\omega = \frac{1}{2}\left|\frac{\mathrm{d}b^2(\varkappa)}{\mathrm{d}(\cos\varkappa)}\right|\mathrm{d}\omega\,. \tag{13.1.71'''}$$

If the diffusion of the beam of particles is not due to a fixed centre of force and to other particles which are initially at rest, then it results that the formulae (13.1.71''), (13.1.71''') take place in the frame $\mathscr{R}_C$. Passing to the frame $\mathscr{R}_L$, the efficacious section $\mathrm{d}\sigma$ is not modified (it is defined as a ratio of numbers of particles, being independent of frame), but the element of solid angle is modified, so that

$$\mathrm{d}\sigma = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_C \mathrm{d}\omega_C = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_L \mathrm{d}\omega_L\,; \tag{13.1.72}$$

we notice that

$$\mathrm{d}\omega_L = 2\pi\sin\varkappa_C\mathrm{d}\varkappa_C\left|\frac{\mathrm{d}(\cos\varkappa_L)}{\mathrm{d}(\cos\varkappa_C)}\right| = \left|\frac{\mathrm{d}(\cos\varkappa_L)}{\mathrm{d}(\cos\varkappa_C)}\right|\mathrm{d}\omega_C\,, \tag{13.1.72'}$$

so that, taking into account (13.1.67) too, we can make a calculus in the frame $\mathscr{R}_L$.

In the case of a diffusion on the "spherical potential hollow", for which

$$U(r) = \begin{cases} U_0 & \text{for} \quad r \le R, \\ 0 & \text{for} \quad r > R, \end{cases} \tag{13.1.73}$$

where U_0 is a positive constant, we get $\sigma = \pi R^2$, hence the area of the central section of the sphere.

In particular, we consider the diffusion of a beam of particles of charge q_1 on the "target" formed by the particles of charge q_2, in a potential field (9.2.4), with

$$k = \frac{q_1 q_2}{4\pi\varepsilon_0}\,, \tag{13.1.74}$$

ε_0 being the permittivity of vacuum in a rationalized system. The diffusion angle is given by $\sin(\varkappa/2) = 1/e$, where $e > 1$ is the eccentricity of the trajectory of a particle of charge q_1 (the trajectories are hyperbolae, as in case of the deviation of the luminous radius, so that we can use the formula (9.2.27'')). Replacing the constants C and k given by (8.1.15) and (9.2.5), we obtain ($\overline{v} = v_\infty$)

$$e^2 = 1 + \left(\frac{mv_\infty^2 b}{k}\right)^2,$$

so that

$$b^2 = \frac{k^2}{m^2 v_\infty^4} \cot^2 \frac{\varkappa}{2};$$

the relation (13.1.71''') leads thus to *Rutherford's formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \left(\frac{k}{2mv_\infty^2} \right)^2 \operatorname{cosec}^4 \frac{\varkappa}{2}, \tag{13.1.75}$$

which gives the efficacious diffusion section in the frame $\mathscr{R}_C$. Using this formula and measuring experimentally, by recording on negatives in the Wilson room, the initial and final directions of the particles, hence the angles of diffraction $\varkappa$, E. Rutherford has put in evidence the existence of the atomic nucleus (charged by a positive electric charge equal to the number order – after Mendeleev's table – of the chemical element which is in Wilson's room and produces this effect). These theoretical results have been verified by numerous experimental researches due to H. Geiger, E. Marsden, van der Boek (1913), J. Chadwick (1920) etc.

It is interesting to see that the above results hold, with a good approximation, in the frame of the model of quantum mechanics too, so far as one can write a conservation theorem of momentum, the diffusion being an elastic one.

13.1.2.2 Plastic Collision of Discrete Mechanical Systems. Space of Plastic Collisions

Let us firstly consider the case of two particles (e.g., two small spheres of masses m_1 and m_2), which are in collision with the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, remaining then glued together and continuing their motion as a single particle of mass $M = m_1 + m_2$, with the common velocity

$$\mathbf{v} = \frac{1}{M}(m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2), \quad M = m_1 + m_2, \tag{13.1.76}$$

obtained from the conservation theorem of momentum (assuming the absence of external percussions). The loss of kinetic energy is

$$(\Delta T)^0 = T' - T'' = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) - \frac{1}{2}(m_1 + m_2)v^2,$$

and the kinetic energy of lost velocities is given by

$$T_0 = \frac{1}{2}\left[m_1 (\mathbf{v} - \mathbf{v}_1)^2 + m_2 (\mathbf{v} - \mathbf{v}_2)^2 \right];$$

in this case, we can write the generalized theorem of Carnot in the form (13.1.23'') (the restitution coefficient k is equal to zero), so that

$$(\Delta T)^0 = T_0 = \frac{1}{2} m (\mathbf{v}_1 - \mathbf{v}_2)^2, \quad \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}, \tag{13.1.77}$$

having $T' \geq T''$.

The phenomenon of plastic collision of two particles leads thus, from a mathematical point of view, to a law of composition of masses and velocities. Following this order of ideas, let be P a particle of mass m and velocity $\mathbf{v}$; we denote the element mass-velocity which characterizes the particle in motion by $S \equiv (m, \mathbf{v})$. We assume that the mass and the velocity are functions of time of class C^1, the mass $m(t)$ being a positive function ($m(t) > 0$). In the set $\mathscr{M} = \{S\}$ of elements S we introduce a relation of equivalence denoted by K; we say thus that *two elements* $S_1, S_2 \in \mathscr{M}$ *are equivalent* (we denote $S_1 \sim S_2$) *if they have the same momentum* ($m_1 \mathbf{v}_1 = m_2 \mathbf{v}_2$). This is written in the form

$$K(S_1) = K(S_2). \tag{13.1.78}$$

One can easily prove that the relation K defined on $\mathscr{M}$ is an *equivalence relation*, because it is *reflexive*, *symmetrical* and *transitive*. The equivalence relation K introduced in the set $\mathscr{M}$ realizes a *partition* of $\mathscr{M}$ into *classes of equivalence*. Let be $\mathscr{M}/K$ this *set of classes of equivalence*; it represents *the quotient set* of $\mathscr{M}$ by K. Thus, corresponding to the relation K, to each element $S \in \mathscr{M}$ will correspond the class of equivalence $\widetilde{S}$, hence

$$S \xrightarrow{K} \widetilde{S} \in \mathscr{M}/K. \tag{13.1.79}$$

In the set $\mathscr{M}/K$ of classes of equivalence, we define a *law of internal composition*, denoted *additively* in the form ($\widetilde{S}_1 \equiv (m_1, \mathbf{v}_1), \widetilde{S}_2 \equiv (m_2, \mathbf{v}_2)$, $\widetilde{S}_1, \widetilde{S}_2 \in \mathscr{M}/K$)

$$\left(\widetilde{S}_1, \widetilde{S}_2\right) \to \widetilde{S}_1 + \widetilde{S}_2 = \left(m_1 + m_2, \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}\right) \equiv (M, \mathbf{v}). \tag{13.1.80}$$

One can verify the property of *commutativity* ($\widetilde{S}_1 + \widetilde{S}_2 = \widetilde{S}_2 + \widetilde{S}_1$) and the property of *associativity* ($\left(\widetilde{S}_1 + \widetilde{S}_2\right) + \widetilde{S}_3 = \widetilde{S}_1 + \left(\widetilde{S}_2 + \widetilde{S}_3\right)$). *The neutral element* $\widetilde{O} \in \mathscr{M}/K$ is given by the relation $\widetilde{O} = (m, \mathbf{0})$, hence by $K\left(\widetilde{O}\right) = \mathbf{0}$. From the definition of that element, it results

$$\widetilde{S}_1 + \widetilde{O} = (m_1, \mathbf{v}_1) + (m, \mathbf{0}) = \left(m_1 + m, \frac{m_1 \mathbf{v}_1}{m_1 + m}\right) = \widetilde{S}_1,$$

because $K\left(\widetilde{S}_1 + \widetilde{O}\right) = m_1 \mathbf{v}_1 = K\left(\widetilde{S}_1\right)$, and the property of null effect of the neutral element is put in evidence. The law of internal composition defines thus an *Abelian group* on $\mathscr{M}/K$.

We define a law of *external composition* in a *multiplicative* notation ($S \equiv (m, \mathbf{v})$, $\alpha \in \mathbb{R}$)

$$\left(\alpha\tilde{S}\right) \to \alpha\tilde{S} = (m, \alpha\mathbf{v})\,. \tag{13.1.80'}$$

The element $\tilde{S}'$, *opposite* to the element $\tilde{S}$ is specified by $\tilde{S}' = (m, -\mathbf{v}) = -(m, \mathbf{v}) = -\tilde{S}$; obviously, we have $\tilde{S} + \tilde{S}' = (m, \mathbf{v}) + (m, -\mathbf{v}) = (2m, \mathbf{0}) = \tilde{O}$, so that $K\left(\tilde{S} + \tilde{S}'\right) = \mathbf{0}$ and we can write $\tilde{S}' = -\tilde{S}$. We notice the property $1\tilde{S} = (m, \mathbf{v}) = \tilde{S}$ for a real number 1; as well, one has the property of *associativity* ($\alpha(\beta\tilde{S}) = \beta(\alpha\tilde{S}) = (\alpha\beta)\tilde{S} = (m, \alpha\beta\mathbf{v})$, $\alpha, \beta \in \mathbb{R}$). Analogously, we can put in evidence a property of *distributivity with respect to addition of real numbers* ($(\alpha + \beta)\tilde{S} = \alpha\tilde{S} + \beta\tilde{S}$, $\alpha, \beta \in \mathbb{R}$) and a property of *distributivity with respect to the law of internal composition* ($\alpha\left(\tilde{S}_1 + \tilde{S}_2\right) = \alpha\tilde{S}_1 + \alpha\tilde{S}_2$, $\alpha \in \mathbb{R}$). We have shown thus that the quotient set $\mathscr{M} / K$, in which we have defined internal and external laws of composition, constitutes a *vector space on the field* $\mathbb{R}$ *of real numbers*.

It is important to observe that the internal composition law introduced above corresponds to the plastic collision, since the mass of the sum element is equal to the sum of the masses of the component elements, while the velocity after collision corresponds to that given by (13.1.76). Also, the quotient set $\mathscr{M} / K$ obtained with the aid of the equivalence law $K(S) = m\mathbf{v}$ corresponds to the mechanical meaning of the collision phenomenon; indeed, if the elements S_1 and S_2 are equivalent, then they have the same momentum too. This justifies the denomination of *space of plastic collisions* given to the vector space thus introduced by W. Kecs and P.P. Teodorescu.

Applying the second principle of mechanics to two equivalent elements S_1 and S_2, we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_1\mathbf{v}_1) = \frac{\mathrm{d}}{\mathrm{d}t}(m_2\mathbf{v}_2) = \mathbf{F}_1 = \mathbf{F}_2\,;$$

hence, the differential equation of motion of two elements of the same class of equivalence is the same, and this illustrates the mechanical sense of the equivalence K.

We can write

$$\alpha_1\tilde{S}_1 + \alpha_2\tilde{S}_2 + \alpha_3\tilde{S}_3 = \left(m_1 + m_2 + m_3, \frac{\alpha_1 m_1\mathbf{v}_1 + \alpha_2 m_2\mathbf{v}_2 + \alpha_3 m_3\mathbf{v}_3}{m_1 + m_2 + m_3}\right);$$

if we take into account the definition of the neutral element, it follows that the relation

$$K\left(\alpha_1\tilde{S}_1 + \alpha_2\tilde{S}_2 + \alpha_3\tilde{S}_3\right) = K\left(\tilde{O}\right) = \mathbf{0}$$

holds if and only if

$$(\alpha_1 m_1)\mathbf{v}_1 + (\alpha_2 m_2)\mathbf{v}_2 + (\alpha_3 m_3)\mathbf{v}_3 = \mathbf{0}.$$

In this way, the relation

$$\alpha_1 \widetilde{S}_1 + \alpha_2 \widetilde{S}_2 + \alpha_3 \widetilde{S}_3 = \widetilde{O}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

cannot occur for arbitrary $\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3$, unless $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

On the other hand, we may have a relation of the form

$$\sum_{i=1}^{n} \alpha_i \widetilde{S}_i = \widetilde{O}, \quad \alpha_i \in \mathbb{R}, \quad i = 1,2,\ldots,n, \quad n > 3,$$

even if not all α_i are zero; it results that *the space of plastic collisions is three-dimensional*.

Let be now a mapping of $\mathscr{M} / K$ onto the real positive half-straight line $\mathbb{R}_+$, defined in the form

$$\widetilde{S} \to \left|\widetilde{S}\right| = \frac{1}{2} m v^2; \tag{13.1.81}$$

we remark that we have thus introduced the kinetic energy of the element. It may be shown that the mapping defined by the relation (13.1.81) satisfies the following properties:

$$\left|\widetilde{S}\right| \geq 0, \tag{13.1.82}$$

$$\left|\widetilde{S}_1 + \widetilde{S}_2\right| \leq \left|\widetilde{S}_1\right| + \left|\widetilde{S}_2\right|, \quad \widetilde{S}_1, \widetilde{S}_2 \in \mathscr{M} / K, \tag{13.1.82'}$$

$$\left|\alpha \widetilde{S}\right| = \alpha^2 \left|\widetilde{S}\right|, \quad \alpha \in \mathbb{R}, \tag{13.1.82''}$$

the equality in relation (13.1.82) involving $\widetilde{S} = \widetilde{O} \in \mathscr{M} / K$. Indeed, the first and the third property follow immediately from the relation of definition (13.1.81); observing that $\left|\widetilde{S}_1\right| + \left|\widetilde{S}_2\right| = T'$ and $\left|\widetilde{S}_1 + \widetilde{S}_2\right| = T''$ and taking into account Carnot's generalized theorem, one obtains the second property too.

We remark that the space of plastic collisions may be normed by introducing a norm defined by the relation $\left\|\widetilde{S}\right\| = \left|m\mathbf{v}\right|$; in this case, the distance in the respective space is given by $\mathrm{d}\left(\widetilde{S}_1, \widetilde{S}_2\right) = \left|m_1 \mathbf{v}_1 - m_2 \mathbf{v}_2\right|$. Although the mapping (13.1.81) does not constitute a norm in the space of plastic collisions, it allows the introduction of the notion of distance in the set $\mathscr{M} / K$, and thus the space of plastic collisions becomes a *metric space*. Therefore, we shall call *distance* in the set $\mathscr{M} / K$ the number

$$\mathrm{d}\left(\widetilde{S}_1, \widetilde{S}_2\right) = \left|\widetilde{S}_1 - \widetilde{S}_2\right| = \frac{1}{2} \frac{(m_1 \mathbf{v}_1 - m_2 \mathbf{v}_2)^2}{m_1 + m_2}; \tag{13.1.83}$$

Taking into account the mapping (13.1.81), it may be easily seen that the distance, as defined above, satisfies the following conditions:

$$\mathrm{d}\left(\widetilde{S},\widetilde{S}\right)=0\,,$$
$$\mathrm{d}\left(\widetilde{S}_1,\widetilde{S}_2\right)=\mathrm{d}\left(\widetilde{S}_2,\widetilde{S}_1\right),\quad \forall\widetilde{S}_1,\widetilde{S}_2\in\mathscr{M}/K\,,$$
$$\mathrm{d}\left(\widetilde{S}_1,\widetilde{S}_3\right)\le\mathrm{d}\left(\widetilde{S}_1,\widetilde{S}_2\right)+\mathrm{d}\left(\widetilde{S}_2,\widetilde{S}_3\right),\quad \forall\widetilde{S}_1,\widetilde{S}_2,\widetilde{S}_3\in\mathscr{M}/K\,,$$

that is the conditions of *reflexivity*, *symmetry* and *triangle relation*, respectively.

Let $\widetilde{S}=(m(t),\mathbf{v}(t))$ be an element of $\mathscr{M}/K$; we assume that $m(t)$ and $\mathbf{v}(t)$ are functions of class C^1 with respect to the time t. If t' is a value belonging to the domain of definition, then there follows

$$\widetilde{S}(t')-\widetilde{S}(t)=\left(m(t')+m(t),\frac{m(t')\mathbf{v}(t')-m(t)\mathbf{v}(t)}{m(t')+m(t)}\right).$$

Multiplying by $1/(t'-t)$, we obtain

$$\frac{\widetilde{S}(t')-\widetilde{S}(t)}{t'-t}=\left(m(t')+m(t),\frac{1}{m(t')+m(t)}\frac{\Delta(m\mathbf{v})}{\Delta t}\right),$$

where $\Delta(m\mathbf{v})=m(t')\mathbf{v}(t')-m(t)\mathbf{v}(t)$ and $\Delta t=t'-t$; passing to the limit, we have

$$\frac{\mathrm{d}\widetilde{S}}{\mathrm{d}t}=\left(2m,\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})\right)=\left(2m,\frac{\mathbf{F}}{2m}\right), \tag{13.1.84}$$

where we took into account the second principle of Newton. The corresponding equivalence relation will be

$$K\left(\frac{\mathrm{d}\widetilde{S}}{\mathrm{d}t}\right)=2m\frac{\mathbf{F}}{2m}=\mathbf{F}\,, \tag{13.1.84'}$$

being thus led to the force acting upon the particle.

Introducing the mapping (13.1.81) for the derived element (13.1.84), it results

$$2\left|\frac{\mathrm{d}\widetilde{S}}{\mathrm{d}t}\right|=2\frac{1}{2}(2m)\left(\frac{\mathbf{F}}{2m}\right)^2=\frac{1}{2}m\left(\frac{\mathbf{F}}{m}\right)^2, \tag{13.1.85}$$

where *the acceleration energy* (corresponding to the case $m=\mathrm{const}$) has been marked out.

We notice that *the derivative* is a linear operator in the space of plastic collisions and verifies the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha\tilde{S}\right) = \alpha\frac{\mathrm{d}\tilde{S}}{\mathrm{d}t}, \quad \alpha = \mathrm{const}\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{S}_1 + \tilde{S}_2\right) = \frac{\mathrm{d}\tilde{S}_1}{\mathrm{d}t} + \frac{\mathrm{d}\tilde{S}_2}{\mathrm{d}t}.$$

If two elements $\tilde{S}_1$ and $\tilde{S}_2$ are acted upon by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively, then the acceleration energy before collision will be

$$\zeta' = \frac{1}{2}m_1\left(\frac{\mathbf{F}_1}{m_1}\right)^2 + \frac{1}{2}m_2\left(\frac{\mathbf{F}_2}{m_2}\right)^2; \tag{13.1.86}$$

the acceleration energy after the plastic collision is given by

$$\zeta'' = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1\dfrac{\mathbf{F}_1}{m_1} + m_2\dfrac{\mathbf{F}_2}{m_2}}{m_1 + m_2}\right)^2 = \frac{1}{2}(m_1 + m_2)\left(\frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2}\right)^2. \tag{13.1.86'}$$

The loss of acceleration energy is, in this case, $(\Delta\zeta)^0 = \zeta' - \zeta''$. *The acceleration energy of the lost accelerations* will be, analogously, given by

$$\zeta_0 = \frac{1}{2}\left[m_1\left(\frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2} - \frac{\mathbf{F}_1}{m_1}\right)^2 + m_2\left(\frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2} - \frac{\mathbf{F}_2}{m_2}\right)^2\right]. \tag{13.1.86''}$$

It results

$$(\Delta\zeta)^0 = \zeta_0 = \frac{1}{2}m\left(\frac{\mathbf{F}_1}{m_1} - \frac{\mathbf{F}_2}{m_2}\right)^2, \quad \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}, \tag{13.1.87}$$

hence $\zeta' \geq \zeta''$, so that the acceleration energy after the plastic collision of two particles is smaller or at most equal to the acceleration energy before collision (*an analogue of the generalized theorem of Carnot*). This relation can be written also in the form

$$\frac{1}{2}\frac{(\mathbf{F}_1 + \mathbf{F}_2)^2}{m_1 + m_2} \leq \frac{1}{2}\frac{\mathbf{F}_1^2}{m_1} + \frac{1}{2}\frac{\mathbf{F}_2^2}{m_2}. \tag{13.1.87'}$$

We have introduced the notion of derivative in the quotient set $\mathscr{M}/K$, but we cannot introduce, analogously, the notion of Riemann integral; instead, we can introduce the concept of *primitive*. Thus, $\tilde{S}^* \in \mathscr{M}/K$ is a primitive for $\tilde{S} = (m, \mathbf{v}) \in \mathscr{M}/K$ if the relation

$$\frac{\mathrm{d}\tilde{S}^*}{\mathrm{d}t} = \tilde{S}, \quad \tilde{S}^* = \int \tilde{S}\,\mathrm{d}t \tag{13.1.88}$$

takes place. In the same order of ideas, we say that the element $\widetilde{S}$ is constant if its momentum is constant ($K\left(\widetilde{S}\right) = m\mathbf{v} = \mathbf{c} = \overrightarrow{\text{const}}$); hence, the element $\widetilde{S}_0 = (m, \mathbf{c}/m)$ is constant in $\mathscr{M}/K$. Corresponding to the relation (13.1.84), we deduce that $K\left(\mathrm{d}\widetilde{S}_0/\mathrm{d}t\right) = \mathrm{d}\mathbf{c}/\mathrm{d}t = \mathbf{0}$, so that for a constant element $\widetilde{S}_0$ we may write $\mathrm{d}\widetilde{S}_0/\mathrm{d}t = \widetilde{O}$. We notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{S}^* + \widetilde{S}_0\right) = \frac{\mathrm{d}\widetilde{S}^*}{\mathrm{d}t} + \frac{\mathrm{d}\widetilde{S}_0}{\mathrm{d}t} = \widetilde{S} + \widetilde{O} = \widetilde{S};$$

hence, if $\widetilde{S}^*$ is a primitive of $\widetilde{S}$, then $\widetilde{S}^* + \widetilde{S}_0$ is a primitive for $\widetilde{S}$ too. Let be $\widetilde{S} = (m(t), \mathbf{v}(t))$, $\widetilde{S}^* = (\overline{m}(t), \overline{\mathbf{v}}(t))$, where $m(t)$ and $\mathbf{v}(t)$ are functions of class C^1; from the first relation (13.1.88) we obtain $(2\overline{m}, (1/2\overline{m})\mathrm{d}(\overline{m}\overline{\mathbf{v}})/\mathrm{d}t) = (m, \mathbf{v})$, so that, by the equality of momenta, we can write $\mathrm{d}(\overline{m}\overline{\mathbf{v}})/\mathrm{d}t = m\mathbf{v}$. Integrating, we get $\overline{m}\overline{\mathbf{v}} = \int m\mathbf{v}\mathrm{d}t$, so that the general form of the primitive $\widetilde{S}^*$ is expressed by

$$\widetilde{S}^* = \int \widetilde{S}\mathrm{d}t = \left(\overline{m}(t), \frac{1}{\overline{m}(t)}\int m(t)\mathbf{v}(t)\mathrm{d}t\right), \tag{13.1.89}$$

where $\overline{m}(t)$ is an arbitrary positive function.

13.2 Dynamics of Mechanical Systems of Variable Mass

The results obtained in Chap. 10, Sect. 3 for the dynamics of the particle of variable mass will be extended, in what follows, to the case of a discrete mechanical system of variable mass; thus, the general theorems of dynamics and some interesting particular problems will be presented. Some cases of continuous mechanical systems will be dealt with too.

13.2.1 Discrete Mechanical Systems

The problem of a particle of variable mass (studied in Chap. 10, Sect. 3.1.1) is considered again in the space of plastic collisions, introduced by W. Kecs and P.P. Teodorescu, modelling the discontinuity of mass in the frame of the theory of distributions; the general problem of a discrete mechanical system is then presented, deducing the corresponding universal theorems.

13.2.1.1 Particle of Variable Mass

Let be an element $\widetilde{S}(t) = (m(t), \mathbf{v}(t)) \in \mathscr{M}/K$, where the momentum corresponds to a particle P; we assume that the functions $m(t)$ and $\mathbf{v}(t)$ are of class C^1. If the mass of the particle P increases, due to a phenomenon of capture at the moment t of an element $\tilde{S}'(t) \equiv (\Delta m, \mathbf{u})$, $\Delta m > 0$, of absolute velocity $\mathbf{u}(t)$ and momentum $K(\tilde{S}') = \Delta m\mathbf{u}$, then the external composition law at that moment leads to

$$\tilde{S}(t) + \tilde{S}'(t) = \left(m + \Delta m, \frac{m\mathbf{v} + \Delta m\mathbf{u}}{m + \Delta m} \right),$$

of momentum $K(\tilde{S} + \tilde{S}') = m\mathbf{v} + \Delta m\mathbf{u}$; at the moment $t + \Delta t$ will exist only one element $\tilde{S}_0 \equiv (m + \Delta m, \mathbf{v} + \Delta'\mathbf{v})$, where $\Delta'\mathbf{v}$ represents the variation of the velocity of the particle P in the time interval Δt, its momentum, after neglecting the terms of higher order, being

$$K(\tilde{S}_0) = (m + \Delta m)(\mathbf{v} + \Delta'\mathbf{v}) = m\mathbf{v} + \Delta m\mathbf{v} + m\Delta'\mathbf{v}.$$

The equivalence $\tilde{S} + \tilde{S}' \sim \tilde{S}_0$ leads to $K(\tilde{S} + \tilde{S}') = K(\tilde{S}_0)$, so that $m\mathbf{v} + \Delta m\mathbf{u} = m\mathbf{v} + \Delta m\mathbf{v} + m\Delta'\mathbf{v}$; introducing also the influence of the forces $\mathbf{F}$, given in the form $m\Delta''\mathbf{v} = \mathbf{F}\Delta t$, where $\Delta''\mathbf{v}$ is the corresponding variation of the velocity of the particle P in the same time interval Δt, and using the principle of the parallelogram, we get the resultant variation of the velocity $\mathbf{v}$ in the form $\Delta\mathbf{v} = \Delta'\mathbf{v} + \Delta''\mathbf{v}$. Passing to limit for $\Delta t \to 0$, we find again Meshcherskiĭ's equation (10.3.3'), where $\mathbf{w} = \mathbf{u} - \mathbf{v}$ is the relative velocity of the element with respect to a non-inertial frame of reference attached to the particle P.

If an element $\tilde{S}''(t) \equiv (-\Delta m, \mathbf{u})$, $\Delta m > 0$, of momentum $K(\tilde{S}'') = -\Delta m\mathbf{u}$ is detached from the particle P at the moment t, then we can write

$$\tilde{S} - \tilde{S}'' = \left(m - \Delta m, \frac{m\mathbf{v} - (-\Delta m\mathbf{u})}{m - \Delta m} \right),$$

the momentum being $K(\tilde{S} - \tilde{S}'') = m\mathbf{v} + \Delta m\mathbf{u}$; at the moment $t + \Delta t$, after emission, one obtains the element $\tilde{S}_0 \equiv (m - (-\Delta m), \mathbf{v} + \Delta'\mathbf{v})$ of momentum $K(\tilde{S}_0) = m\mathbf{v} + \Delta m\mathbf{v} + m\Delta'\mathbf{v}$, neglecting the terms of higher order. Proceeding as before and passing to limit for $\Delta t \to 0$, we find again Meshcherskiĭ's equation (10.3.1).

Analogously, one can obtain the generalized equation (10.3.8) of Meshcherskiĭ; taking into account (10.3.11), one can write this equation also in the form ($\dot{m} = \dot{m}^+ + \dot{m}^-$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F} = \dot{m}^-\mathbf{u}_- + \dot{m}^+\mathbf{u}_+, \quad \dot{m}^- < 0, \quad \dot{m}^+ > 0. \qquad (13.2.1)$$

We assume, in what follows, that the equation (13.2.1) maintains its form in distributions; for distributions of function type, the derivative in the sense of the theory of distributions will be given by (see the formula (1.1.51)

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \frac{\tilde{\mathrm{d}}}{\mathrm{d}t}(m\mathbf{v}) + \sum_{i=1}^{n}(\Delta\mathbf{H})_i\,\delta(t - t_i)\,, \tag{13.2.2}$$

where the sign "tilde" corresponds to the derivative in the usual sense; the momentum $\mathbf{H} = m\mathbf{v}$ has only discontinuities of the first kind, with the jumps

$$(\Delta\mathbf{H})_i = m(t_i + 0)\mathbf{v}(t_i + 0) - m(t_i - 0)\mathbf{v}(t_i - 0)\,, \tag{13.2.2'}$$

the moments of discontinuity being t_i, $i = 1,2,\ldots,n$. If the first n' moments of discontinuity correspond to the mass discontinuities $m^-(t)$, while the other $n - n'$ moments of discontinuities correspond to the mass discontinuities $m^+(t)$, we can write

$$\begin{aligned} \frac{\mathrm{d}m^-(t)}{\mathrm{d}t} &= \frac{\tilde{\mathrm{d}}m^-(t)}{\mathrm{d}t} + \sum_{j=1}^{n'}(\Delta m^-)_j\,\delta(t - t_j)\,, \\ \frac{\mathrm{d}m^+(t)}{\mathrm{d}t} &= \frac{\tilde{\mathrm{d}}m^+(t)}{\mathrm{d}t} + \sum_{k=n'+1}^{n}(\Delta m^+)_k\,\delta(t - t_k)\,, \end{aligned} \tag{13.2.2''}$$

where we have introduced the jumps

$$\begin{aligned} (\Delta m^-)_j &= m^-(t_j + 0) - m^-(t_j - 0)\,, \quad j = 1,2,\ldots,n'\,, \\ (\Delta m^+)_k &= m^+(t_k + 0) - m^+(t_k - 0)\,, \quad k = n' + 1, n' + 2,\ldots,n\,. \end{aligned} \tag{13.2.2'''}$$

The equation (13.2.1) becomes

$$\begin{aligned} &\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}(m\mathbf{v}) + \sum_{i=1}^{n}(\Delta\mathbf{H})_i\,\delta(t - t_i) = \mathbf{F} + \frac{\tilde{\mathrm{d}}m^-}{\mathrm{d}t}\mathbf{u}_- + \frac{\tilde{\mathrm{d}}m^+}{\mathrm{d}t}\mathbf{u}_+ \\ &+ \mathbf{u}^-\sum_{j=1}^{n'}(\Delta m^-)_j\,\delta(t - t_j) + \mathbf{u}^+\sum_{k=n'+1}^{n}(\Delta m^+)_k\,\delta(t - t_k)\,; \end{aligned}$$

finally, this equation may be replaced by the equation

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{F} + \frac{\tilde{\mathrm{d}}m^-}{\mathrm{d}t}\mathbf{u}_- + \frac{\tilde{\mathrm{d}}m^+}{\mathrm{d}t}\mathbf{u}_+\,, \tag{13.2.3}$$

corresponding to the moments of continuity, the derivatives being calculated in the usual sense and by the jump relations

$$\begin{aligned} (\Delta\mathbf{H})_j &= (\Delta m^-)_j\,\mathbf{u}_-\,, \quad j = 1,2,\ldots,n'\,, \\ (\Delta\mathbf{H})_k &= (\Delta m^+)_k\,\mathbf{u}_+\,, \quad k = n' + 1, n' + 2,\ldots,n\,, \end{aligned} \tag{13.2.3'}$$

corresponding to the discontinuities of the mass. One obtains thus *the generalized equation of Meshcherskiĭ for a particle of discontinuous variable mass*.

13.2.1.2 Theorems of Momentum and Moment of Momentum

In the case of a discrete mechanical system $\mathcal{S}$ of variable mass it is convenient to use the universal theorems of mechanics. We assume, in this case, that the constraint percussive forces appear only in case of detachment or capture of some elements, by their contact with the system $\mathcal{S}$ (in fact, by the contact with a subsystem of that one); as well, the elements which have not a relative velocity with respect to this system are attached to it. We suppose that, at a given moment, the particles of the system $\mathcal{S}$ have not a relative velocity with respect to a given frame $\mathcal{R}$, so that the latter one is rigidly linked to the system at the respective moment; thus, we take not in consideration the state previous to that at the moment t. The origin of the movable frame $\mathcal{R}$ will not be taken, in general, at the centre of mass C, because the masses of the particles are variable, so that the position of this centre varies with respect to their positions. Using the notations in Sect. 11.2.2.1, we can express the velocity $\mathbf{v}_i'$ of a particle of position vector $\mathbf{r}_i' = \mathbf{r}_O' + \mathbf{r}_i$, $i = 1,2,\ldots,n$, with respect to the inertial (fixed) frame $\mathcal{R}'$ in the form (see the formula (11.2.10'))

$$\overline{\mathbf{v}}_i' = \mathbf{v}_O' + \boldsymbol{\omega} \times \mathbf{r}_i\,, \quad i = 1,2,\ldots,n\,, \tag{13.2.4}$$

the relative velocity vanishing ($\mathbf{v}_i = \mathbf{0}$ at the moment t); the corresponding acceleration is given by

$$\overline{\mathbf{a}}_i' = \mathbf{a}_O' + \dot{\boldsymbol{\omega}} \times \mathbf{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_i)\,, \quad i = 1,2,\ldots,n\,. \tag{13.2.4'}$$

The absolute velocity of the point which coincides with the centre of mass C at the moment t is given by

$$\overline{\mathbf{v}}_C' = \mathbf{v}_O' + \boldsymbol{\omega} \times \boldsymbol{\rho}\,, \tag{13.2.5}$$

corresponding to the formula (11.2.14) in which we make $\mathbf{v}_C = \mathbf{0}$. The momentum of the mechanical system $\mathcal{S}$ will be given, in this case, by (m_i is the mass of the particle P_i at the moment t, considered not to have a relative velocity with respect to the system $\mathcal{S}$)

$$\mathbf{H}' = \sum_{i=1}^{n} m_i \overline{\mathbf{v}}_i' = M \overline{\mathbf{v}}_C'\,, \tag{13.2.6}$$

where we took into account the formula (3.1.2), which gives the mass centre; hence, the momentum of the discrete mechanical system $\mathcal{S}$ of variable mass, at a given moment, is equal to the momentum of the point which coincides with the mass centre at the respective moment and at which the whole mass of the system would be concentrated.

If the relative velocity of the centre of mass C (with respect to the frame $\mathscr{R}$) is non-zero ($\mathbf{v}_C \neq \mathbf{0}$), then the absolute velocity (with respect to the frame $\mathscr{R}'$) of this centre is given by

$$\mathbf{v}'_C = \overline{\mathbf{v}}'_C + \mathbf{v}_C \tag{13.2.5'}$$

so that

$$\mathbf{H}' = M(\mathbf{v}'_C - \mathbf{v}_C); \tag{13.2.6'}$$

it results that the absolute velocity of the centre of mass C is equal to its relative velocity if the momentum of the mechanical system $\mathscr{S}$ with respect to the fixed frame vanishes ($\mathbf{H}' = \mathbf{0}$).

We assume, to fix the ideas, that the emission phenomenon of some elements of the particles P_j , $j = 1,2,...,\overline{n}$, of relative velocities (with respect to these particles)

$$\mathbf{w}_j = \mathbf{u}_j - \overline{\mathbf{v}}'_j, \quad j = 1,2,...,\overline{n}, \tag{13.2.7}$$

where $\mathbf{u}_j$ are the velocities of these elements with respect to the frame $\mathscr{R}'$. In this case, the equation (10.3.1') allows to write

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_i \overline{\mathbf{v}}'_i) = \mathbf{F}_i + {\sum_{k=1}^{n}}' \mathbf{F}_{ik} + \dot{m}_i \mathbf{u}_i, \quad \dot{m}_i < 0,$$

where $\mathbf{F}_i$ is the resultant of the given external forces which are applied upon the particle P_i , while $\mathbf{F}_{ik}$ are the internal forces due to the action of the particles P_k ; summing for all the particles of the discrete mechanical system $\mathscr{S}$ and observing that the resultant of the internal forces vanishes, we obtain

$$\dot{\mathbf{H}}' = \mathbf{R} + \sum_{j=1}^{\overline{n}} \dot{m}_j \mathbf{u}_j, \quad \dot{m}_j < 0, \tag{13.2.8}$$

stating

Theorem 3.2.1 (*theorem of momentum*). *The derivative with respect to time of the momentum of a free discrete mechanical system of variable mass* (*which emits mass*), *with respect to an inertial frame of reference, at a given moment, is equal to the sum of the resultant of the given external forces which act upon that system and the momentum of the emitted masses in a unity of time, at that moment.*

Taking into account (13.2.7), we can write the relation (13.2.8) in the form

$$\dot{\mathbf{H}}' = \mathbf{R} + \mathbf{R} + \sum_{j=1}^{\overline{n}} \dot{m}_j \overline{\mathbf{v}}'_j, \quad \dot{m}_j < 0, \tag{13.2.8'}$$

too, where

$$\mathsf{R} = \sum_{j=1}^{\bar{n}} \dot{m}_j (\mathbf{u}_j - \mathbf{v}'_j) = \sum_{j=1}^{\bar{n}} \dot{m}_j \mathbf{w}_j\,, \quad \dot{m}_j < 0\,, \tag{13.2.9}$$

is the resultant of the reactive forces; we mention, as well (for the particles which do not emit mass we have $\dot{m}_i = 0$), a theorem of the dynamic resultant in the form

$$\mathbf{A}' = \sum_{i=1}^{n} m_i \bar{\mathbf{a}}'_i = \sum_{i=1}^{n} m_i \dot{\bar{\mathbf{v}}}'_i = \mathbf{R} + \mathsf{R}\,, \tag{13.2.8''}$$

where $\mathbf{A}'$ is the dynamic resultant of the discrete mechanical system $\mathscr{S}$.

If the absolute velocities of the emitted elements vanish ($\mathbf{u}_j = \mathbf{0}$, $j = 1,2,...,\bar{n}$), then the relation (13.2.8) is written in the form

$$\dot{\mathbf{H}}' = \mathbf{R}\,, \tag{13.2.8'''}$$

hence in the same form as in case of a discrete mechanical system of constant mass; as well, if the relative velocities of the emitted elements are zero ($\mathbf{w}_j = \mathbf{0}$, $j = 1,2,...,\bar{n}$), then $\mathsf{R} = \mathbf{0}$, while the relation (13.2.8'') takes the classical form

$$\mathbf{A}' = \mathbf{R}\,. \tag{13.2.8$^{\rm iv}$}$$

These cases have been considered by T. Levi-Civita.

Taking into account (13.2.4'), we notice that

$$\sum_{i=1}^{n} m_i \dot{\bar{\mathbf{v}}}'_i = M[\mathbf{a}'_O + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] = M\bar{\mathbf{a}}'_C\,,$$

so that

$$M\bar{\mathbf{a}}'_C = \mathbf{R} + \mathsf{R}\,; \tag{13.2.10}$$

we thus state

Theorem 13.2.2 (*theorem of motion of the centre of mass*). *The point which coincides with the centre of mass of a free discrete mechanical system of variable mass* (*which emits mass*) *moves, at a given moment, with respect to an inertial frame of reference, as a particle of constant mass, at which would be concentrated the whole mass of the system at that moment and which would be acted upon by the sum of the resultant of the given external forces and the resultant of the reactive forces*.

This equation takes into account that the centre of mass C changes the position with respect to the frame $\mathscr{R}$, due to the variation of the mass of the mechanical system $\mathscr{S}$. As a matter of fact, the acceleration $\bar{\mathbf{a}}'_C$ is the transportation acceleration; taking into account the relative motion, we have

$$\mathbf{a}'_C = \overline{\mathbf{a}}'_C + \mathbf{a}_C + 2\boldsymbol{\omega} \times \mathbf{v}_C , \tag{13.2.11}$$

where $\mathbf{a}_C$ and $2\boldsymbol{\omega} \times \mathbf{v}_C$ are the relative acceleration and the Coriolis acceleration, respectively, corresponding to the mass centre C. We obtain thus the equation

$$M\mathbf{a}'_C = \mathbf{R} + \mathbf{R} + M\mathbf{a}_C + 2M\boldsymbol{\omega} \times \mathbf{v}_C , \tag{13.2.10'}$$

which governs the motion of the mass centre of the discrete mechanical system $\mathscr{S}$ with respect to an inertial frame of reference.

In case of a discrete mechanical system of variable mass, which captures mass, we obtain – analogously – a relation of the form (13.2.8), but for which $\dot{m}_j > 0$, and we can state a theorem of momentum and a theorem of motion of the mass centre in the same form. We may develop a unitary theory too, starting from Meshcherskiĭ's generalized equation, in the form (10.3.8) or in the form (13.2.1). We find thus the theorem of momentum

$$\dot{\mathbf{H}}' = \mathbf{R} + \sum_{j=1}^{\overline{n}} \dot{m}_j^- \mathbf{u}_j^- + \sum_{j=1}^{\overline{n}} \dot{m}_j^+ \mathbf{u}_j^+ , \tag{13.2.12}$$

as well as the theorem of the dynamic resultant

$$\mathbf{A}' = \sum_{i=1}^{n} m_i \dot{\overline{\mathbf{v}}}'_i = \mathbf{R} + \mathbf{R}_- + \mathbf{R}_+ , \tag{13.2.12'}$$

where we have introduced the reactive force

$$\mathbf{R}_- = \sum_{j=1}^{\overline{n}} \dot{m}_j^- (\mathbf{u}_j^- - \overline{\mathbf{v}}'_j) = \sum_{j=1}^{\overline{n}} \dot{m}_j^- \mathbf{w}_j^- \tag{13.2.13}$$

and the braking force

$$\mathbf{R}_+ = \sum_{j=1}^{\overline{n}} \dot{m}_j^+ (\mathbf{u}_j^+ - \overline{\mathbf{v}}'_j) = \sum_{j=1}^{\overline{n}} \dot{m}_j^+ \mathbf{w}_j^+ ; \tag{13.2.13'}$$

the used notations correspond to the previous ones; we suppose that there are $\overline{n} \leq n$ particles which emit and capture mass (eventually, some of those particles can only emit or only capture mass, having $m_j = m_j^0 + m_j^- + m_j^+$, $\dot{m}_j^- < 0$, $\dot{m}_j^+ > 0$, corresponding to the notation (10.3.11)). The theorem of motion of the mass centre becomes

$$M\overline{\mathbf{a}}'_C = \mathbf{R} + \mathbf{R}_- + \mathbf{R}_+ , \tag{13.2.12''}$$

where M is the mass of the discrete mechanical system $\mathscr{S}$ at a given moment.

The moment of momentum of the discrete mechanical system $\mathscr{S}$ of variable mass will be defined in the form

$$\mathbf{K}'_{O'} = \sum_{i=1}^{n} \mathbf{r}'_i \times (m_i \overline{\mathbf{v}}'_i) = \sum_{i=1}^{n} m_i (\mathbf{r}'_O + \mathbf{r}_i) \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r}_i),$$

so that, corresponding to the formula (11.2.16), we obtain

$$\mathbf{K}'_{O'} = \mathbf{K}^O + \mathbf{r}'_O \times (M \overline{\mathbf{v}}'_C) + \boldsymbol{\rho} \times (M \mathbf{v}'_O), \tag{13.2.14}$$

where the pseudomoment of momentum $\mathbf{K}^O$ is given by

$$\mathbf{K}^O = \sum_{i=1}^{n} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{I}_O \boldsymbol{\omega}.$$

If $O \equiv C$, hence if $\boldsymbol{\rho} = \mathbf{0}$, then we obtain a formula of Koenig type of the form (11.2.21). Introducing the frame $\overline{\mathscr{R}}$ with the axes parallel to those of the frame $\mathscr{R}'$, we have $\mathbf{K}^O = \overline{\mathbf{K}}_O$ too.

We can write

$$\mathbf{r}'_i \times \frac{\mathrm{d}}{\mathrm{d}t}(m_i \overline{\mathbf{v}}'_i) = \mathbf{r}'_i \times \mathbf{F}_i + \mathbf{r}'_i \times {\sum_{k=1}^{n}}' \mathbf{F}_{ik} + \mathbf{r}'_i \times (\dot{m}_i \mathbf{u}_i), \quad \dot{m}_i < 0,$$

for a particle P_i, assuming – to fix the ideas – that only a phenomenon of emission takes place. Summing for all the particles of the discrete mechanical system $\mathscr{S}$ and observing that the resultant moment of the inertial forces is equal to zero, it results

$$\dot{\mathbf{K}}'_{O'} = \mathbf{M}_{O'} + \sum_{j=1}^{\overline{n}} \mathbf{r}'_j \times (\dot{m}_j \mathbf{u}_j), \quad \dot{m}_j < 0, \tag{13.2.15}$$

and we can state

Theorem 13.2.3 (*theorem of moment of momentum*). *The derivative with respect to time of the moment of momentum of a free discrete mechanical system of variable mass* (*which emits mass*), *with respect to a fixed pole, in an inertial frame of reference, at a given moment, is equal to the sum of the resultant moment of the given external forces which act upon that system, with respect to the same pole, and the moment of momentum of the emitted masses in a unity of time at that moment, with respect to the mentioned pole.*

Taking into account (13.2.7), we can also write

$$\dot{\mathbf{K}}'_{O'} = \mathbf{M}_{O'} + \mathsf{M}_{O'} + \sum_{j=1}^{\overline{n}} \mathbf{r}'_j \times (\dot{m}_j \overline{\mathbf{v}}'_j), \quad \dot{m}_j < 0, \tag{13.2.15'}$$

where

$$\mathbf{M}_{O'} = \sum_{j=1}^{\overline{n}} \mathbf{r}'_j \times (\dot{m}_j \mathbf{w}_j), \quad \dot{m}_j < 0, \tag{13.2.16}$$

is the resultant moment of the reactive forces with respect to the fixed pole O'. If the absolute velocities of the emitted masses vanish ($\mathbf{u}_j = \mathbf{0}$, $j = 1,2,\ldots,n$), the relation (13.2.15) is written in the form

$$\dot{\mathbf{K}}'_{O'} = \mathbf{M}_{O'}; \tag{13.2.15''}$$

hence, in the case considered by Levi-Civita, one obtains the same form as in the case of a discrete mechanical system of constant mass.

Proceeding as in Sect. 11.2.2.1, we can find for the theorem of moment of momentum an analogue of the formula (11.2.18), in the form (given by C. Agostinelli)

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O + \mathbf{M}_O + \sum_{j=1}^{\overline{n}} \mathbf{r}'_j \times (\dot{m}_j \overline{\mathbf{v}}'_j), \quad \dot{m}_j < 0. \tag{13.2.15'''}$$

In case of capture of mass, we obtain the same formulae (13.2.15–13.2.16), where $\dot{m}_j > 0$. Analogously, we obtain results corresponding to the generalized equation (10.3.8) of Meshcherskiĭ.

We notice that we can group the theorem of momentum and the theorem of moment of momentum, hence the formulae (13.2.8), (13.2.15) in the form of a *theorem of kinetic torsor*; thus, it results

$$\dot{\boldsymbol{\tau}}_{O'}\{\mathbf{H}_i\} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\tau}_{O'}\{\mathbf{H}_i\} = \boldsymbol{\tau}_{O'}\{\mathbf{F}_i\} + \boldsymbol{\tau}_{O'}\{\dot{m}_j \mathbf{u}_j\}. \tag{13.2.17}$$

13.2.1.3 Theorem of Kinetic Energy

The kinetic energy of the discrete mechanical system $\mathscr{S}$ of variable mass is defined in the form

$$T' = \frac{1}{2}\sum_{i=1}^{n} \overline{v}_i'^2 = \frac{1}{2}\sum_{i=1}^{n} m_i (\mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r}_i)^2;$$

corresponding to the formula (11.2.28), we get

$$T' = T^O + \frac{1}{2} M v_O'^2 + M(\mathbf{v}'_O, \boldsymbol{\omega}, \boldsymbol{\rho}), \tag{13.2.18}$$

where the pseudokinetic energy T^O is given by

$$T^O = \frac{1}{2}\sum_{i=1}^{n} m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_O \boldsymbol{\omega}). \tag{13.2.18'}$$

If $O \equiv C$, hence if $\boldsymbol{\rho} = \mathbf{0}$, then we obtain a formula of Koenig type of the form (11.2.37). By introducing the frame $\overline{\mathscr{R}}$, we may write $T^O = \overline{T}$ too.

Starting from the equation of motion of a single particle and effecting a scalar product by $\overline{\mathbf{v}}_i' \mathrm{d}t = \mathrm{d}\mathbf{r}_i'$, we can write

$$\overline{\mathbf{v}}_i' \cdot \mathrm{d}(m_i \overline{\mathbf{v}}_i') = \mathbf{F}_i \cdot \mathrm{d}\mathbf{r}_i' + \sum_{k=1}^{n}{}' \mathbf{F}_{ik} \cdot \mathrm{d}\mathbf{r}_i' + \mathrm{d}m_i \mathbf{u}_i \cdot \mathbf{v}_i', \quad \dot{m}_i < 0,$$

where – to fix the ideas – we have admitted that a phenomenon of emission takes place. We notice that

$$\overline{\mathbf{v}}_i' \cdot \mathrm{d}(m_i \overline{\mathbf{v}}_i') = \mathrm{d}\left(\frac{1}{2} m_i \overline{\mathbf{v}}_i'^2\right) + \frac{1}{2} \overline{\mathbf{v}}_i'^2 \mathrm{d}m_i$$

and

$$\mathrm{d}m_i \mathbf{u}_i \cdot \mathbf{v}_i' = \mathrm{d}m_i (\mathbf{w}_i + \overline{\mathbf{v}}_i') \cdot \overline{\mathbf{v}}_i' = \mathrm{d}m_i v_i'^2 + \mathrm{d}m_i \mathbf{w}_i \cdot \mathbf{v}_i' = \mathrm{d}m_i v_i'^2 + \dot{m}_i \mathbf{w}_i \cdot \mathrm{d}\mathbf{r}_i'.$$

Summing for all the particles of the discrete mechanical system $\mathscr{S}$, it results

$$\mathrm{d}T' = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}} + \mathrm{d}W_{\mathbf{R}} + \frac{1}{2} \sum_{j=1}^{\overline{n}} \mathrm{d}m_j v_j'^2, \quad \dot{m}_j < 0, \tag{13.2.19}$$

where we have introduced *the elementary work of the reactive forces* in the form

$$\mathrm{d}W_{\mathbf{R}} = \sum_{j=1}^{\overline{n}} (\dot{m}_j \mathbf{w}_j) \cdot \mathrm{d}\mathbf{r}_j', \quad \dot{m}_j < 0; \tag{13.2.20}$$

we state thus

Theorem 13.2.4 (*theorem of kinetic energy*). *The differential of the kinetic energy of a free discrete mechanical system of variable mass* (*which emits mass*), *in an inertial frame of reference, at a given moment* t, *is equal to the sum of the elementary work of the given external and internal forces which act upon that system, the work of the reactive forces at the same moment and the kinetic energy of the masses emitted in the interval of time* $\mathrm{d}t$.

The formulae (13.2.19), (13.2.20) hold also in case of capture of mass (we have $\dot{m}_j > 0$); as well, starting from Meshcherskiĭ's generalized equation (10.3.8), we obtain analogous results.

In the particular case considered by Levi-Civita, in which the absolute velocities of the emitted (or captured) masses vanish ($\mathbf{u}_j = \mathbf{0}$, $j = 1, 2, \ldots, \overline{n}$), we can write

$$\mathrm{d}T' = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}} + \frac{1}{2} \mathrm{d}W_{\mathbf{R}} = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}} - \frac{1}{2} \sum_{j=1}^{\overline{n}} \mathrm{d}m_j \overline{\mathbf{v}}_j'^2; \tag{13.2.19'}$$

but if the relative velocities of the emitted masses are null (case considered by Levi-Civita), then we obtain (we have $\mathrm{d}\, W_{\mathbf{R}} = 0$)

$$\mathrm{d}T' = \mathrm{d}W + \mathrm{d}W_{\mathrm{int}} + \frac{1}{2}\sum_{j=1}^{\overline{n}} \mathrm{d}m_j \overline{\mathbf{v}}_j'^2 \,. \tag{13.2.19"}$$

13.2.2 Applications

In what follows, we deal firstly with the problem of the rocket; we consider then the problem of n particles and, in particular, the cases $n = 2$, and $n = 3$, including the problem of motion of an artificial celestial body.

13.2.2.1 The Rocket Problem

In case of a rocket, elements of mass of its body are emitted, hence $\dot{m}_j < 0$, $j = 1, 2, ..., n$; we assume that the relative velocity $\mathbf{w}_j$ is the same (equal to **w**) for all the emitted masses. Among the external forces which act upon the rocket we will render evident the pressures p, of resultant $\mathbf{R}^p$ and resultant moment $\mathbf{M}_O^p$, exerted on the walls by the surrounding air. The theorem of motion of the mass centre and the theorem of moment of momentum are written in the form

$$M[\mathbf{a}_O' + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] = \mathbf{R} + \mathbf{R}^p + \dot{M}\mathbf{w}\,, \quad \dot{M} < 0\,, \tag{13.2.21}$$

$$\begin{aligned} \boldsymbol{\rho} \times (M\mathbf{a}_O') + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O \boldsymbol{\omega}) &= \mathbf{M}_O + \mathbf{M}_O^p + \boldsymbol{\rho}_{\overline{C}} \times (\dot{M}\mathbf{w}) \\ &+ \sum_{j=1}^{\overline{n}} \mathbf{r}_j^{(\overline{C})} \times (\dot{m}_j \mathbf{w}) + \sum_{j=1}^{\overline{n}} \mathbf{r}_j' \times (\dot{m}_j \overline{\mathbf{v}}_j')\,, \quad \dot{m}_j < 0\,, \end{aligned} \tag{13.2.21'}$$

where we took into account that $\mathbf{r}_j = \boldsymbol{\rho} + \mathbf{r}_j^{(\overline{C})}$. The moment of the reactive force with respect to the mass centre $\overline{C}$ of the emitted masses can be neglected, assuming that (the emitted mass is much smaller than the mass of the rocket)

$$\sum_{j=1}^{\overline{n}} \dot{m}_j \mathbf{r}_j^{(\overline{C})} = \dot{M}_j \boldsymbol{\rho}_{\overline{C}}^{(\overline{C})} = \mathbf{0}\,;$$

as well, on the basis of the same considerations, we have

$$\sum_{j=1}^{\overline{n}} \mathbf{r}_j' \times (\dot{m}_j \overline{\mathbf{v}}_j') = \boldsymbol{\rho}_{\overline{C}}' \times (\dot{M}\overline{\mathbf{v}}')\,,$$

where $\overline{\mathbf{v}}'$ is the absolute velocity of the rocket. The equation (13.2.21') becomes thus

$$\boldsymbol{\rho} \times (M\mathbf{a}_O') + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O + \mathbf{M}_O^p + \boldsymbol{\rho}_{\overline{C}} \times (\dot{M}\mathbf{w}) + \boldsymbol{\rho}_{\overline{C}}' \times (\dot{M}\overline{\mathbf{v}}')\,. \tag{13.2.21"}$$

Hence, in the equations of motion of the rocket (the mechanical system $\mathscr{S}$) one introduces the influence of the air pressure and of the reactive force, considered as applied at the mass centre $\overline{C}$ of the emitted masses (Fig. 13.9); in the corresponding equation of the moment of momentum appears also the moment of the force $\dot{M}\overline{\mathbf{v}}'$, with respect to the pole O' of an inertial frame of reference (with respect to which one calculates the acceleration $\mathbf{a}'_O$ too and in which the differentiation with respect to time is performed).

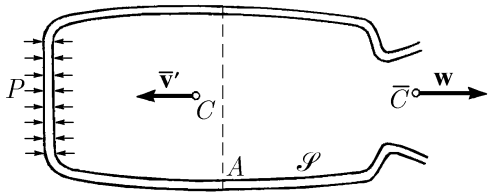

Fig. 13.9 Rocket problem

In this problem, the acceleration $\mathbf{a}'_O = \mathbf{a}'_O(t)$ and the angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ must be determined, supposing that the law of variation of mass $M = M(t)$, deduced from *the law of combustion of fuel*, is known. Due to the emission of mass, the position vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_{\overline{C}}$, as well as the moment of inertia tensor $\mathbf{I}_O$, are functions of time; their variation depends on the mass and on the position of the eliminated elements. Hence, one must assume that the laws of variation of those quantities are given too. In a first approximation, we can consider that $\boldsymbol{\rho}$, $\boldsymbol{\rho}_{\overline{C}}$ and $\mathbf{I}_O$ are constant quantities, at least for a short time interval. We mention that the torsor $\{\mathbf{R}^p, \mathbf{M}_O^p\}$ of *the hydrodynamic forces* corresponds to the pressure of the air, exerted on the external walls of the rocket (a mechanical system $\mathscr{S}$ of variable mass immersed in a fluid), as well as to the pressure of the gases resulting from explosions (on the internal surface of the rocket); thus, there intervene also problems of dynamics of gases, of thermodynamics and even of interactions between the rocket and gases. All these aspects complicate much the problem of the rocket from the mathematical point of view; we assume thus that the action of the pressure is known in time. In this order of ideas, we consider that the rocket can be modelled mathematically as a particle of variable mass.

Let be thus a rocket launched at the Earth surface, at the initial moment, the motion taking place along the local vertical (the Ox-axis is along the ascendent vertical; see Fig. 10.22 too). In the active phase (the rocket moves due to the action of the fuel), the equation (13.2.21) allows to write (the relative velocity $\mathbf{w}$ is directed opposite to the velocity $\mathbf{v}$, hence its component along the Ox-axis is $-\mathbf{w}$; the reactive force $\mathbf{R}$ has the same direction as the velocity $\mathbf{v}$, because $\dot{M} < 0$)

$$M\dot{v} = -\dot{M}w - M_0 g\varphi(v), \tag{13.2.22}$$

where $M_0 g\varphi(v)$ is the resistance of the air (corresponding to the pressure) and where we consider a linear law of variation of mass ($M = M_0(1 - \alpha t)$, $\alpha = \text{const}$).

We assume that $\varphi(v) = (k / g)v^2$, with $k = CA\mu_a / M_0$, where A is the area of the maximal cross section of the rocket, $\mu_a = \mu_a^0 \mathrm{e}^{-\beta x}$, $\mu_a^0 = \mathrm{const}$, $\beta = \mathrm{const}$, is the unit mass of the air, while C is a non-dimensional coefficient of resistance. We notice that $-w = (-u) - v$, where $-u = \mathrm{const}$ is the absolute velocity of gases' elimination; the equation (13.2.22) becomes

$$(1 - \alpha t)\ddot{x} - \alpha\dot{x} + k\mathrm{e}^{-\beta x}\dot{x}^2 = \alpha u\,, \tag{13.2.22'}$$

having a quite complicated non-linear form.

An essential simplification of the problem is obtained by neglecting the resistance of the air; we find thus again Tsiolkovskiĭ's first problem (see Chap. 10, Sect. 3.1.4 too), with the hypothesis $w = u - v \neq \mathrm{const}$. The equation of motion takes the form

$$(1 - \alpha t)\dot{v} = \alpha(v + u), \tag{13.2.23}$$

wherefrom we get

$$v(t) = v_0 + \frac{\alpha u t}{1 - \alpha t}, \tag{13.2.23'}$$

where $v_0 = v(0)$ is the initial velocity. By a new integration, we can write ($x_0 = x(0)$)

$$x(t) = x_0 - (u - v_0)t - \frac{u}{\alpha}\ln(1 - \alpha t)\,. \tag{13.2.23''}$$

Another simplification of the problem can be obtained by adjustment of the law of combustion, so that the rocket have a uniform accelerated motion along the ascendent vertical; the velocity v of the mass centre of the rocket will be thus given by $v^2 = 2a_0 x$, with the acceleration $\dot{v} = a_0 = \mathrm{const}$. In its motion, the rocket must overcome the resistance of the air $M_0 g\varphi(v)$ and the force of attraction of the Earth fMm_E / r^2, $r = R + x$, where m_E is the Earth's mass and we have $fm_E = gR^2$, R being the radius of the Earth, considered to be spherical, while g is the gravity acceleration. In this case, the elementary work corresponding to the displacement along the ascendent vertical is given by

$$\mathrm{d}W = \left[Mg\left(\frac{R}{R + x}\right)^2 + Kx\mathrm{e}^{-\beta x} \right]\mathrm{d}x\,,$$

where $K = 2a_0 A\mu_a^0 C$ is a constant. Integrating between the limits 0 and H, we obtain the work effected by rising the rocket at the height H in the form

$$W = \frac{\bar{M}gRH}{R + H} + \frac{K}{\beta^2}\left[1 - (1 - \beta H)\mathrm{e}^{-\beta H}\right]; \tag{13.2.24}$$

because we do not know the law of motion $x = x(t)$ (we cannot express the mass as a function of x), we will introduce a mean mass $M_0 - \Delta M < \bar{M} < M_0$, where $\Delta M > 0$ is the lost mass of the rocket at the height H. If $\beta = 10^{-6}\,\mathrm{cm}^{-1}$, for $H = 350\mathrm{km} = 3.5 \cdot 10^7\,\mathrm{cm}$, then we have $(1 + \beta H)\mathrm{e}^{-\beta H} = 36\mathrm{e}^{-35} \cong 2.270 \cdot 10^{-14} \ll 1$. Assuming that $a_0 = 10g \cong 9.81 \cdot 10^3\,\mathrm{cm/s^2}$, $A = 1\,\mathrm{m}^2 = 10^4\,\mathrm{cm}^2$, $\mu_a^0 = 10^{-3}\,\mathrm{g/cm^3}$, $C = 6$, it results $K \cong 1.177 \cdot 10^6\,\mathrm{g/s^2}$, while $K / \beta^2 \cong 1.177 \cdot 10^{18}\,\mathrm{g \cdot cm^2/s^2}$; with $g = 9.81 \cdot 10^2\,\mathrm{cm/s^2}$, $R = 6.38 \cdot 10^8\,\mathrm{cm}$ and supposing that $\bar{M} = 5 \cdot 10^6\,\mathrm{g}$, we have $\bar{M}gRH/(R + H) \cong 1.627 \cdot 10^{17}\,\mathrm{g \cdot cm^2/s^2}$. Finally, we obtain $W = 1.340 \cdot 10^{18}\,\mathrm{g \cdot cm^2/s^2} = 1.340 \cdot 10^{18}\,\mathrm{erg} = 1.340 \cdot 10^{11}\,\mathrm{J}$, hence the approximate value of the mechanical energy necessary for the rocket to come out from the terrestrial atmosphere (these data are important to design the motor of the rocket); we notice also that $W \cong K / \beta^2$ (with an error of 12% in the preceding case). The velocity $V = \sqrt{2a_0 H} \cong 8.287 \cdot 10^5\,\mathrm{cm/s} = 8.287\,\mathrm{km/s}$ at the height H, is reached after an interval of time $T = V / a_0 = 84.48$ s; the velocity V is thus greater than the first cosmic velocity at the height H (smaller than at the Earth surface, as it was shown in Chap. 9, Sect. 2.2.2). The duration T is, in fact, greater, the trajectory of the rocket being – in reality – curvilinear; in practice, the active phase is of several minutes.

13.2.2.2 Problem of n Particles

We shall study *the problem of n particles* in the case of the capture phenomenon; we assume thus, for instance, that one has to do with celestial bodies acted upon by internal forces of Newtonian attraction, which are capturing meteorites. We consider to be in the Levi-Civita case (the absolute velocity of the captured masses vanishes), the equation of motion of a particle being of the form (10.3.4).

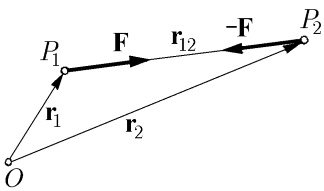

Fig. 13.10 Problem of two particles

In the case $n = 2$ (*the problem of two particles*), studied in 1928, in the frame of the mentioned mathematical model, by Gh. Vrănceanu, let P_1 and P_2 be two particles of position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, with respect to an inertial frame of reference, and of variable masses m_1 and m_2, respectively; we denote $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ (Fig. 13.10), the force of universal attraction (conservative force, which derives from the potential

$U = -fm_1m_2/r$) being, in this case, given by $\mathbf{F} = fm_1m_2\mathbf{r}_{12}/r^3$, $r = |\mathbf{r}_{12}|$, where f is the constant of universal attraction. The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_1\mathbf{v}_1) = f\frac{m_1m_2}{r^3}\mathbf{r}_{12}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(m_2\mathbf{v}_2) = -f\frac{m_1m_2}{r^3}\mathbf{r}_{12}. \tag{13.2.25}$$

Using the theorems of momentum and moment of momentum in the form (13.2.8''') and in the form (13.2.15''), respectively, and observing that the external forces vanish ($\mathbf{R} = \mathbf{0}$, $\mathbf{M}_{O'} = \mathbf{0}$), we can state conservation theorems of momentum and of moment of momentum, respectively; there result the first integrals

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \mathbf{C}_1, \tag{13.2.26}$$

$$\mathbf{r}_1 \times (m_1\mathbf{v}_1) + \mathbf{r}_2 \times (m_2\mathbf{v}_2) = \mathbf{C}_2, \tag{13.2.26'}$$

where $\mathbf{C}_1, \mathbf{C}_2 = \overrightarrow{\mathrm{const}}$. We can write the identity

$$m_1m_2(\mathbf{v}_2 - \mathbf{v}_1) = (m_1 + m_2)m_2\mathbf{v}_2 - m_2(m_1\mathbf{v}_1 + m_2\mathbf{v}_2).$$

Observing that $\mathbf{v}_2 - \mathbf{v}_1 = \dot{\mathbf{r}}_{12}$, taking into account (13.2.26) and eliminating the momentum $m_2\mathbf{v}_2$ between this relation and the second equation (13.2.25), we obtain the vector equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{r}}_{12}) + f\frac{m_1m_2}{r^3}\mathbf{r}_{12} + \mathbf{C}_1\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{m_1}\right) = \mathbf{0}, \tag{13.2.27}$$

where $m = m_1m_2/(m_1 + m_2)$ is the reduced mass; this equation characterizes the motion of the particle P_2 with respect to the particle P_1. Analogously, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{r}}_{21}) + f\frac{m_1m_2}{r^3}\mathbf{r}_{21} + \mathbf{C}_1\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{m_2}\right) = \mathbf{0} \tag{13.2.27'}$$

too, this last equation characterizing the motion of the particle P_1 with respect to the particle P_2. In particular, if the masses m_1 and m_2 are constant in time, we find again the classical equations (8.1.14).

The case $n = 3$ (*the problem of three particles*) has been considered in 1932 in the same conditions, using the above ideas, due to I.I. Plăcințeanu. Let thus be the particles P_i of position vectors $\mathbf{r}_i$, with respect to an inertial frame of reference, and variable masses m_i, $i = 1,2,3$; the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_i\mathbf{v}_i) = f\frac{m_im_j}{r_{ij}^3}\mathbf{r}_{ij} + f\frac{m_im_k}{r_{ik}^3}\mathbf{r}_{ik}, \quad r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|,$$
$$i \neq j \neq k \neq i, \quad i,j,k = 1,2,3. \tag{13.2.28}$$

As in the preceding case, we obtain the first integrals

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + m_3 \mathbf{v}_3 = \mathbf{C}_1 , \tag{13.2.29}$$

$$\mathbf{r}_1 \times (m_1 \mathbf{v}_1) + \mathbf{r}_2 \times (m_2 \mathbf{v}_2) + \mathbf{r}_3 \times (m_3 \mathbf{v}_3) = \mathbf{C}_2 , \tag{13.2.29'}$$

where $\mathbf{C}_1 , \mathbf{C}_2 = \overrightarrow{\mathrm{const}}$. We can write the identities

$$m_1 m_2 (\mathbf{v}_2 - \mathbf{v}_1) = (m_1 + m_2) m_2 \mathbf{v}_2 - m_2 (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2),$$
$$m_1 m_3 (\mathbf{v}_3 - \mathbf{v}_1) = (m_1 + m_3) m_3 \mathbf{v}_3 - m_3 (m_1 \mathbf{v}_1 + m_3 \mathbf{v}_3);$$

taking into account (13.2.28), (13.2.29), introducing the mass $M = m_1 + m_2 + m_3$ of the discrete mechanical system and eliminating the momenta $m_2 \mathbf{v}_2$ and $m_3 \mathbf{v}_3$, we find, as in the case $n = 2$, the equations of motion of the particles P_2 and P_3 with respect to the particle P_1 in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[m_2 \left(1 - \frac{m_2}{M} \right) \dot{\mathbf{r}}_{12} \right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_2 m_3}{M} \dot{\mathbf{r}}_{13} \right) + f \frac{m_1 m_2}{r_{12}^3} \mathbf{r}_{12} \\ &\qquad + f \frac{m_2 m_3}{r_{23}^3} \mathbf{r}_{32} + \mathbf{C}_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_2}{M} \right) = \mathbf{0}, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left[m_3 \left(1 - \frac{m_3}{M} \right) \dot{\mathbf{r}}_{13} \right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_2 m_3}{M} \dot{\mathbf{r}}_{12} \right) + f \frac{m_1 m_3}{r_{13}^3} \mathbf{r}_{13} \\ &\qquad + f \frac{m_2 m_3}{r_{23}^3} \mathbf{r}_{23} + \mathbf{C}_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_3}{M} \right) = \mathbf{0}. \end{aligned} \tag{13.2.30}$$

Summing these two equations, we find (we have $\mathbf{r}_{23} + \mathbf{r}_{32} = \mathbf{0}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_1}{M} (m_2 \dot{\mathbf{r}}_{12} + m_3 \dot{\mathbf{r}}_{13}) \right] + f \frac{m_1 m_2}{r_{12}^3} \mathbf{r}_{12} + f \frac{m_1 m_3}{r_{13}^3} \mathbf{r}_{13} - \mathbf{C}_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_1}{M} \right) = \mathbf{0}; \tag{13.2.30'}$$

this equation, together with one of the equations (13.2.30), constitute a system of two differential equations for the problem of three particles of variable mass. Unlike the equations (13.2.30), the equation (13.2.30') contains only two unknown vector functions ($\mathbf{r}_{12} = \mathbf{r}_{12}(t)$ and $\mathbf{r}_{13} = \mathbf{r}_{13}(t)$); we notice that we can write the latter equation in the form of two equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_1 m_2}{M} \dot{\mathbf{r}}_{12} \right) + f \frac{m_1 m_2}{r_{12}^3} \mathbf{r}_{12} + \mathbf{C}_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_2}{M} \right) = \mathbf{C}(t), \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_1 m_3}{M} \dot{\mathbf{r}}_{13} \right) + f \frac{m_1 m_3}{r_{13}^3} \mathbf{r}_{13} + \mathbf{C}_1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_3}{M} \right) = -\mathbf{C}(t), \end{aligned} \tag{13.2.31}$$

of the form (13.2.27), where the function $\mathbf{C}(t)$ remains to be determined. Subtracting one equation (13.2.30) from the other and taking into account (13.2.31), we get (we notice that $\mathbf{r}_{13} - \mathbf{r}_{12} = \mathbf{r}_{23}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_2 m_3}{M}\dot{\mathbf{r}}_{23}\right) + f\frac{m_2 m_3}{r_{23}^3}\mathbf{r}_{23} = \mathbf{C}(t)\,. \tag{13.2.31'}$$

Thus, the differential equations of the problem will be (13.2.31), (13.2.31'), having a more symmetrical form, by separation of variables.

In the case of an arbitrary number n of particles, we use the equations of motion ($r_{ij} = |\mathbf{r}_{ij}| = |\mathbf{r}_j - \mathbf{r}_j|$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_i \mathbf{v}_i) = f m_i \sum_{j=1}^{n} \frac{m_j}{r_{ij}^3}\mathbf{r}_{ij}\,, \quad j \neq i\,, \quad i = 1,2,\ldots,n\,; \tag{13.2.32}$$

we obtain the first integrals

$$\sum_{i=1}^{n} m_i \mathbf{v}_i = \mathbf{C}_1\,, \tag{13.2.33}$$

$$\sum_{i=1}^{n} \mathbf{r}_i \times (m_i \mathbf{v}_i) = \mathbf{C}_2\,. \tag{13.2.33'}$$

Let us make a change of function and of variable $\mathbf{r}_i(t) = \varphi(\tau)\boldsymbol{\rho}_i(\tau)$, $\mathrm{d}\tau = \psi(t)\mathrm{d}t$ and introduce the notations $\varphi' = \mathrm{d}\varphi/\mathrm{d}\tau$, $\boldsymbol{\rho}_i' = \mathrm{d}\boldsymbol{\rho}_i/\mathrm{d}\tau$, $i = 1,2,\ldots,n$; we can calculate

$$\mathbf{v}_i = \psi(\varphi\boldsymbol{\rho}_i' + \varphi'\boldsymbol{\rho}_i)\,,$$
$$\dot{\mathbf{v}}_i = \psi^2\varphi\boldsymbol{\rho}_i'' + (\dot{\psi}\varphi + 2\psi^2\varphi')\boldsymbol{\rho}_i' + (\dot{\psi}\varphi' + \psi^2\varphi'')\boldsymbol{\rho}_i\,.$$

The equations of motion (13.2.32) become ($\boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_j - \boldsymbol{\rho}_i$)

$$m_i\psi^2\varphi\boldsymbol{\rho}_i'' + [m_i(\dot{\psi}\varphi + 2\psi^2\varphi') + \dot{m}_i\psi\varphi]\boldsymbol{\rho}_i' + [m_i(\psi^2\varphi'' + \dot{\psi}\varphi') + \dot{m}_i\psi\varphi']\boldsymbol{\rho}_i\,,$$
$$= \frac{1}{\rho^2}\sum_{j=1}^{n}\frac{m_i m_j}{\rho_{ij}^3}\boldsymbol{\rho}_{ij}\,, \quad i \neq j\,, \quad i = 1,2,\ldots,n\,;$$

we impose the conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi m_i)\varphi + 2\varphi^2 m_i\varphi' = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\psi m_i)\varphi' + \varphi^2 m_i\varphi'' = 0\,,$$

which can be written also in the form

$$\frac{1}{\varphi}\frac{\mathrm{d}}{\mathrm{d}t}\ln(\psi m_i) + 2\frac{\varphi'}{\varphi} = 0\,, \quad \frac{1}{\psi}\frac{\mathrm{d}}{\mathrm{d}t}\ln(\psi m_i) + \frac{\varphi''}{\varphi'} = 0\,.$$

We notice that we must have $2\varphi'/\varphi = \varphi''/\varphi' = \mathrm{const}$, which can take place only if $\varphi = k$, $k = \mathrm{const}$, $\varphi' = \varphi'' = 0$; in this case, it results $\psi m_i = \mu_i$, $\mu_i = \mathrm{const}$, and we assume that

$$m_i(t) = m_i^0 f(t), \quad f(t) = \frac{1}{\psi(t)}, \quad m_i^0 = \text{const}, \quad i = 1,2,\ldots,n. \tag{13.2.34}$$

The equations of motion (13.2.32) become ($\boldsymbol{\rho}_i = \boldsymbol{\rho}_i(\tau), \boldsymbol{\rho}_{ij} = \boldsymbol{\rho}_{ij}(\tau)$)

$$m_i^0 \boldsymbol{\rho}_i'' = \frac{f^3(t)}{k^3} \sum_{j=1}^{n} \frac{m_i^0 m_j^0}{\rho_{ij}^3} \boldsymbol{\rho}_{ij}, \quad j \neq i, \quad i = 1,2,\ldots,n, \tag{13.2.32'}$$

obtaining the form of classical equations, corresponding to constant masses. It results

$$\sum_{i=1}^{n} m_i^0 \boldsymbol{\rho}_i'' = \mathbf{0},$$

wherefrom

$$\sum_{i=1}^{n} m_i^0 \boldsymbol{\rho}_i = \frac{1}{kf(t)} \sum_{i=1}^{n} m_i \mathbf{r}_i = \frac{1}{kf(t)} M \boldsymbol{\rho}(t) = \frac{\mathbf{C}_1}{k} \tau + \mathbf{C},$$

so that ($\boldsymbol{\rho}$ is the position vector of the centre C)

$$\boldsymbol{\rho} = \frac{\mathbf{C}_1 \tau + \mathbf{C} k}{M_0}, \quad \frac{1}{f(t)} M = M_0 = \sum_{i=1}^{n} m_i^0, \tag{13.2.35}$$

the centre of mass having a rectilinear trajectory; we have

$$\mathbf{v}_C = \frac{1}{f(t)} \boldsymbol{\rho}' = \frac{1}{M_0} \mathbf{C}_1 \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{M_0 f(t)} \mathbf{C}_1 = \frac{\mathbf{C}_1}{M(t)}, \tag{13.2.35'}$$

the velocity $\mathbf{v}_C = \mathbf{v}_C(t)$ being constant only if $f(t) = \text{const}$, and $\mathbf{C}_1$ being the constant of the first integral of the momentum (13.2.33). Hence, we can state that, in case of a discrete mechanical system $\mathscr{S}$ of masses having the same variation in time ($m_i(t) = m_i^0 f(t)$, $m_i^0 = \text{const}$, $i = 1,2,\ldots,n$), *the centre of mass C has a rectilinear and non-uniform motion*; the motion of the centre C is uniform only in case of constant masses.

13.3.2.3 Motion of an Artificial Celestial Body

Let be, for instance, an artificial celestial body B, which is launched from the Earth E and which moves away from our planet; this body can enter in the zone of attraction of another body of the solar system, e.g., in the attraction zone of the Sun S. We have thus a problem of three particles (Earth E, Sun S and artificial celestial body B); if the mass m of the body B can be neglected with respect to the mass m_E of the Earth

($m \ll m_E$) and to the mass m_S of the Sun ($m \ll m_S$), then we can assume some approximations of computation.

To fix the ideas, we will consider the motion of the body B on the straight line which joins the centre of mass E of the Earth to the centre of mass S of the Sun, assuming that the two celestial bodies are fixed (Fig. 13.11); the equation of motion is of the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{BE} + F_{BS} = -f\frac{mm_E}{x^2} + f\frac{mm_S}{(s-x)^2}, \tag{13.2.36}$$

where s is the distance from the Earth to the Sun, while x is the abscissa of the centre of mass of the body B with respect to the Earth E, chosen as origin. Observing that $\mathrm{d}v/\mathrm{d}t = v\mathrm{d}v/\mathrm{d}x$ and integrating, we obtain the first integral of the mechanical energy

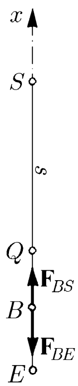

Fig. 13.11 Motion of an artificial celestial body

$$v^2 = 2f\left(\frac{m_E}{x} + \frac{m_S}{s-x}\right) + h, \quad h = v^2 - 2f\left(\frac{m_E}{x_0} + \frac{m_S}{s-x_0}\right), \tag{13.2.36'}$$

where $x_0 = x(0)$, $v_0 = v(0)$, corresponding to the initial moment $t = 0$. From (13.2.36) one observes that for x sufficiently small we have $\dot{v} < 0$, hence the velocity decreases till the body B reaches a point Q, the abscissa of which is given by $\left[x_Q/(s-x_Q)\right]^2 = m_E/m_S$, hence by

$$x_Q = \frac{s}{\sqrt{\frac{m_S}{m_E}}+1} = \frac{s\varkappa}{1+\varkappa} \cong s\varkappa, \quad \varkappa^2 = \frac{m_E}{m_S}, \tag{13.2.37}$$

taking into account that $\varkappa^2 \cong 3\cdot 10^{-6}$, hence $\varkappa \cong 1.732\cdot 10^{-3}$; because $s = 2.348\cdot 10^4 R$, where $R = 6.38\cdot 10^8\ \mathrm{cm}$ is the radius of the Earth, we obtain $x_Q \cong 1.729\cdot 10^{-3}\,\mathrm{s} \cong 40.579R \cong 2.590\cdot 10^{10}\ \mathrm{cm} = 2.59\cdot 10^5\ \mathrm{km}$.

If the initial velocity v_0 is too small, the velocity of the body B can vanish before reaching the point Q; in this case, the velocity changes of sign, so that the body returns on the Earth. Observing that the second cosmic velocity is given by $v_{II}^2 = 2fm_E / x_0 \cong 1.249700 \cdot 10^8 \, \mathrm{m}^2 / \mathrm{s}^2$, we can write the relation (13.2.36') also in the form

$$v^2 = v_0^2 + x_0 v_{II}^2 \left[\frac{1}{x} - \frac{1}{x_0} + \frac{1}{\varkappa^2} \left(\frac{1}{s - x} - \frac{1}{s - x_0} \right) \right]; \tag{13.2.36''}$$

for numerical data, it results the approximate formula

$$v_Q^2 = v_0^2 - 0.951 v_{II}^2 = v_0^2 - V^2 , \tag{13.2.38}$$

where $V^2 = 0.951 v_{II}^2$, hence $V \cong 0.975 v_{II} \cong 10.902 \mathrm{km/s}$. If $v_0 < V$, then the body B does not reach Q, returning on the Earth, while if $v_0 > V$, then the body B reaches Q with a non-zero velocity $v_Q > 0$ and passes through this position, continuing its way towards the Sun with a monotone increasing velocity.

If, in particular, $v_0 = V$, then the body B reaches Q with a null velocity ($v_Q = 0$); the point Q represents thus a position of equilibrium, namely a labile position of equilibrium, because an arbitrary perturbation of the position of equilibrium (towards E or towards S), moves away the body B from this position. The moment at which the body B reaches Q is given by

$$t_Q = \int_{x_0}^{x_Q} \frac{\mathrm{d}x}{v(x)}; \tag{13.2.39}$$

observing that $x = x_0$ is a double root of $v(x) = 0$, we can write

$$t_Q = \int_{x_0}^{x_Q} \frac{f(x)}{(x - x_Q)^2} \mathrm{d}x \tag{13.2.39'}$$

too, where $f(x)$ is a regular function in the neighbourhood of the point $x = x_0$. The integral (13.2.39') is improper, so that the time t_Q is infinite; the body B comes near to the position Q, but never reaches it.

Obviously, in the above modelling, the hypothesis of rectilinearity represents an approximation; as well, we have assumed that the masses are constant. However, the results thus obtained are useful from a qualitative point of view.

13.2.3 Continuous Mechanical Systems

One meets, frequently, interesting problems where one deals with continuous mechanical systems of variable mass, e.g.: the problem of a captive balloon, the problem of a glacier or of an iceberg the mass of which diminishes by melting, the problem of an airplane which flies during snowfall (the mass of the snow-flakes is added to the mass of the airplane) etc.; more difficult are the problems in which one must take into account the deformability of the mechanical system. In what follows, we consider two such problems: Cayley's problem, which has merely a historical interest, and the problem of the winch, which has a particular practical interest, using various approximations of calculation.

13.2.3.1 Cayley's Problem

In 1857, Cayley considered the problem of a heavy homogeneous chain, wrapped up on the cylinder C, at rest on a horizontal table; we assume that the chain falls along the vertical, due to its own weight. Let be x the abscissa of the movable end P of the chain, the origin being taken at the level of the table, while the Ox-axis is directed towards the descendent vertical (Fig. 13.12); the equation of motion of this point is $x = x(t)$ and, approximating the whole chain by a particle of variable mass, we can use the theorem of momentum in the form (13.2.8''') (the Levi-Civita case). We write thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\gamma}{g}x\right)\dot{x}\right] = \gamma x\,, \tag{13.2.40}$$

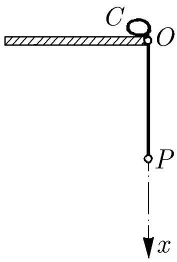

Fig. 13.12 Cayley's problem

where γ is the unit weight, while $\dot{x} = \dot{x}(t)$ is the velocity of the chain (the same for all its elements). Observing that $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$, we can write $x\dot{x}\mathrm{d}(x\dot{x})/\mathrm{d}x = gx^2$, wherefrom, by integration,

$$\frac{1}{2}(x\dot{x})^2 = \frac{g}{3}x^3\,,$$

the integration constant vanishing (we assume that the chain begins to wrap up from the state of rest, so that $x(0) = 0$, $\dot{x}(0) = 0$).

We deduce ($x \neq 0$)

$$\frac{\mathrm{d}x}{\sqrt{x}} = \sqrt{\frac{2g}{3}}\mathrm{d}t\,,$$

so that $\sqrt{x} = \sqrt{2g/3}\;t/2$; hence,

$$x(t) = \frac{g}{6}t^2\,,\quad v(t) = \frac{g}{3}t\,,\quad a(t) = \frac{g}{3}\,, \tag{13.2.40'}$$

the motion of the elements of the chain being uniformly accelerated.

13.2.3.2 The Winch

The winch is a simple machine formed by a homogeneous cylinder of radius R, on which is wrapped up an inextensible and non-torsionable homogeneous cable, of own weight γ; we assume that the winch rotates with an angular velocity ω around a horizontal axle passing through O, at the end of the cable being tied a weight $\mathbf{G} = m\mathbf{g}$, modelled as a particle, the equation of motion of which is $x = x(t)$ (the Ox-axis is along the descendent vertical; Fig. 13.13). The moment of momentum with respect to the fixed pole O is given at the moment t by (the phenomenon being unidimensional, we consider only the non-zero component of the moment of momentum)

$$K_O = I_O\omega + mRv + K'\,,$$

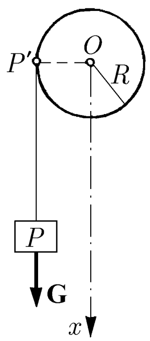

Fig. 13.13 Winch

where $I_O = I_O(t)$ is the moment of inertia of the cylinder with respect to its axis at the moment t, $R(mv)$ is the moment of momentum of the weight $\mathbf{G}$, while K' is the moment of momentum of the unwrapped cable (the cable $P'P$), given by

$$K' = \frac{\gamma}{g}R\int_0^x v\mathrm{d}\xi = \frac{\gamma}{g}Rvx\,.$$

We notice that

$$I_O(t) = I_O^0(t) - \left(\frac{\gamma}{g}x\right)R^2 \,,$$

where I_O^0 is the moment of inertia of the cylinder on which is wrapped the cable (at the initial moment $t = 0$), while $(\gamma x / g)R^2$ is the moment of inertia of the cable which was unwrapped; assuming that the unwrapping of the cable is without sliding friction, we have $v = R\omega$, so that it results

$$K_O = \left(I_O^0 + mR^2\right)\omega \,, \tag{13.2.41}$$

as the whole cable would be wrapped on the cylinder.

Applying the theorem of moment of momentum in the form (13.2.15") (case considered by Levi-Civita), we can write

$$\left(I_O^0 + mR^2\right)\dot{\omega} = (\gamma x)R + (mg)R + M + \overline{M} \,, \tag{13.2.42}$$

where M is the moment of the driving couple, while $\overline{M}$ is the moment of the friction couple of the winch with the axle about which it rotates. Supposing that

$$\overline{M} = -\alpha \operatorname{sign}\omega - 2\beta\omega \,, \quad \alpha, \beta > 0 \,, \tag{13.2.43}$$

we put into evidence the Coulombian friction (case $\beta = 0$), as well as the hydrodynamic friction (case $\alpha = 0$, where a lubricant is used). The differential equation of the motion (13.2.42) takes the form ($\omega = v / R = \dot{x} / R$, $\dot{\omega} = \dot{v} / R = \ddot{x} / R$)

$$\ddot{x} + 2a\dot{x} - bx = c \,, \quad a, b, c = \text{const} \,, \quad a, b > 0 \,, \tag{13.2.44}$$

with the notations

$$a = \frac{\beta}{I_O^0 + mR^2} \,, \quad b = \frac{\gamma R^2}{I_O^0 + mR^2} \,, \quad c = \frac{R(mgR + M - \alpha \operatorname{sign}\omega)}{I_O^0 + mR^2} \,. \tag{13.2.44'}$$

We obtain thus (the winch begins to move at the initial moment with the weight G in the upper position, hence with $x(0) = 0$, $\dot{x}(0) = 0$)

$$x(t) = \frac{c}{b}\left[\frac{1}{a'}\mathrm{e}^{-at}\left(a' \cosh a't + a \sinh a't\right) - 1\right], \quad a' = \sqrt{a^2 + b} \,, \tag{13.2.45}$$

as well as

$$\dot{x}(t) = \frac{c}{a'}\mathrm{e}^{-at} \sinh a't \,, \tag{13.2.45'}$$

$$\ddot{x}(t) = \frac{c}{a'} \mathrm{e}^{-at} \left(a' \cosh a't - a \sinh a't \right), \tag{13.2.45''}$$

all these quantities being positive for $t > 0$. The motion of the winch is thus completely determined.

Chapter 14

Dynamics of the Rigid Solid

If for any input (for any system of external forces) or for any interactions (system of internal forces) all the pairs of points (particles) of a mechanical system remain at a mutual invariant distance in time, then we have to do with a *non-deformable mechanical system*; in case of a continuous non-deformable medium we use the denomination of *rigid solid*. This notion represents an idealization of physical reality, because any body subjected to the action of forces is deformed; but often these deformations are very small with respect to the dimensions of the body. Thus, the rigid solid (body) represents a mathematical model of the bodies the deformation of which can be neglected in a first approximation. Sometimes, in case of special problems concerning the rigid solids, there can appear contradictions and it is necessary to complete the mathematical model, assuming that the mechanical system is no more rigid (partially or in its totality); this happens, for instance, in the calculation of the constraint forces which appear in case of hyperstatic mechanical systems, in some cases of friction or in other problems concerning the collision of bodies. The problems put in mechanics of rigid solids form what is called stereomechanics.

We present firstly the problems concerning the free or constrained rigid solid; starting from these results, we consider then the motion of the rigid solid about a fixed axis, as well as the plane-parallel motion of it.

14.1 Motion of a Free or Constrained Rigid Solid

In what follows, we will give results and will establish general theorems for the motion of the free or constrained rigid solid; the results thus obtained will be applied to various particular cases.

14.1.1 Motion of the Free Rigid Solid

After some preliminary considerations concerning the representation of the rigid displacement of a solid, the geometric and mechanical quantities which appear in case of a free rigid solid are specified; the corresponding general theorems are then stated, in these conditions.

14.1.1.1 Finite Rototranslations

We have seen that a rigid solid $\mathscr{S}$ is characterized by the relation

P.P. Teodorescu, *Mechanical Systems, Classical Models*,

$$\left|\overrightarrow{P_i P_j}\right| = \left|\mathbf{r}'_j - \mathbf{r}'_i\right| = \text{const}\,, \tag{14.1.1}$$

where $\mathbf{r}'_i$ and $\mathbf{r}'_j$ are the position vectors of two arbitrary points, P_i and P_j, respectively, with respect to the inertial frame $\mathcal{R}'$. We have shown in Chap. 3, Sect. 2.2.3 that a free rigid solid has six degrees of freedom, so that its position with respect to a fixed frame of reference can be represented by six parameters, which can be, e.g.: the co-ordinates x'_{O1}, x'_{O2}, x'_{O3} of a point O of the rigid solid and *Euler's angles* (*the angle of precession* ψ, $0 \le \psi < 2\pi$, *the angle of nutation* θ, $0 \le \theta \le \pi$ and *the angle of proper rotation* φ, $0 \le \varphi < 2\pi$), which specify the rotation of the rigid body with respect to the point O (the orientation of a non-inertial frame $\mathcal{R}$ rigidly linked to the solid and with the pole at O, with respect to a non-inertial frame $\overline{\mathcal{R}}$, with the pole at O and with the axes parallel to the axes of an inertial frame $\mathcal{R}'$) (Fig. 14.1).

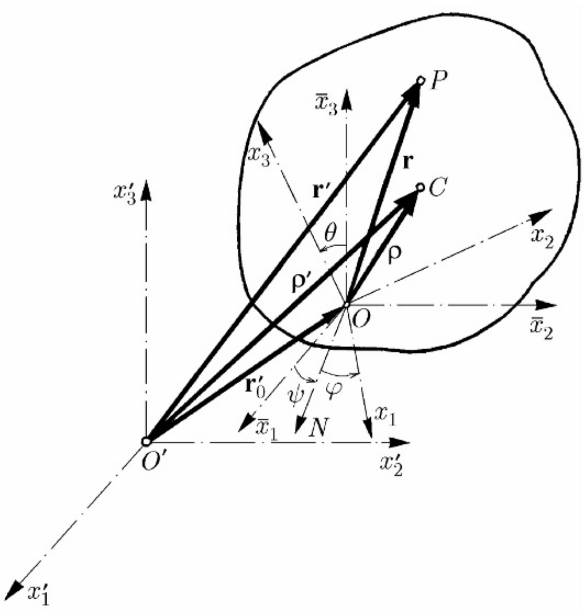

Fig. 14.1 The rigid solid in an inertial frame of reference $\mathcal{R}'$ and in non-inertial ones $\mathcal{R}$ and $\overline{\mathcal{R}}$

Let be a square matrix $\mathbf{M}$ with complex elements. The matrix $\mathbf{M}^+ = \overline{\mathbf{M}}^{\mathrm{T}}$, where $\mathbf{M}^{\mathrm{T}}$ is *the transpose* of the matrix $\mathbf{M}$ (obtained by replacing the lines by the columns), while $\overline{\mathbf{M}}$ is *the conjugate matrix* of the matrix $\mathbf{M}$ (obtained by replacing its elements by the corresponding complex conjugate elements), is called *the adjoint matrix* of the matrix $\mathbf{M}$. If $\mathbf{M} = \overline{\mathbf{M}}$, then the matrix $\mathbf{M}$ is *real* (all its elements are real), while if $\mathbf{M} = -\overline{\mathbf{M}}$, then the matrix $\mathbf{M}$ is *purely imaginary*. A square matrix $\mathbf{S}$ is called *symmetric* or *antisymmetric* (*skew-symmetric*) as we have $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ or $\mathbf{S} = -\mathbf{S}^{\mathrm{T}}$, respectively. A square matrix $\mathbf{H}$ is called *Hermitian* (*self-adjoint*) or *antiHermitian* if $\mathbf{H} = \mathbf{H}^{+}$ or $\mathbf{H} = -\mathbf{H}^{+}$, respectively; we notice that a real and symmetric matrix is Hermitian. If a square matrix $\mathbf{O}$ satisfies the relation $\mathbf{O}^{\mathrm{T}} = \mathbf{O}^{-1}$, where $\mathbf{O}^{-1}$ is *the inverse* of the matrix $\mathbf{O}$ ($\mathbf{O}\mathbf{O}^{-1} = \mathbf{O}^{-1}\mathbf{O} = \mathbf{E}$, $\mathbf{E}$ being *the unit matrix*), that is $\mathbf{O}\mathbf{O}^{\mathrm{T}} = \mathbf{O}^{\mathrm{T}}\mathbf{O} = \mathbf{E}$, then $\mathbf{O}$ is called *orthogonal complex matrix*, while if a square matrix $\mathbf{U}$ satisfies the relation $\mathbf{U}^{+} = \mathbf{U}^{-1}$ ($\mathbf{U}\mathbf{U}^{+} = \mathbf{U}^{+}\mathbf{U} = \mathbf{E}$), then it is called *unitary matrix*. If $\mathbf{R}^{+} = \mathbf{R}^{-1}$ and $\mathbf{R} = \overline{\mathbf{R}}$ (the square matrix $\mathbf{R}$ is unitary and real), then $\mathbf{R}^{\mathrm{T}} = \overline{\mathbf{R}}^{\mathrm{T}} = \mathbf{R}^{+} = \mathbf{R}^{-}$, that is $\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$; in this case, the matrix $\mathbf{R}$ is called *orthogonal real matrix* (or only *orthogonal*). The sum of the elements of the principal diagonal of a square matrix $\mathbf{M}$ represents *the trace of the matrix* (denoted by $\mathbf{tr\,M}$), being an invariant to a linear transformation of the matrix elements.

The matrix $\boldsymbol{\alpha} = \boldsymbol{\Phi\Theta\Psi}$ specified by (3.2.11''') allows to pass from the frame $\overline{\mathscr{R}}$ (or from the frame $\mathscr{R}'$) to the frame $\mathscr{R}$ by the transformation relation (3.2.11'') of the form $\mathbf{i} = \boldsymbol{\alpha}\mathbf{i}'$; in other words, the transformation matrix $\boldsymbol{\alpha}$ may be conceived as an operator which, acting on the frame $\overline{\mathscr{R}}$, transforms that one in the frame $\mathscr{R}$. If the matrix $\boldsymbol{\alpha}$ operates on the components of a vector $\bar{\mathbf{r}}$ in the frame $\overline{\mathscr{R}}$, then we obtain the components of the vector $\mathbf{r} = \bar{\mathbf{r}}$ in the frame $\mathscr{R}$ (the vector does not change); we can, as well, consider the relation $\mathbf{r}^{*} = \boldsymbol{\alpha}\mathbf{r}$, which transforms a vector $\mathbf{r}$ in a vector $\mathbf{r}^{*}$ in the same frame $\mathscr{R}$. In the first case, the matrix $\boldsymbol{\alpha}$ corresponds to a counterclockwise rotation, while in the second case it corresponds to a clockwise one. The matrix $\boldsymbol{\alpha}$ is an orthogonal one, the trace of which does not vanish, in general.

Because we can determine, at any moment, the position of the rigid solid by the position of the frame $\mathscr{R}$ with respect to the frame $\overline{\mathscr{R}}$, hence by the parameters which specify this position, the transformation matrix will be of the form $\boldsymbol{\alpha} = \boldsymbol{\alpha}(t)$; if at the initial moment $t = t_0$ we have $\mathscr{R} \equiv \overline{\mathscr{R}}$, then it results $\boldsymbol{\alpha}(t_0) = \mathbf{E}$, coinciding with the unit matrix. The motion being continuous, the matrix $\boldsymbol{\alpha}(t)$ must be a continuous function of time and we can state that it is obtained by continuity from the identical transformation. Taking into account the rigidity condition (14.1.1), it results that the matrix $\boldsymbol{\alpha}$ is orthogonal.

We will assume now that the pole O, common to the frames $\overline{\mathscr{R}}$ and $\mathscr{R}$ is fixed. If the motion of the frame $\mathscr{R}$ about O is a motion of rotation, then there exists a direction which corresponds to the axis of rotation and which is not affected by the operator $\boldsymbol{\alpha}$, a vector along this direction having the same components in the two frames. To put in evidence the existence of such a direction we will show that there exists a vector $\mathbf{r}$ which has the property $\mathbf{r} = \boldsymbol{\alpha}\mathbf{r}$. On the other hand, the equation $\boldsymbol{\alpha}\mathbf{r} = \lambda\mathbf{r}$, λ scalar, has a solution for the eigenvalues λ of the matrix $\boldsymbol{\alpha}$; we will try to show that between these eigenvalues is also the eigenvalue $\lambda = 1$. The equation $(\boldsymbol{\alpha} - \lambda\mathbf{E}) = \mathbf{0}$ leads to the characteristic equation $\det[\boldsymbol{\alpha} - \lambda\mathbf{E}] = 0$, which gives the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (see

Chap. 3, Sect. 1.2.3 too). Due to the orthogonality of the matrix $\boldsymbol{\alpha}$, the modulus of a vector $\mathbf{r}$ remains invariant by the mentioned transformation. The characteristic equation (an algebraic equation of third degree with real coefficients) has at least a real root and can have also complex solutions, the corresponding eigenvectors being, in this case, complex (they do not exist in the real physical space). The modulus of such a vector is, in the general case, $|\mathbf{r}|^2 = \mathbf{r} \cdot \overline{\mathbf{r}}$, where we have put in evidence the conjugate eigenvector $\overline{\mathbf{r}}$; by transformation, one obtains $(\lambda \mathbf{r}) \cdot (\overline{\lambda} \overline{\mathbf{r}})$, so that we must have $\lambda \overline{\lambda} = 1$ (if λ is an eigenvalue, then $\overline{\lambda}$ is an eigenvalue too). The real root can be only $\lambda = \pm \mathbf{1}$. We notice that $\det \boldsymbol{\alpha} = \lambda_1 \lambda_2 \lambda_3$ and can be equal to $\pm \mathbf{1}$. Because one cannot pass by a jump (the motion is continuous) from $\det \boldsymbol{\alpha}(t_0) = \det \mathbf{E} = 1$ (corresponding to a *proper rotation*) to $\det \boldsymbol{\alpha} = -1$ (corresponding to an *improper rotation*), it results that one can have only $\det \boldsymbol{\alpha}(t) = 1$. A transformation matrix of the form

$$\tilde{\boldsymbol{\alpha}} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\mathbf{E}, \quad \det \tilde{\boldsymbol{\alpha}} = -1,$$

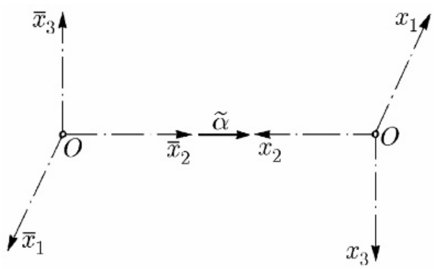

Fig. 14.2 Impossibility to transform a right-handed frame of reference in a left-handed one

would correspond to an *inversion* (*reflection*) of the axes of the frame $\overline{\mathscr{R}}$, but there does not exist any *rigid motion* which could transform a right-handed frame of reference into a left-handed one or conversely (to do this, one must pass by a four-dimensional space) (Fig. 14.2); hence, an inversion can never correspond to a real displacement of a rigid solid. We obtain the same conclusion for any transformation matrix for which $\det \boldsymbol{\alpha} = -1$, including thus the inversion operation, because such a matrix can be written in the form $\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} \boldsymbol{\alpha}_1$, $\det \boldsymbol{\alpha}_1 = 1$. Obviously, on this way we obtain the same conclusion as above. The three eigenvalues of the matrix cannot be distinct, if they are real, because $\lambda_j = \pm 1$, $j = 1,2,3$. If two of the eigenvalues are equal, then they cannot be real and equal to $-\mathbf{1}$ (the third of the eigenvalues must be equal to 1, to can have $\lambda_1 \lambda_2 \lambda_3 = 1$); one obtains the same conclusion if two of the eigenvalues are complex conjugate, because $\lambda \overline{\lambda} = 1$. The trivial case in which all three roots are real ($\lambda_1 = \lambda_2 = \lambda_3 = 1$) corresponds to *the identical transformation*. Hence, one can state that, excluding the above mentioned trivial case, to any rigid motion corresponds only a single eigenvalue equal to 1. For the general displacement of the rigid solid by which that one passes from a position to another one, with respect to a given frame of reference, we can state

Theorem 14.1.1 (*L. Euler*) *The general displacement of a rigid solid with a fixed point is a finite rotation about an axis which passes through this point and is uniquely determined.*

We can always transform the matrix $\boldsymbol{\alpha}$ so as to obtain a new matrix $\boldsymbol{\alpha}^*$ leading to a frame $\mathscr{R}$ ($\mathbf{i} = \boldsymbol{\alpha}^* \mathbf{i}'$), with the axis Ox_3 along the rotation axis; in this case

$$\boldsymbol{\alpha}^* = \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{14.1.2}$$

where χ is the rotation angle. We notice that $\operatorname{tr}\boldsymbol{\alpha}^* = 1 + 2\cos\chi$; knowing that the trace of the matrix is invariant with respect to the transformation thus effected, we have also $\operatorname{tr}\boldsymbol{\alpha} = 1 + 2\cos\chi$. In case of the matrix $\boldsymbol{\alpha}$ of the form (3.2.11'''), we can express the angle χ as a function of the Euler angles by the relation

$$\cos\chi = \cos^2\frac{\theta}{2}\cos(\psi + \varphi) - \sin^2\frac{\theta}{2}. \tag{14.1.3}$$

If we suppress the link imposed to the rigid solid (the fixed point) and if we introduce the three degrees of freedom corresponding to the translation of the origin O of the frames $\overline{\mathscr{R}}$ and $\mathscr{R}$, then we can state

Theorem 14.1.2 (*Chasles*) *The general displacement of a free rigid solid is a finite rototranslation.*

14.1.1.2 Eulerian Parameters. Quaternions. Stereographic Parameters

Besides the representation of the rotation of the rigid solid by means of *Euler's angles*, one can imagine other representations too, useful in various cases. Thus, the finite rotation of angle χ about an axis of unit vector $\mathbf{u}$, which passes through the pole $\boldsymbol{O}$, can be characterized by the set $\{\mathbf{u}, \chi + 2n\pi\} = \{-\mathbf{u}, -\chi + 2n\pi\}$, $n \in \mathbb{Z}$; but this representation is multiform. We can reduce this multiplicity by introducing a vector $\mathbf{V}$, of components λ, μ, ν, and a scalar ρ in the form

$$\mathbf{V} = \mathbf{u}\sin\frac{\chi}{2}, \quad \rho = \cos\frac{\chi}{2}, \quad \lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1. \tag{14.1.4}$$

We see easily that the parameters λ, μ, ν, ρ which satisfy this relation determine a unique rotation; but to a given rotation correspond the parameters $-\lambda$, $-\mu$, $-\nu$, $-\rho$ too, hence two sets of parameters. Hence, the parameters λ, μ, ν, ρ, called *Eulerian parameters*, can describe – analogously – the rotation of the rigid solid; if we consider these parameters as Cartesian co-ordinates of a point in the E_4-space, then we can state: (i) any point of the hypersphere (14.1.4) can specify the actual (final) configuration (position) of the rigid solid; (ii) any actual configuration of the rigid solid determines two diametrical opposite points on the hypersphere; (iii) there exists a one-to-one correspondence between the actual configuration of the rigid solid and the straight lines passing through the origin of the space E_4.

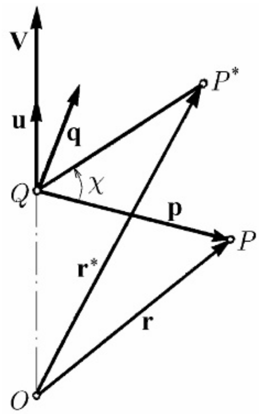

Fig. 14.3 Eulerian parameters

Let be $P(\mathbf{r})$ and $P^*(\mathbf{r})$ the initial and the actual positions of a point of the rigid solid, respectively, which are rotated by an angle χ about an axis which passes through the pole O and is specified by the unit vector **u** (or by the vector **V**) and let be Q their common projection on this axis; at the point Q we consider a right-handed orthogonal frame, determined by the vectors $\overrightarrow{QP} = \mathbf{p}$, $\mathbf{q}$, $|\mathbf{q}| = 1$, and **V** (Fig. 14.3). We can write

$$\mathbf{r}^* = \overrightarrow{OQ} + \overrightarrow{QP^*} = \overrightarrow{OQ} + \mathbf{p}\cos\chi + \mathbf{q}\sin\chi\,;$$

but

$$\mathbf{p} = \mathbf{r} - \overrightarrow{OQ}\,, \quad \mathbf{q} = \frac{\mathbf{V}\times\mathbf{p}}{Vp} = \frac{\mathbf{V}\times\mathbf{r}}{Vp}\,,$$

so that

$$\mathbf{r}^* = \mathbf{r}\cos\chi + \overrightarrow{OQ}(1-\cos\chi) + \frac{\sin\chi}{V}\mathbf{V}\times\mathbf{r} = \left(\rho^2 - V^2\right)\mathbf{r} + 2V^2\overrightarrow{OQ} + 2\rho\mathbf{V}\times\mathbf{r}\,.$$

Because $V^2\overrightarrow{OQ} = (\mathbf{V}\cdot\mathbf{r})\mathbf{V}$, we have, finally,

$$\mathbf{r}^* = \left(\rho^2 - V^2\right)\mathbf{r} + 2(\mathbf{V}\cdot\mathbf{r})\mathbf{V} + 2\rho\mathbf{V}\times\mathbf{r}\,; \tag{14.1.5}$$

projecting on the axes of the frame $\mathscr{R}$, we obtain

$$x_j^* = \alpha_{ij}x_j\,, \quad \alpha_{ij} = \left(\rho^2 - V_kV_k\right)\delta_{ij} + 2V_iV_j - 2\rho\in_{ijk}V_k\,, \tag{14.1.5'}$$

so that the transformation matrix is

$$\boldsymbol{\alpha} = \begin{bmatrix} \rho^2 + \lambda^2 - \mu^2 - \nu^2 & 2(\lambda\mu - \nu\rho) & 2(\nu\lambda + \mu\rho) \\ 2(\lambda\mu + \nu\rho) & \rho^2 + \mu^2 - \nu^2 - \lambda^2 & 2(\mu\nu - \lambda\rho) \\ 2(\nu\lambda - \mu\rho) & 2(\mu\nu + \lambda\rho) & \rho^2 + \nu^2 - \lambda^2 - \mu^2 \end{bmatrix}, \tag{14.1.6}$$

$$\lambda^2 + \mu^2 + \nu^2 = \sin^2\frac{\chi}{2}, \quad \lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1\,, \tag{14.1.6'}$$

corresponding to *Olinde Rodrigues's formulae*.

Obviously, this matrix remains invariant if one changes the signs of Euler's parameters. By comparison with the matrix (3.2.11''') which depends on Euler's angles, we find easily (we consider one of the two determinations)

$$\begin{aligned} \lambda &= \sin\frac{\theta}{2}\cos\frac{\psi - \varphi}{2}, \quad \mu = \sin\frac{\theta}{2}\sin\frac{\psi - \varphi}{2}, \\ \nu &= \cos\frac{\theta}{2}\sin\frac{\psi + \varphi}{2}, \quad \rho = -\cos\frac{\theta}{2}\cos\frac{\psi + \varphi}{2}. \end{aligned} \tag{14.1.6''}$$

A *quaternion* q is defined in the form $\mathrm{q} = a\mathrm{i} + b\mathrm{j} + c\mathrm{k} + d$, $a,b,c,d \in \mathbb{R}$, where i,j,k are *quaternion units*, which satisfy the relations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$, $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$. *The vector part* $\mathbf{V}\mathrm{q}$, *the scalar part* $S\mathrm{q}$, *the conjugate quaternion* $K\mathrm{q}$, *the norm* $N\mathrm{q}$ and *the reciprocal quaternion* q^{-1} are defined by the relations

$$\begin{gathered} \mathbf{V}\mathrm{q} = a\mathrm{i} + b\mathrm{j} + c\mathrm{k}\,, \quad S\mathrm{q} = d\,, \quad \mathrm{q} = \mathbf{V}\mathrm{q} + S\mathrm{q}\,, \quad K\mathrm{q} = -\mathbf{V}\mathrm{q} + S\mathrm{q}\,, \\ N\mathrm{q} = \sqrt{\mathrm{q}K\mathrm{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}\,, \quad \mathrm{q}^{-1} = \frac{K\mathrm{q}}{(N\mathrm{q})^2} = \frac{K\mathrm{q}}{\mathrm{q}K\mathrm{q}}. \end{gathered}$$

The vector part $\mathbf{V}\mathrm{q}$ can be considered as a usual vector; thus, if $S\mathrm{q} = 0$, then the quaternion q degenerates, becoming a vector. A quaternion q determines a number $h > 0$, a unit vector p and an angle χ, $0 \le \chi < 2\pi$, by the relation $\mathrm{q} = h[\cos(\chi/2) + \mathrm{p}\sin(\chi/2)]$; in this case, $N\mathrm{q} = h$, while $\mathrm{q}^{-1} = h^{-1}[\cos(\chi/2) - \mathrm{p}\sin(\chi/2)]$.

Let be the quaternion $\mathrm{q} = \lambda\mathrm{i} + \mu\mathrm{j} + \nu\mathrm{k} + \rho$, $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$, hence with $N\mathrm{q} = 1$; we have $\mathrm{q}^{-1} = -\lambda\mathrm{i} - \mu\mathrm{j} - \nu\mathrm{k} + \rho$. We introduce also the degenerate quaternions $\mathrm{r} = x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}$, $\mathrm{r}^* = x_1^*\mathrm{i} + x_2^*\mathrm{j} + x_3^*\mathrm{k}$ with $S\mathrm{r} = S\mathrm{r}^* = 0$. The relation $\mathrm{r}^* = \mathrm{q}\mathrm{r}\mathrm{q}^{-1}$ defines a transformation which represents a rotation of angle χ about the axis p. Passing to a matric notation, we find again the matrix (14.1.6), λ, μ, ν and ρ being thus Eulerian parameters; we obtain, in a quaternion notation, a new form of the respective representation. Observing that $h = 1$, it results

$$\mathbf{V}\mathrm{q} = \mathrm{p}\sin\frac{\chi}{2}, \quad (N\mathbf{V}\mathrm{q})^2 = \lambda^2 + \mu^2 + \nu^2 = \sin^2\frac{\chi}{2}, \quad S\mathrm{q} = \rho = \cos\frac{\chi}{2}; \tag{14.1.7}$$

hence, the vector $\mathbf{V}\mathrm{q}$ is the vector $\mathbf{V}$ (along the rotation axis), the angle χ of the quaternion being the rotation angle.

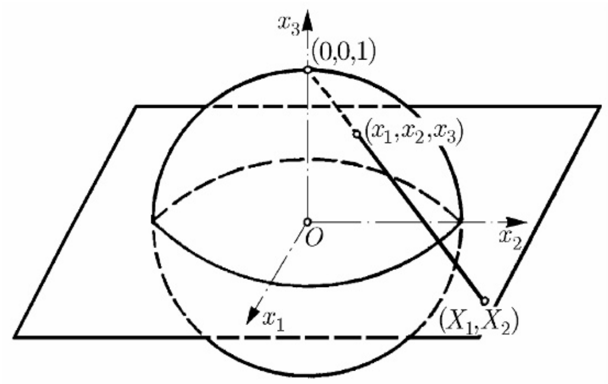

Fig. 14.4 Stereographic projection

Projecting the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ from the point $(0,0,1)$ on the plane $x_3 = 0$, one obtains a *stereographic projection*. Let (X_1, X_2) be thus the projection of the point (x_1, x_2, x_3) (Fig. 14.4); we will have

$$X_1 = \frac{x_1}{1 - x_3}, \quad X_2 = \frac{x_2}{1 - x_3}, \quad x_1 = \frac{2X_1}{X_1^2 + X_2^2 + 1},$$
$$x_2 = \frac{2X_2}{X_1^2 + X_2^2 + 1}, \quad x_3 = \frac{X_1^2 + X_2^2 - 1}{X_1^2 + X_2^2 + 1},$$

as well as

$$\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 = 4\frac{\mathrm{d}X_1^2 + \mathrm{d}X_2^2}{(X_1^2 + X_2^2 + 1)^2} = 4\frac{\mathrm{d}Z\mathrm{d}\overline{Z}}{(Z\overline{Z} + 1)^2},$$

where we have denoted $Z = X_1 + \mathrm{i}X_2$, $\overline{Z}$ being the complex conjugate. Any transformation $Z \to Z^*$ induces a transformation $(x_1, x_2, x_3) \to (x_1^*, x_2^*, x_3^*)$ of the unit sphere in itself, which will be rigid if the sum $\mathrm{d}x_j \mathrm{d}x_j$ is conserved, hence if

$$\frac{\mathrm{d}Z^* \, \mathrm{d}\overline{Z}^*}{\left(Z^* \overline{Z}^* + 1\right)^2} = \frac{\mathrm{d}Z \, \mathrm{d}\overline{Z}}{\left(Z\overline{Z} + 1\right)^2}.$$

One can show that this condition is satisfied by the transformations

$$Z^* = \frac{pZ + q}{-\overline{q}Z + p}, \quad p\overline{p} + q\overline{q} = 1, \tag{14.1.8}$$

where *the stereographic parameters* p and q are complex numbers to which correspond three degrees of freedom. Observing that

$$\begin{aligned} x_1^* + \mathrm{i}x_2^* &= \frac{2Z^*}{Z^* \overline{Z}^* + 1} = p^2 (x_1 + \mathrm{i}x_2) - q^2 (x_1 + \mathrm{i}x_2) - 2pqx_3, \\ x_1^* - \mathrm{i}x_2^* &= \overline{p}^2 (x_1 - \mathrm{i}x_2) - \overline{q}^2 (x_1 + \mathrm{i}x_2) - 2\overline{pq}x_3, \\ x_3^* &= \frac{Z^* \overline{Z}^* - 1}{Z^* \overline{Z}^* + 1} = p\overline{q}(x_1 + \mathrm{i}x_2) + \overline{p}q(x_1 - \mathrm{i}x_2) + (p\overline{p} - q\overline{q})x_3, \end{aligned}$$

we find the transformation matrix

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{1}{2}(p^2 + \overline{p}^2 - q^2 - \overline{q}^2) & \frac{\mathrm{i}}{2}(p^2 - \overline{p}^2 + q^2 - \overline{q}^2) & -(pq + \overline{pq}) \\ \frac{\mathrm{i}}{2}(\overline{p}^2 - p^2 + q^2 - \overline{q}^2) & \frac{1}{2}(p^2 + \overline{p}^2 + q^2 + \overline{q}^2) & \mathrm{i}(pq - \overline{pq}) \\ p\overline{q} + \overline{p}q & \mathrm{i}(p\overline{q} - \overline{p}q) & p\overline{p} - q\overline{q} \end{bmatrix}, \tag{14.1.9}$$

hence a new representation of the motion of rotation (not only of the rigid motion of the unit sphere about its centre) by means of the stereographic parameters.

Comparing the expressions (14.1.6) and (14.1.9) of the matrix $\boldsymbol{\alpha}$, taking into account (14.1.6") and by a choice of sign, we find the connection between the stereographic parameters, the Eulerian parameters and Euler's angles in the form

$$\begin{aligned} p &= \rho + \mathrm{i}\nu = -\cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\psi+\varphi)/2}, \\ q &= -\mu + \mathrm{i}\lambda = \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\psi-\varphi)/2}. \end{aligned} \tag{14.1.10}$$

14.1.1.3 The Cayley-Klein Parameters. Pauli's Matrices

Starting from the above results, we define the complex numbers

$$\begin{aligned} \alpha &= -\overline{p} = -\rho + \mathrm{i}\nu = \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\psi+\varphi)/2}, \\ \beta &= -\overline{q} = \mu + \mathrm{i}\lambda = \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\psi-\varphi)/2}, \end{aligned} \tag{14.1.11}$$

$$\begin{aligned} \gamma &= q = -\mu + \mathrm{i}\lambda = \mathrm{i}\sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}(\psi-\varphi)/2}, \\ \delta &= -p = -\rho - \mathrm{i}\nu = \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}(\psi+\varphi)/2}, \end{aligned} \tag{14.1.11'}$$

which satisfy the condition of unimodularity, being connected by the relations

$$\gamma = -\overline{\beta}, \quad \delta = \overline{\alpha}, \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \alpha\delta - \beta\gamma = \alpha\overline{\alpha} + \beta\overline{\beta} = 1, \tag{14.1.11''}$$

corresponding to the relation (14.1.8) between the stereographic parameters. These numbers are *the Cayley-Klein parameters* of the motion of rotation and constitute a new representation of it. The corresponding transformation matrix will be

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{1}{2}(\alpha^2 - \beta^2 - \gamma^2 + \delta^2) & \frac{\mathrm{i}}{2}(\gamma^2 + \delta^2 - \alpha^2 - \beta^2) & \gamma\delta - \alpha\beta \\ \frac{\mathrm{i}}{2}(\alpha^2 - \beta^2 + \gamma^2 - \delta^2) & \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) & -\mathrm{i}(\alpha\beta + \gamma\delta) \\ \beta\delta - \alpha\gamma & \mathrm{i}(\alpha\gamma + \beta\delta) & \alpha\delta + \beta\gamma \end{bmatrix}. \tag{14.1.12}$$

The above given parametric representations correspond to *the group of proper rotations* $O(3)^+$; but they can be put in connection with *the two-dimensional special unitary group* $SU(2)$, homomorphic with the group $O(3)^+$ too. Thus, *the Cayley-Klein representation* is characterized by the matrix

$$\mathbf{Q} \equiv \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{14.1.13}$$

where α and β are *the complex Cayley-Klein parameters*, while λ, μ, ν, ρ, $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$, specified by the relations (14.1.11), (14.1.11'), are *the real Cayley-Klein parameters* of the $SU(2)$ group (they coincide with Euler's parameters). We can choose as independent parameters the real numbers λ, μ, ν and $\operatorname{sgn}\rho = \rho/|\rho|$, the magnitude of ρ being given by the last relation (14.1.6'). The elements of the matrix $\mathbf{Q}$ being defined by (14.1.11), (14.1.11'), it results $|\alpha| \leq 1$, $|\beta| \leq 1$, so that we can choose $|\alpha| = \cos(\theta/2)$ and $|\beta| = \sin(\theta/2)$, where θ, $0 \le \theta \le \pi$ is uniquely determined by α and β; the angles φ and ψ are introduced as functions of the arguments of the complex numbers α and β in the form $\arg\alpha = (\psi + \varphi)/2$, $\arg\beta = -(\psi - \varphi - \pi)/2$, wherefrom we obtain $\varphi = \arg\alpha + \arg\beta - \pi/2$, $\psi = \arg\alpha - \arg\beta + \pi/2$. Because $0 \le \arg\alpha < 2\pi$ and $0 \le \arg\beta < 2\pi$, the domain of variation of the parameters φ and ψ is specified by $-\pi/2 \le \varphi < 7\pi/2$, $-3\pi/2 < \psi < 5\pi/2$, φ and ψ being defined till a multiple of 4π. In this case, the matrix $\mathbf{Q}$ is of the form

$$\mathbf{Q} = \begin{bmatrix} -\rho + \mathrm{i}\nu & \mu + \mathrm{i}\lambda \\ -\mu + \mathrm{i}\lambda & -\rho - \mathrm{i}\nu \end{bmatrix} = -\rho \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathrm{i} \begin{bmatrix} \nu & \lambda - \mathrm{i}\mu \\ \lambda + \mathrm{i}\mu & -\nu \end{bmatrix}$$
$$= \begin{bmatrix} \cos\frac{\theta}{2} \mathrm{e}^{\mathrm{i}(\psi+\varphi)/2} & \mathrm{i}\sin\frac{\theta}{2} \mathrm{e}^{-\mathrm{i}(\psi-\varphi)/2} \\ \mathrm{i}\sin\frac{\theta}{2} \mathrm{e}^{\mathrm{i}(\psi-\varphi)/2} & \cos\frac{\theta}{2} \mathrm{e}^{-\mathrm{i}(\psi+\varphi)/2} \end{bmatrix}. \qquad (14.1.13')$$

If the parameters α and β are given, then one can find an infinity of systems (ψ, θ, φ); but taking into account the intervals of definition of these angles, the solution is unique. Conversely, a given system (ψ, θ, φ) determines uniquely the parameters α and β, because between these angles and the Cayley-Klein parameters there exists a one-to-one correspondence; hence, the latter parameters determine, also, univocally, the rotation of the rigid solid with respect to a point of it.

We notice that a matrix $\mathbf{Q}$ with α, β, γ, δ arbitrary complex numbers can be univocally represented in the form

$$\mathbf{Q} = \frac{1}{2}(\alpha + \delta)\mathbf{E} + \frac{1}{2}(\beta + \gamma)\boldsymbol{\sigma}_1 + \frac{\mathrm{i}}{2}(\beta - \gamma)\boldsymbol{\sigma}_2 + \frac{1}{2}(\alpha - \delta)\boldsymbol{\sigma}_3, \qquad (14.1.14)$$

$$\mathbf{E} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\sigma}_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_2 \equiv \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (14.1.14')$$

The matrix $\mathbf{E}$ is the unit matrix, while $\boldsymbol{\sigma}_j$, $j = 1,2,3$, are *the Pauli spin matrices* (Hermitian and unitary matrices, the traces of which vanish). If the trace of the matrix $\mathbf{Q}$ vanishes ($\alpha + \delta = 0$), then the first term of the sum (14.1.14) disappears, while if the matrix $\mathbf{Q}$ is Hermitian (α, δ real numbers, $\gamma = \bar{\beta}$), then the coefficients of Pauli's matrices are real. We notice that Pauli's matrices verify the relations $\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = \mathbf{E}$, $\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k = \mathrm{i} \in_{jkl} \boldsymbol{\sigma}_l$, $j \ne k$, $j,k = 1,2,3$. Analogously, the matrices $\boldsymbol{\tau}_j = -\mathrm{i}\boldsymbol{\sigma}_j$, $j = 1,2,3$, verify the multiplication rules of the quaternion units.

The matrix (14.1.13') will be thus represented in the form

$$\mathbf{Q} = -\rho\mathbf{E} + \mathrm{i}\mathbf{P}\,, \quad \mathbf{P} = \lambda\boldsymbol{\sigma}_1 + \mu\boldsymbol{\sigma}_2 + \nu\boldsymbol{\sigma}_3\,. \tag{14.1.13''}$$

We notice that, through the relation $\mathbf{P}(x_1, x_2, x_3) = x_j\boldsymbol{\sigma}_j$, each point (x_1, x_2, x_3) defines a matrix **P** and, reciprocally, each Hermitian matrix of zero trace defines a point (x_1, x_2, x_3). Let be a unitary matrix **U**, non-Hermitian ($\mathbf{U}^+ \neq \mathbf{U}$), in general, by means of which we define the matrix $\mathbf{P} = \mathbf{U}\mathbf{P}\mathbf{U}^+$, which is Hermitian and of null trace; we obtain thus a transformation $(x_1, x_2, x_3) \to (x_1^*, x_2^*, x_3^*)$ of the space E_3 in itself. Because $\mathbf{U}\mathbf{U}^+ = \mathbf{E}$, we have $\det\mathbf{U}\det\mathbf{U}^+ = 1$, so that $\det\mathbf{P}^* = \det\mathbf{P}$ and we may write $x_j^* x_j^* = x_j x_j$; as well, the relation $\mathrm{d}\mathbf{P}^* = \mathbf{U}\mathrm{d}\mathbf{P}\mathbf{U}^+$ leads to $\mathrm{d}x_j^*\mathrm{d}x_j^* = \mathrm{d}x_j\mathrm{d}x_j$, the transformation being thus a rigid rotation about the origin.

Let be the unitary matrices $\mathbf{U}_j(\chi) = \mathbf{E}\cos(\chi/2) + \mathrm{i}\boldsymbol{\sigma}_j\sin(\chi/2)$, $j = 1,2,3$, χ real.

The transformation

$$\mathbf{P}^* = x_1^*\boldsymbol{\sigma}_1 + x_2^*\boldsymbol{\sigma}_2 + x_3^*\boldsymbol{\sigma}_3 = \mathbf{U}_3\mathbf{P}\mathbf{U}_3^+ = \left[\mathbf{E}\cos\frac{\chi}{2} + \mathrm{i}\boldsymbol{\sigma}_3\sin\frac{\chi}{2}\right]$$
$$\times(x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3)\left[\mathbf{E}\cos\frac{\chi}{2} - \mathrm{i}\boldsymbol{\sigma}_3\sin\frac{\chi}{2}\right]$$

leads to

$$x_1^* = x_1\cos\chi + x_2\sin\chi\,, \quad x_2^* = -x_1\sin\chi + x_2\cos\chi\,, \quad x_3^* = x_3\,, \tag{14.1.2'}$$

where we took into account the relations verified by Pauli's matrices; this transformation (relative to a fixed frame of reference) corresponds to a rotation of angle χ about the axis Ox_3 and is characterized by the matrix $\boldsymbol{\alpha}^*$ given by (14.1.2). Because of symmetry reasons, the matrices $\mathbf{U}_1(\chi)$ and $\mathbf{U}_2(\chi)$ lead, analogously, to rotations about the axes Ox_1 and Ox_2, respectively; each Pauli's spin matrix is thus associated to a rotation about a co-ordinate axis.

Let us consider the matrices

$$\mathbf{Q}_\psi \equiv \begin{bmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{bmatrix} = \mathbf{E}\cos\frac{\psi}{2} + \mathrm{i}\boldsymbol{\sigma}_3\sin\frac{\psi}{2}\,,$$
$$\mathbf{Q}_\theta \equiv \begin{bmatrix} \cos\dfrac{\theta}{2} & \mathrm{i}\sin\dfrac{\theta}{2} \\ \mathrm{i}\sin\dfrac{\theta}{2} & \cos\dfrac{\theta}{2} \end{bmatrix} = \mathbf{E}\cos\frac{\theta}{2} + \mathrm{i}\boldsymbol{\sigma}_1\sin\frac{\theta}{2}\,, \tag{14.1.13'''}$$
$$\mathbf{Q}_\varphi \equiv \begin{bmatrix} \mathrm{e}^{\mathrm{i}\varphi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi/2} \end{bmatrix} = \mathbf{E}\cos\frac{\varphi}{2} + \mathrm{i}\boldsymbol{\sigma}_3\sin\frac{\varphi}{2}\,,$$

corresponding to rotations by Euler's angles ψ, θ, φ about the axes Ox_3', ON and Ox_3, respectively (Fig. 14.1); the motion of rotation of the rigid solid will be thus characterized by the matrix

$$\mathbf{Q} = \mathbf{Q}_\varphi \mathbf{Q}_\theta \mathbf{Q}_\psi \,, \tag{14.1.13$^{\text{iv}}$}$$

corresponding to the formulae (14.1.13') and (14.1.13").

One observes that the Cayley-Klein parameters (the matrices $\mathbf{Q}$ too) are characterized by the semi-angles of rotation, unlike the matrices $\boldsymbol{\alpha}$ where appear just these angles. Thus, for $\chi = 0$ or for $\chi = 2\pi$, the matrix $\boldsymbol{\alpha}^*$ specified by (14.1.2) is reduced to the unit matrix; in exchange, for $\chi = 0$, e.g., the matrix $\mathbf{Q}_\psi$ is reduced to the unit matrix $\mathbf{E}$, while for $\chi = 2\pi$ it becomes the matrix $-\mathbf{E}$. In general, to a matrix $\boldsymbol{\alpha}$ corresponds a couple of matrices $\{\mathbf{Q}, -\mathbf{Q}\}$, the matrix $\mathbf{Q}$ being thus a bivalent function of the matrix $\boldsymbol{\alpha}$; as a matter of fact, we have made an arbitrary choice of sign in the relations established between various representations. We mention that the relations between the matrices $\boldsymbol{\alpha}$ and $\mathbf{Q}$ correspond to the relations which are established between the real space E_3 and a two-dimensional space corresponding to a matrix of $\mathbf{Q}$ type; a two-dimensional complex vector will be called *spinor*, the corresponding space being a *spinor space*. This space is more adequate to the physical reality in the quantum model of mechanics, the wave function of a part of it having a spinorial character; indeed, the semi-angles and the property of bivalence are closely connected to the fact that the electron spin is a semi-integer.

14.1.1.4 Kinematic Considerations

If α and β are the matrices corresponding to two finite rotations, then we notice that $\boldsymbol{\alpha\beta} \neq \boldsymbol{\beta\alpha}$, because the product of matrices is not commutative; hence, *the sum of two finite rotations depends on the order in which they are effected*. In case of *infinitesimal rotations*, there correspond *infinitesimal orthogonal transformations* of matrix $\boldsymbol{\alpha} = \mathbf{E} + \boldsymbol{\varepsilon}$, where the product of two infinitesimal matrices of $\boldsymbol{\varepsilon}$ type is neglected with respect to such a matrix; in this case,

$$(\mathbf{E} + \boldsymbol{\varepsilon}_1)(\mathbf{E} + \boldsymbol{\varepsilon}_2) = \mathbf{E}^2 + \mathbf{E}\boldsymbol{\varepsilon}_1 + \mathbf{E}\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 = \mathbf{E} + \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 \,,$$

so that the product of two such matrices is commutative. Hence, in case of infinitesimal rotations the order of their application is immaterial. We notice also that to a rotation of angle $\mathrm{d}\chi$ about an axis of rotation there corresponds the matrix

$$\boldsymbol{\alpha} = \mathbf{E} + \boldsymbol{\varepsilon} = \begin{bmatrix} 1 & \mathrm{d}\chi & 0 \\ -\mathrm{d}\chi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E} + \mathrm{d}\chi \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14.1.2"}$$

Starting from these considerations, we can find again the results obtained in Chap. 5, §2 concerning the kinematics of the rigid solid. We mention, especially, the Theorem 5.2.4 which characterizes the general motion of the rigid solid with the aid of the *fixed* and *movable axoids*.

The rotation velocity of the movable frame of reference $\mathscr{R}$, rigidly linked to the rigid solid and with the pole at O, with respect to the movable frame $\overline{\mathscr{R}}$ and to the fixed frame $\mathscr{R}'$, is characterized by the angular velocity vector $\boldsymbol{\omega}$, which is expressed as function of *Euler's angles* in the matric form (5.2.34) and, in components, with respect to the frame $\overline{\mathscr{R}}$ or to the frame $\mathscr{R}'$, by means of the kinematic relations of Euler (5.2.35) or (5.2.35'), respectively. Starting from the latter relations, we may express the angular velocities corresponding to Euler's angles ψ, θ, φ in the form (see Chap. 3, Sect. 2.2.3 too) (Fig. 14.1)

$$\begin{aligned} \dot{\psi} &= (\omega_1 \sin\varphi + \omega_2 \cos\varphi)\operatorname{cosec}\theta\,, \\ \dot{\theta} &= \omega_1 \cos\varphi - \omega_2 \sin\varphi\,, \\ \dot{\varphi} &= \omega_3 - (\omega_1 \sin\varphi + \omega_2 \cos\varphi)\cot\theta \end{aligned} \tag{14.1.15}$$

or in the form

$$\begin{aligned} \dot{\psi} &= \omega_3' - (\omega_1' \sin\psi - \omega_2' \cos\psi)\cot\theta\,, \\ \dot{\theta} &= \omega_1' \cos\psi + \omega_2' \sin\psi\,, \\ \dot{\varphi} &= (\omega_1' \sin\psi - \omega_2' \cos\psi)\operatorname{cosec}\theta\,, \end{aligned} \tag{14.1.15'}$$

respectively. We remark also some interesting differential relations, e.g.,

$$\frac{\mathrm{d}\,\omega_1}{\mathrm{d}\,\varphi} = \omega_2\,, \quad \frac{\mathrm{d}\,\omega_2}{\mathrm{d}\,\varphi} = -\omega_1\,, \quad \frac{\mathrm{d}^2\,\omega_k}{\mathrm{d}\,\varphi^2} + \omega_k = 0\,, \quad k = 1,2\,. \tag{14.1.15''}$$

Analogously, Euler's angles which specify the position of the frame $\overline{\mathscr{R}}$ or of the frame $\mathscr{R}'$ with respect to the frame $\mathscr{R}$ are $\psi' = -\varphi$, $\theta' = -\theta$, $\varphi' = -\psi$; as well, $\boldsymbol{\omega}' = -\boldsymbol{\omega}$ is the rotation angular velocity vector of the inertial frame with respect to the non-inertial one. We can thus pass from the relations (5.2.35) to the relations (5.2.35') or from the relations (14.1.15) to the relations (14.1.15').

To pass from $\omega_i(t)$, $i = 1,2,3$, to Euler's angles $\psi(t)$, $\theta(t)$ and $\varphi(t)$, hence to integrate the system (14.1.15) with respect to the latter unknown functions, one can introduce the intermediate unknown functions $\alpha_j(t)$, $j = 1,2,3$, which are the direction cosines of the Ox_3'-axis with respect to the movable frame $\mathscr{R}$; one obtains thus the relations (5.2.36) which lead to

$$\cos\theta = \alpha_3\,, \quad \tan\varphi = \frac{\alpha_1}{\alpha_2}\,, \tag{14.1.16}$$

the angle ψ resulting by a quadrature from the first relation (14.1.15). The functions α_j, $j = 1,2,3$, are connected by the differential relations (5.2.37').

The velocities distribution is given by Euler's formula (5.2.3), written in the form

$$\mathbf{v}' = \mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r}, \tag{14.1.17}$$

and *the accelerations distribution* by the formula (5.2.6), i.e.

$$\mathbf{a}' = \mathbf{a}'_O + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \tag{14.1.17'}$$

where we have put in evidence the quantities related to the frames $\mathscr{R}'$ and $\mathscr{R}$, respectively. Differentiating *the rigidity condition* (14.1.1) with respect to time, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\mathbf{r}'_j - \mathbf{r}'_i)^2 \right] = 2(\mathbf{r}'_j - \mathbf{r}'_i) \cdot (\mathbf{v}'_j - \mathbf{v}'_i) = 2\overrightarrow{P_i P_j} \cdot (\mathbf{v}'_j - \mathbf{v}'_i) = 0\,; \tag{14.1.18}$$

noting that the relation (14.1.17) can be written in the form $\mathbf{v}'_j - \mathbf{v}'_i = \boldsymbol{\omega} \times \overrightarrow{P_i P_j}$, it results that Euler's formula verifies this condition. *The compatibility condition of velocities* may be thus written in the form

$$\mathbf{v}'_j \cdot \overrightarrow{P_i P_j} = \mathbf{v}'_i \cdot \overrightarrow{P_i P_j} \tag{14.1.18'}$$

too, corresponding to the relation (5.2.4). Starting from the relation (14.1.18'), written successively for the couples of points (P_2, P_3), (P_3, P_1) and (P_1, P_2), as well as for (P, P_1), (P, P_2) and (P, P_3) (the point P non-coplanar with the points P_1, P_2, P_3), R. Voinaroski and L. Livovschi have found again the relation (14.1.17). Because it can be stated from hypotheses of rigidity, it results that the respective relation has an intrinsic character; on the other hand, the angular velocity vector $\boldsymbol{\omega}$ can be obtained as an axial vector associated to an antisymmetric tensor of second order, defined by means of the velocities of three non-collinear points of the rigid solid (hence, independent on the movable frame $\mathscr{R}$). *The condition of compatibility of the accelerations* is of the form

$$(\mathbf{a}'_i - \mathbf{a}'_j) \cdot \overrightarrow{P_i P_j} = (\mathbf{v}'_i - \mathbf{v}'_j)^2 = \left(\boldsymbol{\omega} \times \overrightarrow{P_i P_j}\right)^2, \tag{14.1.18''}$$

corresponding to the relation (5.2.10). As a matter of fact, one can use all the results contained in Chap. 5, §2.

With the aid of the results obtained in Chap. 5, Sec. 3.2 concerning the relative motion of the rigid solid, one can state *the group character of the rigid motions*. Corresponding to the Theorem 5.3.3, we can thus state that the set of translations of a free rigid solid forms an Abelian group, while from the Theorem 5.3.4 it results that the set of rotations of a free rigid solid about concurrent axes of rotation form an Abelian group too; as well, the Theorem 5.3.5 allows to state that the set of rotations of a free rigid solid about parallel axes of rotation, so that the resultant angular velocity vector is non-zero, constitutes also an Abelian group.

Applying the divergence and curl operators to the velocities (14.1.17), taking into account the formulae (A.2.31'), (A.2.31") and observing that the vectors $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ are constant with respect to the vector $\mathbf{r}$ and that $\operatorname{div}\mathbf{r} = 3$, $\operatorname{curl}\mathbf{r} = \mathbf{0}$, $(\boldsymbol{\omega}\cdot\nabla)\mathbf{r} = \boldsymbol{\omega}$, we may write

$$\operatorname{div}\mathbf{v}' = 0, \quad \operatorname{curl}\mathbf{v}' = 2\boldsymbol{\omega}, \tag{14.1.19}$$

in the frame $\mathcal{R}$; replacing $\mathbf{r} = \mathbf{r}' - \mathbf{r}'_O$, we can make the same affirmation for the frame $\mathcal{R}'$ too. We state thus

Theorem 14.1.3 *The velocities' field of a free rigid solid with respect to an inertial frame of reference is a solenoidal one* (*a field of curls*), *its curl being equal to the double of the rotation angular velocity vector with respect to this frame.*

In the case in which the non-inertial frame of reference $\mathcal{R}$ is not rigidly linked to the rigid solid, intervenes a supplementary rotation vector $\mathbf{\Omega}$, which characterizes the motion of this frame with respect to the rigid solid; obviously, the results which are obtained are more complicated, intervening the difference $\boldsymbol{\omega} - \mathbf{\Omega}$, but they are not of a particular practical interest.

14.1.1.5 Momentum and Moment of Momentum in Case of a Rigid Solid

Various mechanical quantities which appear in the study of continuous mechanical systems get particular forms in case of a rigid solid; we will express these quantities with respect to an inertial frame of reference $\mathcal{R}'$ with the pole at a point O' and with respect to a non-inertial frame $\mathcal{R}$ with the pole at a point O, rigidly linked to the considered rigid solid, the connection between the two frames being specified by

$$\mathbf{r}' = \mathbf{r}'_O + \mathbf{r} \tag{14.1.20}$$

for a particle P of the rigid solid. We introduce, as well, the frame $\overline{\mathcal{R}}$ too, with the pole at the same point O and with the axes parallel to the axes of the frame $\mathcal{R}'$; the connection between the frames $\mathcal{R}'$ and $\overline{\mathcal{R}}$ is of the same form (14.1.20). Defining *the momentum of the rigid solid* in the form

$$\mathbf{H}' = \iiint_V \mu(\mathbf{r})\mathbf{v}'\mathrm{d}V, \tag{14.1.21}$$

where V is the volume, and taking into account the expression (14.1.17) of the velocity and the relations

$$\mathbf{M} = \iiint_V \mu(\mathbf{r})\mathrm{d}V, \quad M\boldsymbol{\rho} = \iiint_V \mu(\mathbf{r})\mathbf{r}\mathrm{d}V, \tag{14.1.22}$$

where $\boldsymbol{P}$ is the position vector of the mass centre in the frame $\mathcal{R}$, we obtain the formula

$$\mathbf{H}' = M(\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}), \tag{14.1.21'}$$

which is a particular case of the formula (11.2.11) (for $\mathbf{H} = \mathbf{0}$); observing that the velocity of the mass centre is given by

$$\mathbf{v}'_C = \mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}, \tag{14.1.17''}$$

we can write

$$\mathbf{H}' = M\mathbf{v}'_C \tag{14.1.21''}$$

too, and we state

Theorem 14.1.4 *The momentum of a rigid solid with respect to a given frame of reference is equal to the momentum of its mass centre with respect to the same frame, assuming that the whole mass of it is concentrated at this centre.*

The moment of momentum of the rigid solid is defined in the form

$$\mathbf{K}'_{O'} = \iiint_V \mathbf{r}' \times [\mu(\mathbf{r})\mathbf{v}'] \mathrm{d}V. \tag{14.1.23}$$

Taking into account (14.1.20), (14.1.17), we may assume

$$\mathbf{K}'_{O'} = \iiint_V \mu(\mathbf{r})(\mathbf{r}'_O + \mathbf{r}) \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}V = \mathbf{r}'_O \times \mathbf{v}'_O \iiint_V \mu(\mathbf{r}) \mathrm{d}V$$
$$+ \mathbf{r}'_O \times \left[\boldsymbol{\omega} \times \iiint_V \mu(\mathbf{r})\mathbf{r} \mathrm{d}V \right] - \mathbf{v}'_O \times \iiint_V \mu(\mathbf{r})\mathbf{r} \mathrm{d}V + \iiint_V \mu(\mathbf{r})\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}V,$$

the relations (14.1.22) leading thus to

$$\mathbf{K}'_{O'} = \mathbf{K}^O + M(\mathbf{r}'_O \times \mathbf{v}'_C - \mathbf{v}'_O \times \boldsymbol{\rho}), \tag{14.1.23'}$$

where *the pseudomoment of momentum* is given by

$$\mathbf{K}^O = \iiint_V \mu(\mathbf{r})\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \mathrm{d}V = \iiint_V \mu(\mathbf{r})\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \mathrm{d}V. \tag{14.1.24}$$

We find thus again the formulae (11.2.16), (11.2.16'), where $\mathbf{v}_i = \mathbf{0}$. To $\lambda\boldsymbol{\omega}$, λ scalar, corresponds the homologous vector $\lambda\mathbf{K}^O$, while if $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ have as homologous vectors $\mathbf{K}_1^O$ and $\mathbf{K}_2^O$, respectively, then to the vector $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ corresponds the vector $\mathbf{K}_1^O + \mathbf{K}_2^O$, due to the distributivity of the vector product with respect to the addition of vectors; hence, the linear transformation (14.1.24) is a *vector homographic transformation*, called *inertial homography*. Noting that

$$\mathbf{K}^O = \iiint_V \mu(\mathbf{r})[r^2\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r})\mathbf{r}] \mathrm{d}V,$$

introducing the moment of inertia tensor $\mathbf{I}_O$ of components (3.1.81) and taking into account the relation (3.1.83), we obtain the remarkable relation

$$\overline{\mathbf{K}}_O = \mathbf{K}^O = \mathbf{I}_O \boldsymbol{\omega}, \tag{14.1.24'}$$

which corresponds to the relations (11.2.17), (11.2.17') where we make $\mathbf{K}_O = \mathbf{0}$; $\overline{\mathbf{K}}_O$ represents the moment of momentum with respect to the pole O in the non-inertial frame and – in case of the rigid solid – is reduced to the pseudomoment of momentum considered above. We state thus

Theorem 14.1.5 *The moment of momentum of a rigid solid with respect to a pole O' of a given inertial frame of reference $\mathscr{R}'$, in this frame, is equal to the sum of the pseudomoment of momentum of the rigid solid with respect to an arbitrary pole O, rigidly linked to the rigid solid* (*the contracted product of the moment of inertia tensor with respect to the same pole by the rotation angular velocity vector of a non-inertial frame $\mathscr{R}$ with the pole at O, rigidly linked to the rigid solid, with respect to the inertial frame*), *the moment of momentum of the centre of mass, translated at the pole O, where the whole mass of the rigid solid is considered to be concentrated, taken with respect to the pole O', in the frame $\mathscr{R}'$, and the moment of momentum of the pole O, translated at the centre of mass, where it is assumed that the whole mass of the rigid solid is concentrated, calculated with respect to the pole O, in the inertial frame $\mathscr{R}'$ too.*

In the particular case in which the frame of reference $\mathscr{R}$ is of Koenig type ($\boldsymbol{\omega} = \mathbf{0}$, hence $\mathbf{K}^O = \mathbf{0}$), there results the formula

$$\mathbf{K}'_{O'} = \mathbf{r}'_O \times (M\mathbf{v}'_C) + \boldsymbol{\rho} \times (M\mathbf{v}'_O). \tag{14.1.25}$$

Taking into account that this frame is rigidly linked to the rigid solid, it results that the latter one will have a motion of translation (we are in a particular case of motion).

If the pole of the non-inertial frame coincides with the centre of mass of the rigid solid ($O \equiv C, \boldsymbol{\rho} = \mathbf{0}$), then we obtain a formula of Koenig type (in which, instead of the moment of momentum with respect to the centre of mass, in the frame $\mathscr{R}$, which is equal to zero, appears the corresponding pseudomoment of momentum)

$$\mathbf{K}'_{O'} = \mathbf{I}_C \boldsymbol{\omega} + \boldsymbol{\rho}' \times (M\mathbf{v}'_C), \tag{14.1.25'}$$

where $\mathbf{I}_C$ is the central moment of inertia tensor; thus, the motion is not particularized and has – further – a general character. If we have also $\boldsymbol{\omega} = \mathbf{0}$, then the non-inertial frame $\mathscr{R}$ is a Koenig frame and we get

$$\mathbf{K}'_{O'} = \boldsymbol{\rho}' \times (M\mathbf{v}'_C). \tag{14.1.25''}$$

Hence, *the moment of momentum of a rigid solid in motion of translation, with respect to a given frame of reference, is equal to the moment of momentum of its centre of mass with respect to this frame, assuming that its whole mass is concentrated at this centre.*

As in Sect. 11.2.2.1, starting from (14.1.23') and noting that $\mathbf{K}'_O = \mathbf{K}'_{O'} - \mathbf{r}'_O \times (M\mathbf{v}'_C)$, we obtain

$$\mathbf{K}'_O = \mathbf{I}_O \boldsymbol{\omega} + \boldsymbol{\rho} \times (M\mathbf{v}'_O), \tag{14.1.26}$$

a formula analogue to (11.2.23), the moment of momentum $\mathbf{K}'_O$ being calculated with respect to the inertial frame $\mathscr{R}'$; in particular, if $O \equiv C$, then we can write

$$\mathbf{K}'_C = \mathbf{I}_C(\mathscr{S})\boldsymbol{\omega}, \tag{14.1.26'}$$

corresponding to the relation (11.2.23'), where we make $\mathbf{K}_C = \mathbf{0}$. We state thus

Theorem 14.1.6 *The moment of momentum of a rigid solid with respect to the centre of mass, in an inertial frame of reference, is equal to the contracted product of the central moment of inertia tensor by the rotation angular velocity vector of a non-inertial frame rigidly linked to the rigid solid, with the pole at the centre of mass too, with respect to the inertial frame.*

This result has a general character and takes place for any motion of the rigid solid; it represents the most simple formula of kinetic nature corresponding to such a motion. It can be obtained also directly, starting from the formula (14.1.24) and making $O \equiv C$; it results

$$\begin{aligned}\mathbf{I}_C\boldsymbol{\omega} &= \iiint_V \mu(\mathbf{r})\overrightarrow{CP} \times \frac{\mathrm{d}\overrightarrow{CP}}{\mathrm{d}t}\mathrm{d}V = \iiint_V \mu(\mathbf{r})(\mathbf{r}' - \boldsymbol{\rho}') \times (\mathbf{v}' - \mathbf{v}'_C)\mathrm{d}V \\ &= \iiint_V (\mathbf{r}' - \boldsymbol{\rho}') \times [\mu(\mathbf{r})\mathbf{v}']\mathrm{d}V + \mathbf{v}'_C \times \iiint_V (\mathbf{r}' - \boldsymbol{\rho}')\mathrm{d}V = \mathbf{K}'_{C'},\end{aligned}$$

because the last integral represents the statical moment of the rigid solid with respect to the mass centre, so that it vanishes.

It is thus put in evidence the necessity to introduce the moment of inertia tensor, studied thoroughly in Chap. 3, Sec. 1.2.

We can define *the dynamic resultant of the rigid solid* in the form

$$\mathbf{A}' = \iiint_V \mu(\mathbf{r})\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t}\mathrm{d}V. \tag{14.1.27}$$

Taking into account (14.1.17') and (14.1.22), we obtain

$$\mathbf{A}' = M\mathbf{a}'_O + M\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + M\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}), \tag{14.1.27'}$$

corresponding to the relation (11.2.11') for $\mathbf{H} = \mathbf{0}$ and $\partial\mathbf{r}/\partial t = \mathbf{0}$. As well, *the dynamic moment of the rigid solid*, defined in the form

$$\mathbf{D}'_{O'} = \iiint_V \mathbf{r}' \times \left[\mu(\mathbf{r})\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t}\right]\mathrm{d}V, \tag{14.1.28}$$

leads to

$$\mathbf{D}'_{O'} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O\boldsymbol{\omega}) + \boldsymbol{\rho} \times (M\mathbf{a}'_O), \tag{14.1.28'}$$

corresponding to the relation (11.2.25), where we take into account (14.1.26) and (14.1.17').

14.1.1.6 Kinetic energy and work in case of a rigid solid

The kinetic energy of the rigid solid is defined in the form

$$T' = \frac{1}{2}\iiint_V \mu(\mathbf{r}) v'^2 \mathrm{d}V . \tag{14.1.29}$$

With the aid of the relation (14.1.17), we can write

$$T' = \frac{1}{2} v_{O'}^2 \iiint_V \mu(\mathbf{r}) \mathrm{d}V + \mathbf{v}'_{O'} \cdot \left[\boldsymbol{\omega} \times \iiint_V \mu(\mathbf{r}) \mathbf{r} \,\mathrm{d}V \right] + \frac{1}{2}\iiint_V \mu(\mathbf{r})(\boldsymbol{\omega}\times\mathbf{r})^2 \,\mathrm{d}V ,$$

wherefrom, using the relations (14.1.22), we obtain

$$T' = T^O + \frac{1}{2} M v_O'^2 + M(\mathbf{v}'_O, \boldsymbol{\omega}, \boldsymbol{\rho}) = T^O - \frac{1}{2} M v_O'^2 + M \mathbf{v}'_O \cdot \mathbf{v}'_C , \tag{14.1.29'}$$

corresponding to the relations (11.2.28), where we make $\mathbf{v}_C = \mathbf{0}$. We have introduced *the pseudokinetic energy of the rigid solid*

$$\begin{aligned} T^O &= \frac{1}{2}\iiint_V \mu(\mathbf{r})(\boldsymbol{\omega}\times\mathbf{r})^2 \,\mathrm{d}V = \frac{1}{2}\iiint_V \mu(\mathbf{r})\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)^2 \mathrm{d}V \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot \iiint_V \mu(\mathbf{r})\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})\mathrm{d}V = \frac{1}{2}\boldsymbol{\omega}\cdot \iiint_V \mu(\mathbf{r})\mathbf{r}\times\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}V . \end{aligned} \tag{14.1.30}$$

Taking into account (14.1.24) and (14.1.24'), we can also write

$$\overline{T} = T^O = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{K}^O = \frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{I}_O\boldsymbol{\omega}), \tag{14.1.30'}$$

corresponding to the relations (11.2.28"), (11.2.29), where we make $\mathbf{K}_O = \mathbf{0}$ and $T = 0$; $\overline{T}$ represents the kinetic energy with respect to the non-inertial frame $\overline{\mathcal{R}}$, which – in case of the rigid solid – is reduced to the pseudokinetic energy considered above. We may thus state

Theorem 14.1.7 *The kinetic energy of a rigid solid with respect to a given inertial frame of reference $\mathcal{R}'$ is equal to the sum of the pseudokinetic energy of this solid with respect to an arbitrary non-inertial frame $\mathcal{R}$ with the pole at O, rigidly linked to the rigid solid (the semi-scalar product of the rotation angular velocity vector by the contracted product of the moment of inertia tensor with respect to the pole O by the rotation angular velocity vector of the frame $\mathcal{R}$ with respect to the frame $\mathcal{R}'$) and the scalar product of the velocity of the pole O with respect to the frame $\mathcal{R}'$ by the momentum of the rigid solid with respect to the same frame $\mathcal{R}$, from which is subtracted the kinetic energy of the pole O at which is considered to be concentrated the whole mass of the rigid solid, in the frame $\mathcal{R}'$.*

We can write the relations (11.2.29), (11.2.29') too, introducing the axial moment of inertia with respect to the instantaneous axis of rotation $\varDelta$; thus, the axial moment of inertia plays the rôle of a mass in the instantaneous motion of rotation. Noting that $\mathrm{d}\mathbf{K}^O$ is the vector homologous to $\mathbf{d\omega}$, we may write

$$\mathbf{K}^O \cdot \mathrm{d}\boldsymbol{\omega} = \iiint_V \mu(\mathbf{r})\mathrm{d}\boldsymbol{\omega} \cdot [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\mathrm{d}V = \iiint_V \mu(\mathbf{r})(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\mathrm{d}\boldsymbol{\omega} \times \mathbf{r})\mathrm{d}V$$
$$= \iiint_V \mu(\mathbf{r})\boldsymbol{\omega} \cdot [\mathbf{r} \times (\mathrm{d}\boldsymbol{\omega} \times \mathbf{r})]\mathrm{d}V ,$$

whence the remarkable relation

$$\mathbf{K}^O \cdot \mathrm{d}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathrm{d}\,\mathbf{K}^O . \tag{14.1.31}$$

Taking into account (11.2.29') and (14.1.31), we have

$$\mathbf{K}^O \cdot \mathrm{d}\boldsymbol{\omega} = \frac{1}{2}\mathrm{d}(\mathbf{K}^O \cdot \boldsymbol{\omega}) = \mathrm{d}T^O = \frac{1}{2}\mathrm{d}(I_\varDelta \omega^2) = \frac{1}{2}\mathrm{d}(\omega\sqrt{I_\varDelta})^2 = \omega\sqrt{I_\varDelta}\,\mathrm{d}(\omega\sqrt{I_\varDelta}) .$$

In Chap. 3, Sect. 1.2.6 we have introduced the ellipsoid of inertia as locus of the points P for which

$$\overrightarrow{OP} = \frac{K}{\sqrt{I_\varDelta}}\frac{\boldsymbol{\omega}}{\omega} = \frac{K}{\sqrt{I_\varDelta}}\operatorname{vers}\boldsymbol{\omega} . \tag{14.1.32}$$

In this case

$$\mathrm{d}\overrightarrow{OP} = \frac{K}{\sqrt{I_\varDelta}}\frac{\mathrm{d}\boldsymbol{\omega}}{\omega} - K\boldsymbol{\omega}\frac{\mathrm{d}(\omega\sqrt{I_\varDelta})}{\omega^2 I_\varDelta} ,$$

so that, using the above results,

$$\mathbf{K}^O \cdot \mathrm{d}\overrightarrow{OP} = K\left(\mathbf{K}^O \cdot \frac{\mathrm{d}\boldsymbol{\omega}}{\omega\sqrt{I_\varDelta}}\right) - K\frac{\mathbf{K}^O \cdot \boldsymbol{\omega}}{\omega^2 I_\varDelta}\mathrm{d}(\omega\sqrt{I_\varDelta}) = 0 .$$

Hence, the *pseudomoment of momentum* $\mathbf{K}^O$ *is along the normal* OQ *at* O *to the plane* Π, *tangent at* P *to the ellipsoid of inertia* (see Fig. 3.9) too).

We define a vector

$$\mathbf{J}^O = \frac{\mathbf{K}^O}{\sqrt{2T^O}} = \frac{\mathbf{K}^O}{\sqrt{2\overline{T}}} = \frac{\mathbf{I}_O\boldsymbol{\omega}}{\omega\sqrt{I_\varDelta}} = \frac{1}{\sqrt{I_\varDelta}}\mathbf{I}_O \operatorname{vers}\boldsymbol{\omega} , \tag{14.1.33}$$

which is situated along the same normal and is associated to the moment of inertia tensor $\mathbf{I}_O$. Taking into account (3.1.100), we can write, in components,

$$J_i^O = \frac{1}{K} I_{ij} x_j, \quad i = 1,2,3, \tag{14.1.33'}$$

where x_i are the co-ordinates of the point P of the ellipsoid of inertia; we have, with respect to the principal axes of inertia,

$$\frac{J_1^O}{I_1 x_1} = \frac{J_2^O}{I_2 x_2} = \frac{J_3^O}{I_3 x_3} = \frac{1}{K}. \tag{14.1.33"}$$

Using the relation (3.1.82'), we get

$$\mathbf{J}^O \cdot \operatorname{vers} \boldsymbol{\omega} = \sqrt{I_\Delta}\,. \tag{14.1.34}$$

We notice that ($Q \in \Pi$)

$$\mathbf{J}^O \cdot \overrightarrow{OP} = J^O\, \overline{OQ} = K \frac{\mathbf{K}^O \cdot \boldsymbol{\omega}}{I_\Delta \omega^2} = K\,. \tag{14.1.34'}$$

If we take $K = R^2 \sqrt{M}$ (see Chap. 3, Sect. 1.2.6), then the extremity of the vector $\mathbf{J}^O / \sqrt{M}$ applied at O is the point P' (the inverse of the point Q with respect to a sphere of centre O and radius R), the locus of which is the gyration ellipsoid; we obtain this result also by comparing the relations (3.1.104') and (14.1.33"). It results thus a graphic method to determine the pseudomoment of momentum $\mathbf{K}^O$.

The relation (14.1.26') shows that the central moment of inertia tensor $\mathbf{I}_C$ plays an important rôle. In this order of ideas, let be an axis which passes through the pole O and let be $\bar{O}$ another pole on $\boldsymbol{\omega}$, of position vector $\mathbf{r}_{\bar{O}} = \alpha\boldsymbol{\omega}$, α scalar, with respect to the pole O. The relation $\mathbf{r} = \alpha\boldsymbol{\omega} + \bar{\mathbf{r}}$ between the position vectors of a point P of the rigid solid with respect to the poles O and $\bar{O}$, respectively, allows to write

$$\mathbf{K}^O = \iiint_V \mu(\mathbf{r})(\alpha\boldsymbol{\omega} + \bar{\mathbf{r}}) \times [\boldsymbol{\omega} \times (\alpha\boldsymbol{\omega} + \bar{\mathbf{r}})]\,\mathrm{d}V = \mathbf{K}^{\bar{O}} + \alpha\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \iiint_V \mu(\mathbf{r})\bar{\mathbf{r}}\,\mathrm{d}V\right]$$
$$= \mathbf{K}^{\bar{O}} + \alpha\boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \iiint_V \mu(\mathbf{r})\mathbf{r}\,\mathrm{d}V\right] = \mathbf{K}^{\bar{O}} + \alpha M \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = K^{\bar{O}} + \alpha M [(\boldsymbol{\omega} \cdot \boldsymbol{\rho})\boldsymbol{\omega} - \omega^2 \boldsymbol{\rho}]\,.$$

If the axis $\boldsymbol{\omega}$ is the principal axis of inertia with respect to the pole O, then $\mathbf{J}^O$ must be collinear with $\boldsymbol{\omega}$, hence we must have $\mathbf{K}^O = \lambda\boldsymbol{\omega}$, λ scalar; if we wish that $\boldsymbol{\omega}$ be a principal axis of inertia with respect to the pole O too, analogously we must have $\mathbf{K}^{\bar{O}} = \bar{\lambda}\boldsymbol{\omega}$, $\bar{\lambda}$ scalar. In this case,

$$\lambda\boldsymbol{\omega} = \bar{\lambda}\boldsymbol{\omega} + \alpha M (\boldsymbol{\omega} \cdot \boldsymbol{\rho})\boldsymbol{\omega} - \alpha M \omega^2 \boldsymbol{\rho}\,,$$

wherefrom it results $\boldsymbol{\rho} = \beta\boldsymbol{\omega}$, β scalar. We can thus state that *a principal axis of inertia with respect to a given pole is a principal axis of inertia with respect to any point of it if and only if it is a central axis of inertia*. Analogously, let be a central principal plane of inertia determined by the axes $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ through O and a point $\bar{O}$ in this plane, specified by the position vector $\alpha\boldsymbol{\omega} + \alpha'\boldsymbol{\omega}'$, α, α' scalars; a point P of the rigid solid will be specified by the position vectors $\mathbf{r}$ and $\bar{\mathbf{r}}$ with respect to the poles O and $\bar{O}$, respectively, so that $\mathbf{r} = \alpha\boldsymbol{\omega} + \alpha'\boldsymbol{\omega}' + \bar{\mathbf{r}}$. Putting the condition that the axes which pass through $\bar{O}$ and are parallel to the axes $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ be principal axes of inertia if the axes $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ have this property, then we find – as above – that $\boldsymbol{\rho} = \beta\boldsymbol{\omega} + \beta'\boldsymbol{\omega}'$, β, β' scalars; hence, *a principal plane of inertia with respect to a given pole is a principal plane of inertia with respect to any point of it if and only if it is a central principal plane of inertia*.

If $O \equiv C$, then the relation (14.1.29') takes the form (11.2.37), where the pseudokinetic energy is given by

$$T^C = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{K}^C = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{K}'_C = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_C\boldsymbol{\omega}). \tag{14.1.35}$$

If we eliminate the pseudomoment of momentum $\mathbf{K}^O$ and the pseudokinetic energy T^O between the relations (14.1.23'), (14.1.29') and (14.1.30') and take into account (14.1.17'), then we find the remarkable relation

$$T' = \frac{1}{2}\mathbf{K}'_O \cdot \boldsymbol{\omega} + \frac{1}{2}M\mathbf{v}'_C \cdot (\mathbf{v}'_O - \boldsymbol{\omega} \times \mathbf{r}'_O) = \frac{1}{2}\mathbf{K}'_O \cdot \boldsymbol{\omega} + \frac{1}{2}M\mathbf{v}'_C \cdot (\mathbf{v}'_C - \boldsymbol{\omega} \times \boldsymbol{\rho}') \tag{14.1.36}$$

analogous to the relation (14.1.17').

Introducing *the transportation kinetic energy of the rigid solid* $\mathscr{S}$, in the non-inertial frame of reference $\mathscr{R}$, with respect to the inertial frame $\mathscr{R}'$, given by (11.2.41), we notice that

$$T' = T'(\mathrm{tr}\, O), \tag{14.1.29''}$$

corresponding to the formula (14.1.29'). In the case of the rigid solid we have $T = 0$, *the comoment* given by the kinematic torsor with the torsor of momenta vanishing too ($(\mathscr{T}, \tau) = 0$); in this case, the formula (11.2.43) leads to

$$T' = \frac{1}{2}(\mathscr{T}', \tau'). \tag{14.1.29'''}$$

In case of the free rigid solid $\mathscr{S}$, the elementary work of the given internal forces vanishes. *The elementary work of the given external forces* is expressed in the form (we assume that a system of n given forces is acting)

$$\mathrm{d}W' = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathrm{d}\mathbf{r}_i' = \left(\sum_{i=1}^{n} \mathbf{F}_i \right) \cdot \mathrm{d}\mathbf{r}_O' + \sum_{i=1}^{n} (\boldsymbol{\omega} \times \mathbf{r}_i) \cdot \mathbf{F}_i \mathrm{d}t ,$$

where we have used the formula (11.2.10') and we have made $\mathrm{d}\mathbf{r}_i = \mathbf{0}$ and $\mathbf{v}_i = \mathbf{0}$. Introducing the torsor of the given external forces ($\tau_O\{\mathbf{F}_i\} = \{\mathbf{R}, \mathbf{M}_O\}$) and taking into account the properties of the triple scalar product, we can write

$$\mathrm{d}W' = \mathbf{R} \cdot \mathrm{d}\mathbf{r}_O' + \mathbf{M}_O \cdot \boldsymbol{\omega}\mathrm{d}t , \tag{14.1.37}$$

obtaining thus the expression of the elementary work of the given external forces in case of a free rigid solid; we get the same result if we make $\mathrm{d}W_{\mathrm{int}}' = \mathrm{d}W = \mathrm{d}W_{\mathrm{int}} = 0$ and equate to zero the quantities connected to the constraint forces in the relation (11.2.31). For the power of the given forces we have (corresponding, analogously, to the formula (11.2.31'))

$$P' = \mathbf{R} \cdot \mathbf{v}_O' + \mathbf{M}_O \cdot \boldsymbol{\omega} , \tag{14.1.37'}$$

which allows to state

Theorem 14.1.8 *The power of the given external forces which act upon a free rigid solid, with respect to a given frame of reference, is equal to the power of the torsor of these forces at an arbitrary pole, rigidly linked to the rigid solid, with respect to the same frame.*

Introducing the kinematic torsor (11.2.33), we can express the relation (14.1.37') also in the form

$$P' = (\mathscr{T}', \tau\{\mathbf{F}_i\}) . \tag{14.1.37''}$$

14.1.1.7 General Theorems

Starting from the theorem of momentum written in the form (12.1.33) for a continuous mechanical system and using the expression (14.1.21'), (14.1.21'') for the momentum of the rigid solid $\mathscr{S}$, we may write

$$\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(M\mathbf{v}_C') = M\mathbf{a}_O' + M\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + M\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \mathbf{R} ; \tag{14.1.38}$$

this result can be expressed also in the form ($\mathbf{R}' = \mathbf{R}$, the resultant being invariant to a change of pole)

$$\mathbf{F}_t^{(c)} + \mathbf{R} = \mathbf{0} , \tag{14.1.38'}$$

where we have introduced *the transportation complementary force*

$$\mathbf{F}_t^{(c)} = -M[\mathbf{a}_O' + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})], \tag{14.1.38''}$$

corresponding to the mass centre. The above formulae can be obtained also from the formulae (11.2.11'), (11.2.12), where we make $\mathbf{H} = \mathbf{0}$, $\partial \boldsymbol{\rho} / \partial t = \mathbf{0}$, $\mathbf{F}_C^{(C)} = \mathbf{0}$, $\bar{\mathbf{R}} = \mathbf{0}$. Introducing the acceleration of the mass centre with respect to the inertial frame of reference $\mathscr{R}'$, it results

$$M\mathbf{a}'_C = \mathbf{R} . \tag{14.1.39}$$

We can thus state:

Theorem 14.1.9 (*theorem of momentum*) *The derivative with respect to time of the momentum of a free rigid solid, in an inertial frame of reference, is equal to the resultant of the given external forces which act upon that solid.*

Theorem 14.1.9' (*theorem of motion of the mass centre*) *The centre of mass of a free rigid solid moves, with respect to an inertial frame of reference, as a free particle at which would be concentrated the whole mass of the solid and which would be acted upon by the resultant of the given external forces.*

Analogously, we use the theorem of moment of momentum written in the form (12.1.33') for a continuous mechanical system, the moment of momentum of the rigid solid $\mathscr{S}$ being given by (14.1.23'); we obtain

$$\frac{\mathrm{d}\mathbf{K}'_{O'}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{K}^O}{\mathrm{d}t} + M[\mathbf{v}'_O \times \mathbf{v}'_C + \mathbf{r}'_O \times \mathbf{a}'_C - \mathbf{a}'_O \times \boldsymbol{\rho} - \mathbf{v}'_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] = \mathbf{M}_{O'},$$

the moment of the given external forces being $\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{r}'_O \times \mathbf{R}$. Taking into account (14.1.17') and (14.1.39), it results

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) + \frac{\mathrm{d}\mathbf{K}^O}{\mathrm{d}t} = \mathbf{M}_O , \tag{14.1.40}$$

a relation of the form (11.2.18), where we make $\overline{\mathbf{M}}_O = \mathbf{0}$. We state

Theorem 14.1.10 (*theorem of moment of momentum*) *The derivative with respect to time of the pseudomoment of momentum of a free rigid solid with respect to an arbitrary pole O, rigidly connected to the solid, in an inertial frame of reference, is equal to the resultant moment of the given external forces which act upon this solid, with respect to that pole, from which we subtract the dynamic moment, in the inertial frame of the considered pole, translated at the centre of mass of the rigid solid, at which is assumed to be concentrated the whole mass of this solid, taken with respect to the pole O.*

The relation (14.1.24') allows also to write

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O . \tag{14.1.40'}$$

If the non-inertial frame of reference $\mathscr{R}$ is of Koenig type ($\boldsymbol{\omega} = \mathbf{0}$), then the rigid solid $\mathscr{S}$ has a motion of translation and we remain with the relation

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) = \mathbf{M}_O. \tag{14.1.41}$$

If we have $O \equiv C$ (hence, $\boldsymbol{\rho} = \mathbf{0}$), then it results that the non-inertial frame of reference is a Koenig frame; we can have a motion of translation of the free rigid solid if and only if $\mathbf{M}_C = \mathbf{0}$. We notice that *we can have* $\mathbf{M}_O = \mathbf{0}$ even if the pole O does not coincide with the centre of mass, *if this pole has a uniform and rectilinear motion with respect to an inertial frame or if the support of the acceleration* $\mathbf{a}'_O$ *of the pole* O *with respect to an inertial frame passes through the mass centre* C. But if $O \equiv C$ with $\boldsymbol{\omega} \neq \mathbf{0}$, then the theorem of moment of momentum becomes

$$\frac{\mathrm{d}K^C}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{I}_C(\mathscr{S})\boldsymbol{\omega}] = \mathbf{M}_C, \tag{14.1.41'}$$

without any loss of generality. We obtain the same result starting from the relation (11.2.24), where we make $\mathbf{K}_C = \mathbf{0}$ and $\mathbf{M}_C = \mathbf{0}$. Taking into account the relation (14.1.26') too, we state, as well,

Theorem 14.1.10' *The derivative with respect to time of the moment of momentum of a free rigid solid, with respect to its mass centre, in an inertial frame of reference, with respect to this frame, is equal to the resultant moment of the given external forces which act upon this solid, with respect to the mass centre.*

Starting from the theorem of kinetic energy stated in, Sect. 11.2.2.2, we obtain

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = \mathbf{R} \cdot \mathbf{v}'_O + \mathbf{M}_O \cdot \boldsymbol{\omega}. \tag{14.1.42}$$

There results

Theorem 14.1.11 (*theorem of kinetic energy*). *The derivative with respect to time of the kinetic energy of a free rigid solid, in an inertial frame of reference, is equal to the power of the given external forces considered to be applied at an arbitrary pole, rigidly linked to the solid, with respect to the given frame.*

Taking into account (14.1.29'), we are led to

$$\frac{\mathrm{d}T^O}{\mathrm{d}t} + M\mathbf{v}'_O \cdot \mathbf{a}'_O + M[(\mathbf{a}'_O, \boldsymbol{\omega}, \boldsymbol{\rho}) + (\mathbf{v}'_O, \dot{\boldsymbol{\omega}}, \boldsymbol{\rho}) + (\mathbf{v}'_O, \boldsymbol{\omega}, \boldsymbol{\omega} \times \boldsymbol{\rho})] = \mathbf{R} \cdot \mathbf{v}'_O + \mathbf{M}_O \cdot \boldsymbol{\omega},$$

and the formula (14.1.38) allows to write the theorem of kinetic energy in the form

$$\frac{\mathrm{d}T^O}{\mathrm{d}t} + M(\mathbf{a}'_O, \boldsymbol{\omega}, \boldsymbol{\rho}) = \mathbf{M}_O \cdot \boldsymbol{\omega}, \tag{14.1.43}$$

corresponding to the relation (11.2.34); starting from the relation (14.1.30'), we are led to

$$\frac{\mathrm{d}T^O}{\mathrm{d}t} = \frac{1}{2}[\mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega})] \cdot \boldsymbol{\omega} + \frac{1}{2}(\mathbf{I}_O\boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}} = \frac{1}{2}(\mathbf{I}_O\dot{\boldsymbol{\omega}}) \cdot \boldsymbol{\omega} + \frac{1}{2}(\mathbf{I}_O\boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}},$$

wherefrom, taking into account (14.1.31), it results

$$\frac{\mathrm{d}T^O}{\mathrm{d}t} = (\mathbf{I}_O\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega} = (\mathbf{I}_O\boldsymbol{\omega})\cdot\dot{\boldsymbol{\omega}}. \tag{14.1.43'}$$

If we write the relation (14.1.43') in the form $\mathrm{d}T^O = \mathbf{K}^O \cdot \mathrm{d}\boldsymbol{\omega}$, then we notice that we have

$$\mathbf{K}^O = \operatorname{grad}_{\boldsymbol{\omega}} T^O \tag{14.1.44}$$

too, where $T^O = T^O(\boldsymbol{\omega})$. Analogously, starting from the relation (14.1.29'), considering successively that $T' = T'(\mathbf{v}'_O)$, $T' = T'(\boldsymbol{\omega})$ and taking into account (14.1.21'), (14.1.26), we may write $\mathrm{d}T' = \mathbf{H}'\cdot\mathrm{d}\mathbf{v}'_O$ and $\mathrm{d}T' = \mathbf{K}'_O\cdot\mathrm{d}\boldsymbol{\omega}$, respectively, so that

$$\mathbf{H}' = \operatorname{grad}_{\mathbf{v}'_O} T', \quad \mathbf{K}'_O = \operatorname{grad}_{\boldsymbol{\omega}} T'. \tag{14.1.44'}$$

Observing that the kinetic energy T' is a homogeneous function of second degree with respect to the components of the vectors $\mathbf{v}'_O$ and $\boldsymbol{\omega}$, Euler's theorem allows to write

$$2T' = \mathbf{v}'_O\cdot\operatorname{grad}_{\mathbf{v}'_O} T' + \boldsymbol{\omega}\cdot\operatorname{grad}_{\boldsymbol{\omega}} T' = \mathbf{v}'_O\cdot\mathbf{H}' + \boldsymbol{\omega}\cdot\mathbf{K}'_O, \tag{14.1.44''}$$

finding thus again the relation (14.1.29''').

If $O \equiv C$, then the relation (14.1.43) becomes

$$\frac{\mathrm{d}T^C}{\mathrm{d}t} = [\mathbf{I}_C(\mathscr{S})\boldsymbol{\omega}]\cdot\dot{\boldsymbol{\omega}} = [\mathbf{I}_C(\mathscr{S})\dot{\boldsymbol{\omega}}]\cdot\boldsymbol{\omega} = \mathbf{M}_C\cdot\boldsymbol{\omega}, \tag{14.1.43''}$$

without any loss of generality concerning the result.

A scalar product of the formula (14.1.41') by $\boldsymbol{\omega}$ leads to the formula (14.1.43''); obviously, by a scalar product of the formula (14.1.40) by $\boldsymbol{\omega}$, we get, analogously, the relation (14.1.43). Because one can obtain the formula (14.1.43) taking into account the theorems of momentum and of motion of the mass centre, respectively, we can state that *the theorem of kinetic energy is obtained as a linear combination of the theorems of momentum and moment of momentum*; consequently, it represents a supplementary non-independent equation of motion. However, the theorem of kinetic energy is important by itself; it can be used directly in various applications (e.g., in the study of motion of the rigid solid with a fixed axis). One can also show that, starting from the theorem of kinetic energy, we find again the theorems of momentum and moment of momentum, because the relation (14.1.42) must hold for any rototranslation $\mathscr{T}'_O \equiv (\boldsymbol{\omega}, \mathbf{v}'_O)$, according to the relation (14.1.29''').

14.1.1.8 General Equations of Motion

The theorems of momentum and moment of momentum constitute together the theorem of torsor for a free rigid solid $\mathscr{S}$; these theorems lead to two vector differential equations or to six scalar differential equations and describe, entirely, the motion of the rigid solid, which has six degrees of freedom. Indeed, taking into account the principle of variation of the kinetic torsor for a continuous mechanical system, which can be applied to an arbitrary subsystem $S \subset \mathscr{S}$, and observing that if we know the motion of the subsystem S, then we know the motion of the whole rigid solid $\mathscr{S}$, we obtain the same result. In vector form, these equations are

$$M[\mathbf{a}'_O + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})] = \mathbf{R}, \tag{14.1.45}$$

$$M\boldsymbol{\rho} \times \mathbf{a}'_O + \mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O, \tag{14.1.46}$$

while, in components, we obtain

$$M\left(a'_{Oi} + \in_{ijk} \dot{\omega}_j \rho_k + \in_{ijk} \in_{klm} \omega_j \omega_l \rho_m\right) = R_i, \quad i = 1,2,3, \tag{14.1.45'}$$

$$\in_{ijk} M\rho_j a'_{Ok} + I_{ij} \dot{\omega}_j + \in_{ijk} I_{kl} \omega_j \omega_l = M_{Oi}, \quad i = 1,2,3; \tag{14.1.46'}$$

these equations can be written also in the form

$$M\left(a'_{Oi} + \in_{ijk} \dot{\omega}_j \rho_k + 2\omega_j \omega_{[i} \rho_{j]}\right) = R_i, \quad i = 1,2,3, \tag{14.1.45''}$$

$$\in_{ijk} M\rho_j a'_{Ok} + I_{kl}\left(\delta_{ik} \dot{\omega}_l + \in_{ijk} \omega_j \omega_l\right) = M_{Oi}, \quad i = 1,2,3. \tag{14.1.46''}$$

With respect to the principal axes of inertia, we get

$$\begin{aligned} M(\rho_2 a'_{O3} - \rho_3 a'_{O2}) + I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 &= M_{O1}, \\ M(\rho_3 a'_{O1} - \rho_1 a'_{O3}) + I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_3 \omega_1 &= M_{O2}, \\ M(\rho_1 a'_{O2} - \rho_2 a'_{O1}) + I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 &= M_{O3}, \end{aligned} \tag{14.1.46'''}$$

where $I_1 \geq I_2 \geq I_3$ are the principal moments of inertia relative to the pole O.

Assuming that $O \equiv C$, the equations of motion of the free rigid solid become

$$M\mathbf{a}'_C = M\frac{\mathrm{d}\mathbf{v}'_C}{\mathrm{d}t} = \mathbf{R}, \tag{14.1.47}$$

$$I_C \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_C \boldsymbol{\omega}) = \mathbf{M}_C; \tag{14.1.48}$$

with respect to the central principal axes of inertia, we get

$$Ma'_{Ci} = R_i, \quad i = 1,2,3, \tag{14.1.47'}$$

$$\begin{aligned} I_1^{(C)} \dot{\omega}_1 + \left(I_3^{(C)} - I_2^{(C)}\right)\omega_2 \omega_3 &= M_{C1}, \\ I_2^{(C)} \dot{\omega}_2 + \left(I_1^{(C)} - I_3^{(C)}\right)\omega_3 \omega_1 &= M_{C2}, \\ I_3^{(C)} \dot{\omega}_3 + \left(I_2^{(C)} - I_1^{(C)}\right)\omega_1 \omega_2 &= M_{C3}, \end{aligned} \tag{14.1.48'}$$

where $I_1^{(C)} \geq I_2^{(C)} \geq I_3^{(C)}$ are the central principal moments of inertia. The equations (14.1.46''') take the form (14.1.48') with respect to an arbitrary point O of the rigid solid too, if this point has a uniform and rectilinear motion ($\mathbf{a}'_O = \mathbf{0}$) with respect to an inertial frame of reference or if the support of the acceleration of this point passes at any time through the mass centre C; this form of the equations of the motion of rotation of the rigid solid $\mathscr{S}$ about a point of it is due to L. Euler.

We have seen that the theorem of kinetic energy is a linear consequence of the theorem of torsor; this theorem can be used to obtain a first integral of the system of equations of motion. In the non-inertial frame of reference $\mathscr{R}$, the relation (14.1.43) takes the form

$$\frac{1}{2}(I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3) + M \in_{ijk} a'_{Oi}\omega_j\rho_k = M_{O1}\omega_1 + M_{O2}\omega_2 + M_{O3}\omega_3, \tag{14.1.49}$$

with respect to the principal axes of inertia; if $O \equiv C$, then the relation (14.1.49) becomes (corresponding to the relation (14.1.43'''))

$$\frac{1}{2}\left(I_1^{(C)}\omega_1\dot{\omega}_1 + I_2^{(C)}\omega_2\dot{\omega}_2 + I_3^{(C)}\omega_3\dot{\omega}_3\right) = M_{C1}\omega_1 + M_{C2}\omega_2 + M_{C3}\omega_3. \tag{14.1.49'}$$

Because all the above equations are written with respect to the non-inertial frame of reference $\mathscr{R}$, rigidly linked to the solid, the acceleration of the pole of this frame must be expressed analogously. Hence,

$$\mathbf{a}'_O = \frac{\mathrm{d}\mathbf{v}'_O}{\mathrm{d}t} = \dot{\mathbf{v}}_O + \boldsymbol{\omega}\times\mathbf{v}'_O, \quad \mathbf{a}'_C = \frac{\mathrm{d}\mathbf{v}'_C}{\mathrm{d}t} = \dot{\mathbf{v}}_C + \boldsymbol{\omega}\times\mathbf{v}'_C, \tag{14.1.50}$$

where we have put into evidence the derivatives $\dot{\mathbf{v}}'_O = \partial\mathbf{v}'_O/\partial t$ and $\dot{\mathbf{v}}'_C = \partial\mathbf{v}'_C/\partial t$ of the velocities $\mathbf{v}'_O$ and $\mathbf{v}'_C$ with respect to time, respectively. The equations (14.1.45), (14.1.46) take the form

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega}\times(\mathbf{v}'_O + \boldsymbol{\omega}\times\boldsymbol{\rho}) + \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}] = \mathbf{R}, \tag{14.1.51}$$

$$M\boldsymbol{\rho}\times(\dot{\mathbf{v}}'_O + \boldsymbol{\omega}\times\mathbf{v}'_O) + \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O, \tag{14.1.51'}$$

while the equation (14.1.47) becomes

$$M(\dot{\mathbf{v}}'_C + \boldsymbol{\omega}\times\mathbf{v}'_C) = \mathbf{R}. \tag{14.1.52}$$

From the above considerations, we notice that the motion of the free rigid solid does not depend on the action of each external force, taken separately, but only on the torsor of all the forces with respect to an arbitrary point of it (the pole of the non-inertial frame of reference $\mathscr{R}$); hence, two equivalent systems of forces (modelled as sliding vectors) applied upon a free rigid solid lead to the same motion of it. In the modelling as a particle of the rigid solid, its motion is reduced to the motion of translation of its mass centre C acted upon by the resultant $\mathbf{R}$, the number of degrees of freedom being reduced from six to three. If one cannot neglect the rotation of the solid about the centre C, then we introduce also the action of the resultant moment $\mathbf{M}_C$, which specifies the motion. Obviously, to can obtain such a decomposition, we must assume that $\mathbf{R} = \mathbf{R}(\boldsymbol{\rho}', \mathbf{v}'_C; t)$, non-depending on $\boldsymbol{\omega}$ and on Euler's angles ψ, θ, φ; we can use thus the equation (14.1.47) as a Newton equation of motion for a particle.

In general, we have

$$\begin{aligned} R_i &= R_i(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t), \\ M_{Oi} &= M_{Oi}(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t), \quad i = 1,2,3, \end{aligned}$$

the rigid solid being *non-autonomous*; if the time does not intervene explicitly, then the solid is *autonomous* (or *dynamic*). The 12 unknown scalar functions (the co-ordinates $x'_{Oi} = x'_{Oi}(t)$ of the pole O, the components $v'_{Oi} = v'_{Oi}(t)$ of the velocity of the pole O with respect to the inertial frame of reference $\mathcal{R}'$, Euler's angles $\psi = \psi(t)$, $\theta = \theta(t)$, $\varphi = \varphi(t)$ and the components $\omega_i = \omega_i(t)$, $i = 1,2,3$, of the instantaneous angular velocity vector of the rigid solid with respect to the non-inertial frame of reference $\overline{\mathcal{R}}$) are determined by the system of first order differential equations (14.1.45"), (14.1.46") and (14.1.15), written in the normal form (the coefficients of the equations (14.1.53), (14.1.53') are constant and we can solve this linear system with respect to the derivatives of first order)

$$M \frac{\mathrm{d}v'_{Oi}}{\mathrm{d}t} + M \in_{ijk} \rho_k \frac{\mathrm{d}\omega_j}{\mathrm{d}t} = \widetilde{R}_i = R_i - 2M\omega_j \omega_{[i}\rho_{j]}, \quad i = 1,2,3, \tag{14.1.53}$$

$$M \in_{ijk} \rho_j \frac{\mathrm{d}v'_{Ok}}{\mathrm{d}t} + I_{ij} \frac{\mathrm{d}\omega_j}{\mathrm{d}t} = \widetilde{M}_{Oi} = M_{Oi} - \in_{ijk} I_{kl}\omega_j\omega_l, \quad i = 1,2,3, \tag{14.1.53'}$$

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}t} &= f_1(\theta, \varphi, \omega_1, \omega_2) = (\omega_1 \sin\varphi + \omega_2 \cos\varphi)\operatorname{cosec}\theta, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= f_2(\varphi, \omega_1, \omega_2) = \omega_1 \cos\varphi - \omega_2 \sin\varphi, \\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} &= f_3(\theta, \varphi, \omega_1, \omega_2, \omega_3) = \omega_3 - (\omega_1 \sin\varphi + \omega_2 \cos\varphi)\cot\theta, \end{aligned} \tag{14.1.53"}$$

to which we associate the equations

$$\frac{\mathrm{d}x'_{Oi}}{\mathrm{d}t} = v'_{Oi} = v'_{Oi}(x'_{O1}, x'_{O2}, x'_{O3}; t), \quad i = 1,2,3, \tag{14.1.53'''}$$

the derivatives being taken in the inertial frame of reference $\mathcal{R}'$, as well as the initial conditions (at the moment $t = t_0$) of Cauchy type

$$\begin{aligned} x'_{Oi}(t_0) = x'^0_{Oi}, \quad v'_{Oi}(t_0) = v'^0_{Oi}, \quad \psi(t_0) = \psi^0, \quad \theta(t_0) = \theta^0, \\ \varphi(t_0) = \varphi^0, \quad \omega_i(t_0) = \omega_i^0, \quad i = 1,2,3. \end{aligned} \tag{14.1.54}$$

For the boundary value problem thus formulated, one can show

Theorem 14.1.12 (*of existence and uniqueness*; *Cauchy-Lipschitz*). *If the functions* $\widetilde{R}_i$, $\widetilde{M}_{Oi}$, f_i *and* v'_{Oi}, $i = 1,2,3$, *are continuous on the* 13-*dimensional interval* $\mathscr{D}$, *specified by*

$$\begin{gathered}
x'^0_{Oi} - X^0_i \le x'_{Oi} \le x'^0_{Oi} + X^0_i\,,\quad v'^0_{Oi} - V^0_i \le v'_{Oi} \le v'^0_{Oi} + V^0_i\,,\\
\psi^0 - \Psi^0 \le \psi \le \psi^0 + \Psi^0\,,\quad \theta^0 - \Theta^0 \le \theta \le \theta^0 + \Theta^0\,,\\
0 \notin \left[\theta^0 - \Theta^0, \theta^0 + \Theta^0\right],\quad \varphi^0 - \Phi^0 \le \varphi \le \varphi^0 + \Phi^0\,,\\
\omega^0_i - \Omega^0_i \le \omega_i \le \omega^0_i + \Omega^0_i\,,\quad t_0 - T_0 \le t \le t_0 + T_0\,,\\
X^0_i, V^0_i, \Psi^0, \Theta^0, \Omega^0_i, T_0 = \text{const}\,,\quad i = 1,2,3\,,
\end{gathered}$$

and defined on the space Cartesian product of the phase space (*of canonical co-ordinates* $x'_{O1}, x'_{O2}, x'_{O3}$, ψ, θ, φ, $Mv'_{O1}, Mv'_{O2}, Mv'_{O3}, M\omega_1, M\omega_2, M\omega_3$) *by the time space* (*of co-ordinate* t), *and if Lipschitz's conditions*

$$\begin{gathered}
\Big|\widetilde{R}_i\left(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t\right)\\
-\widetilde{R}_i\left(\overline{x}'_{O1}, \overline{x}'_{O2}, \overline{x}'_{O3}, \overline{v}'_{O1}, \overline{v}'_{O2}, \overline{v}'_{O3}, \overline{\psi}, \overline{\theta}, \overline{\varphi}, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3; t\right)\Big|\\
\le \frac{M}{\mathscr{T}}\left\{\sum_{j=1}^{3}\left[\frac{1}{\tau}\left|x'_{Oj} - \overline{x}'_{Oj}\right| + \left|v'_{Oj} - \overline{v}'_{Oj}\right| + \lambda\left|\omega_j - \overline{\omega}_j\right|\right] + \frac{\lambda}{\tau}\left(\left|\psi - \overline{\psi}\right| + \left|\theta - \overline{\theta}\right| + \left|\varphi - \overline{\varphi}\right|\right)\right\},\\
\Big|\widetilde{M}_{Oi}\left(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t\right)\\
-\widetilde{M}_{Oi}\left(\overline{x}'_{O1}, \overline{x}'_{O2}, \overline{x}'_{O3}, \overline{v}'_{O1}, \overline{v}'_{O2}, \overline{v}'_{O3}, \overline{\psi}, \overline{\theta}, \overline{\varphi}, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3; t\right)\Big|\\
\le M\frac{\mathscr{L}}{\mathscr{T}}\left\{\sum_{j=1}^{3}\left[\frac{1}{\tau}\left|x'_{Oj} - \overline{x}'_{Oj}\right| + \left|v'_{Oj} - \overline{v}'_{Oj}\right| + \lambda\left|\omega_j - \overline{\omega}_j\right|\right] + \frac{\lambda}{\tau}\left(\left|\psi - \overline{\psi}\right| + \left|\theta - \overline{\theta}\right| + \left|\varphi - \overline{\varphi}\right|\right)\right\},\\
\left|f_i\left(\theta, \varphi, \omega_1, \omega_2, \omega_3\right) - f_i\left(\overline{\theta}, \overline{\varphi}, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\right)\right| \le \frac{1}{\mathscr{T}}\left(\tau\sum_{j=1}^{3}\left|\omega_j - \overline{\omega}_j\right| + \left|\theta - \overline{\theta}\right| + \left|\varphi - \overline{\varphi}\right|\right),\\
\left|v'_{Oi}\left(x'_{O1}, x'_{O2}, x'_{O3}; t\right) - v'_{Oi}\left(\overline{x}'_{O1}\overline{x}'_{O2}, \overline{x}'_{O3}; t\right)\right| \le \frac{1}{\mathscr{T}}\sum_{j=1}^{3}\left|x'_{Oj} - \overline{x}'_{Oj}\right|,
\end{gathered}$$

where $\mathscr{T}$ *and* $\mathscr{L}$ *are positive* (*temporal and spatial, respectively*) *constants, independent on* x'_{Oi}, v'_{Oi}, ψ, θ, φ, ω_i *and* t, *while* τ *and* λ *are* (*temporal and spatial, respectively*) *constants equal to unity, for* $i = 1,2,3$, *then there exists a unique solution* $x'_{Oi} = x'_{Oi}(t)$, $v'_{Oi} = v'_{Oi}(t)$, $\psi = \psi(t), \theta = \theta(t), \varphi = \varphi(t)$, $\omega_i = \omega_i(t)$, $i = 1,2,3$, *of the system* (14.1.53)-(14.1.53"), *which satisfies the initial conditions* (14.1.54) *and is defined on the interval* $t_0 - T \le t \le t_0 + T$, *where*

$$T \le \min\left[T_0, \frac{X^0_i}{\mathscr{V}}, \tau\left(\frac{V^0_i}{\mathscr{V}}\right), \lambda\left(\frac{\Psi^0}{\mathscr{V}}\right), \lambda\left(\frac{\Theta^0}{\mathscr{V}}\right), \lambda\left(\frac{\Phi^0}{\mathscr{V}}\right), \lambda\tau\left(\frac{\Omega^0_i}{\mathscr{V}}\right), \mathscr{T}\right],$$

$$\mathscr{V} = \max\left[\tau\left(\frac{\widetilde{R}_i}{M}\right), \frac{\tau}{\lambda}\left(\frac{\widetilde{M}_{Oi}}{M}\right), \lambda f_i, v'_{Oi} \quad \text{in } \mathscr{D}\right].$$

According to *the theorem of Peano*, the continuity of the functions $\widetilde{R}_i$, $\widetilde{M}_{Oi}$, f_i and v'_{Oi} on the interval $\mathscr{D}$ ensures the existence of the solution; for its uniqueness, *the conditions of Lipschitz* must be fulfilled too. As a matter of fact, the conditions in the Theorem 14.1.12 are *sufficient conditions* of existence and uniqueness, *which are not also necessary*. As in the case of only one particle (see Chap. 6, Subsec. 1.2.1) or of a discrete mechanical system (see Sect. 11.1.1.5), we can make a prolongation of the solution for $t \in [t_1, t_2]$, outside the interval $[t_0 - T, t_0 + T]$ (the time t_0 must not be necessarily the initial one, but can be a time arbitrarily chosen). As well, we can state a *theorem on the continuous dependence of the solution on a parameter* (analogous to the Theorems 6.1.3 and 11.1.2), *a theorem* of Poincaré type *on the analytical dependence of the solution on a parameter* (analogous to the Theorems 6.1.4 and 11.1.3) and *a theorem on the derivability of the solutions* (analogous to the Theorems 6.1.5 and 11.1.4). Analogously, one can prove theorems concerning the continuous dependence of the solution on the initial conditions or on several parameters.

One can use also another representation of the motion than that based on Euler's angles; thus, if

$$\begin{aligned} R_i &= R_i\,(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \alpha_1, \alpha_2, \alpha_3, \omega_1, \omega_2, \omega_3; t), \\ M_{Oi} &= M_{Oi}\,(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \alpha_1, \alpha_2, \alpha_3, \omega_1, \omega_2, \omega_3; t), \end{aligned}$$

then we replace the subsystem (14.1.53") by the subsystem (5.2.37'), written in the form

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = -\in_{ijk} \omega_j \alpha_k, \quad i = 1,2,3, \tag{14.1.54}$$

passing to Euler's angles by means of the relations (14.1.15), (14.1.16). The theorem of existence and uniqueness is stated analogously for the system of equations (14.1.53), (14.1.53'), (14.1.53'"), (14.1.54).

If, in particular, the given functions R_i and M_{Oi}, $i = 1,2,3$, depend on a smaller number of variables, then the system of differential equations (14.1.53)-(14.1.53'") can be decomposed in subsystems. Thus, if

$$\begin{aligned} R_i &= R_i\,(v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t), \\ M_{Oi} &= M_{Oi}\,(v'_{O1}, v'_{O2}, v'_{O3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t), \end{aligned}$$

then the system (14.1.53)-(14.1.53'") is decomposed in the subsystem (14.1.53'") and the subsystem (14.1.53)-(14.1.53"), while if

$$\begin{aligned} R_i &= R_i\,(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \omega_1, \omega_2, \omega_3; t), \\ M_{Oi} &= M_{Oi}\,(x'_{O1}, x'_{O2}, x'_{O3}, v'_{O1}, v'_{O2}, v'_{O3}, \omega_1, \omega_2, \omega_3; t), \end{aligned}$$

then this system is decomposed in the subsystem (14.1.53), (14.1.53'), (14.1.53''') and the subsystem (14.1.53''). If

$$R_i = R_i\left(v'_{O1}, v'_{O2}, v'_{O3}, \omega_1, \omega_2, \omega_3; t\right),$$
$$M_{Oi} = M_{Oi}\left(v'_{O1}, v'_{O2}, v'_{O3}, \omega_1, \omega_2, \omega_3; t\right),$$

then the system (14.1.53)-(14.1.53''') is decomposed in the subsystem (14.1.53), (14.1.53'), which specifies the velocity of the rigid solid, the subsystem (14.1.53'''), which determines the motion of translation of the solid, and the subsystem (14.1.53''), which puts in evidence its motion of rotation. Obviously, to each subsystem we add initial conditions of Cauchy type in the form (14.1.54). We can state theorems of existence and uniqueness for each of these subsystems.

We notice that one can replace the equations (14.1.53') by the equations (14.1.46'''), passing to the principal axes of inertia.

As well, without any loss of generality, we can use the equations (14.1.47'), (14.1.48'), (14.1.53'') in the form

$$\frac{\mathrm{d}v'_{Ci}}{\mathrm{d}t} = \frac{1}{M} R_i, \quad i = 1,2,3, \tag{14.1.55}$$

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{1}{I_1^{(C)}} \widetilde{M}_{C1} = \frac{1}{I_1^{(C)}}\left[M_{C1} + \left(I_2^{(C)} - I_3^{(C)}\right)\omega_2\omega_3\right],$$
$$\frac{\mathrm{d}\omega_2}{\mathrm{d}t} = \frac{1}{I_2^{(C)}} \widetilde{M}_{C2} = \frac{1}{I_2^{(C)}}\left[M_{C2} + \left(I_3^{(C)} - I_1^{(C)}\right)\omega_3\omega_1\right], \tag{14.1.55'}$$
$$\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = \frac{1}{I_3^{(C)}} \widetilde{M}_{C3} = \frac{1}{I_3^{(C)}}\left[M_{C3} + \left(I_1^{(C)} - I_2^{(C)}\right)\omega_1\omega_2\right],$$

$$\frac{\mathrm{d}\rho'_i}{\mathrm{d}t} = v'_{Ci}, \quad i = 1,2,3, \tag{14.1.55''}$$

to which we associate the equations (14.1.53''); thus, the importance of the mass centre C is put in evidence. Observing that

$$R_i = R_i\left(\rho'_1, \rho'_2, \rho'_3, v'_{C1}, v'_{C2}, v'_{C3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t\right),$$
$$M_{Ci} = M_{Ci}\left(\rho'_1, \rho'_2, \rho'_3, v'_{C1}, v'_{C2}, v'_{C3}, \psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t\right),$$
$$v'_{Ci} = v'_{Ci}\left(\rho'_1, \rho'_2, \rho'_3; t\right), \quad i = 1,2,3,$$

we can adapt the Theorem 14.1.12, correspondingly. As above, if R_i and M_{Oi}, $i = 1,2,3$, depend on a smaller number of variables, then the considered system of differential equations can be decomposed, conveniently, in subsystems. Thus, if

$$R_i = R_i\left(\rho'_1, \rho'_2, \rho'_3, v'_{C1}, v'_{C2}, v'_{C3}; t\right), \; M_{Ci} = M_{Ci}\left(\psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t\right), \; i = 1,2,3,$$

then the system (14.1.53''), (14.1.55)-(14.1.55'') is decomposed in the subsystem (14.1.55), (14.1.55''), which specifies the motion of the mass centre C with respect to the inertial frame of reference $\mathscr{R}'$ (as a single particle, *justifying the modelling of the rigid solid as a particle*), and the subsystem (14.1.53"), (14.1.55'), which determines the motion of rotation of the rigid solid about the centre C (the rotation of the frame $\mathscr{R}$ with respect to the frame $\overline{\mathscr{R}}$), considered as a fixed point; this allows to study the motion of the free rigid solid in two successive steps, representing an important simplification of calculation. To the whole system, as well as to each subsystem, we associate initial conditions of Cauchy type (at the moment $t = t_0$), of the form

$$\rho_i'(t_0) = \rho_i'^0, \quad v_{Ci}'(t_0) = v_{Ci}'^0, \quad \psi(t_0) = \psi^0, \quad \theta(t_0) = \theta^0,$$
$$\varphi(t_0) = \varphi^0, \quad \omega_i(t_0) = \omega_i^0, \quad i = 1,2,3. \tag{14.1.56}$$

For each of these subsystems one can state a theorem of existence and uniqueness. We notice also that the subsystem (14.1.55) corresponds to the theorem of momentum with respect to the frame $\overline{\mathscr{R}}$ (or to the frame $\mathscr{R}'$), while the subsystem (14.1.55') corresponds to the theorem of moment of momentum, with respect to the same frame $\overline{\mathscr{R}}$. In what follows, we suppose to be in the latter case of decomposition. We can determine first integrals for the first subsystem, obtaining conservation theorems by means of the considerations in Chap. 6, Subsec. 1.2.2; for the second subsystem, we use the study contained in Chap. 15, §1 for the rigid solid with a fixed point.

The equations of motion used above have been written with respect to the non-inertial frame of reference $\mathscr{R}$. Starting from the theorem of torsor written for a free rigid solid, by means of the relations (14.1.44'), and observing that $\mathbf{K}_{O'}' = \mathbf{K}_O' + \mathbf{r}_O' \times \mathbf{H}'$, we can write these equations with respect to the inertial frame $\mathscr{R}'$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T'}{\partial v_{Oi}'}\right) = R_i, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial T'}{\partial \omega_i'} + \in_{ijk} x_{Oj}' \frac{\partial T'}{\partial v_{Ok}'}\right] = M_{O'i}, \quad i = 1,2,3 \tag{14.1.57}$$

too. Passing from the frame $\mathscr{R}'$ to the frame $\mathscr{R}$, we find again the above results.

14.1.1.9 State of Rest

If $\mathbf{R} = \mathbf{0}$ on an interval of time $[t_0, t_1]$, t_0 being the initial moment, then the mass centre C has a uniform and rectilinear motion, so that $v_{Ci}'(t) = v_{Ci}'^0$, while $\rho_i'(t) = v_{Ci}'^0(t - t_0) + \rho_i'^0$, $i = 1,2,3$, for $t \in [t_0, t_1]$; in particular, if $v_{Ci}'^0 = 0$, $i = 1,2,3$, hence if the mass centre C is at rest with respect to the inertial frame of reference $\mathscr{R}'$ at a given moment t (e.g., at the initial moment), then this point remains at rest in the whole interval $[t_0, t_1]$. If $\mathbf{M}_C = \mathbf{0}$ in the time interval $[t_0, t_1]$, then the solution of the subsystem of equations (14.1.55') corresponds to an Euler-Poinsot motion of the free rigid solid (particular case of rotation of a rigid solid about the mass centre C, considered to be fixed; see Sect. 15.1.2), which depends of the initial condition $\boldsymbol{\omega}(t_0) = \boldsymbol{\omega}^0$; if, in particular, $\omega_i^0 = 0$, $i = 1,2,3$, hence if the free rigid solid (the frame $\mathscr{R}$ too) does not rotate with respect to the frame $\overline{\mathscr{R}}$ at a moment t (e.g., the initial moment), it will not rotate in the whole interval of time $[t_0, t_1]$. We can thus state

Theorem 14.1.13 *A free rigid solid subjected to the action of a system of given external forces of null torsor has an inertial motion* (*a uniform rectilinear motion of translation, with respect to an inertial frame of reference, associated to an Euler-Poinsot motion with respect to the mass centre, considered as a fixed point*).

This result can be considered also as a *principle*, generalizing *the principle of inertia* (from a free particle to a free rigid solid).

If, in particular, the kinematic torsor $\{\boldsymbol{\omega}^0, \mathbf{v}_C'^0\}$ vanishes, then we can state that *the free rigid solid subjected to the action of a given system of external forces of null torsor remains at rest* (*in equilibrium*) *with respect to an inertial frame of reference if that state has been its initial one*; we find thus again the Theorem 4.2.1, where we have a priori neglected the possibility of existence of homogeneous initial conditions. As well, *the free rigid solid subjected to the action of two given external forces, of equal magnitude and contrary directions, having the same support* (of null torsor), *conserves its initial state of rest or motion with respect to an inertial frame of reference*; we find thus again the modelling as sliding vectors of the forces which act upon a rigid solid (see Chap. 2, Subsec. 2.2.2). We mention also (what was stated before too) that *two systems of given external forces having the same torsor* (systems of equivalent forces) *have the same effect* (motion or rest), *with respect to an inertial frame of reference, upon a free rigid solid* (we apply the Theorem 14.1.13, the difference of the initial conditions vanishing); e.g., the field of gravity forces which act upon a free rigid solid can be replaced by a resultant (the own weight of the solid), applied at the mass centre C.

14.1.1.10 Principle of Virtual Work

The Theorem 4.1.8 allows to state

Theorem 14.1.14 (*theorem of virtual work*) *The necessary and sufficient condition of equilibrium of a free rigid solid, acted upon by a system of given external forces, is given by the vanishing of the virtual work of these forces for any system of virtual displacements of the respective solid.*

Using the form (4.1.58') of this theorem, which implies the introduction of virtual velocities with respect to the frame of reference $\mathscr{R}'$, of the form

$$\mathbf{v}_i^* = \mathbf{v}_O^* + \boldsymbol{\omega}^* \times \mathbf{r}_i\,, \quad i = 1,2,3\,, \tag{14.1.58}$$

corresponding to Euler's formula (5.2.3), we can write

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i^* = \sum_{i=1}^{n} \mathbf{F}_i \cdot (\mathbf{v}_O^* + \boldsymbol{\omega}^* \times \mathbf{r}_i)$$
$$= \mathbf{v}_O^* \cdot \sum_{i=1}^{n} \mathbf{F}_i + \boldsymbol{\omega}^* \cdot \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \mathbf{v}_O^* \cdot \mathbf{R} + \boldsymbol{\omega}^* \cdot \mathbf{M}_O = 0\,.$$

Taking into account that the virtual quantities $\mathbf{v}_O^*$ and $\boldsymbol{\omega}^*$ are arbitrary, it results that the torsor of the external forces at the pole O (in this case, the frame $\mathscr{R}$ is inertial too) must be equal to zero for equilibrium, hence the same result as above. Obviously, as in the case considered in Chap. 4, Subsec. 1.2.3, we can start from the Theorem 14.1.14 considered as a principle, finding again the results from which we started in the state of rest problem.

In the dynamic case, the Theorem 11.1.28 allows to state

Theorem 14.1.15 (*theorem of virtual work*; *d'Alembert-Lagrange*) *The motion of a free rigid solid takes place so that the virtual work of the lost volume forces of d'Alembert which act upon it vanishes for any system of virtual displacements of the respective solid.*

We notice that, in this case, we have to do with a continuous mechanical system, so that we can no more use Newton's equation (hence the lost forces of d'Alembert) in the classical form, being necessary to introduce the lost volume forces of d'Alembert. Using the virtual velocities

$$\mathbf{v}^* = \mathbf{v}_O^* + \boldsymbol{\omega}^* \times \mathbf{r}\,, \tag{14.1.58'}$$

we can write

$$\iiint_V \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \cdot \mathbf{v}^* \mathrm{d}V = 0\,, \tag{14.1.59}$$

so that

$$\iiint_V \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \cdot (\mathbf{v}_O^* + \boldsymbol{\omega}^* \times \mathbf{r}) \mathrm{d}V$$
$$= \mathbf{v}_O^* \cdot \iiint_V \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \mathrm{d}V + \boldsymbol{\omega}^* \cdot \iiint_V \mathbf{r} \times \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \mathrm{d}V = 0\,,$$

wherefrom

$$\iiint_V \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \mathrm{d}V = 0\,, \quad \iiint_V \mathbf{r} \times \left[\mathbf{F} - \mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \mathrm{d}V = 0\,,$$

being thus led to the theorem of torsor for a free rigid solid. Obviously, the Theorem 14.1.15 can be enounced also in the form of a principle.

If the rigid solid is acted upon by the given external forces $\mathbf{F}_i$, $i = 1,2,...,n$, we can write the theorem of virtual velocities in the form

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i^* + \iiint_V \left[-\mu(\mathbf{r}) \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \right] \cdot \mathbf{v}^* \mathrm{d}V = 0\,. \tag{14.1.59'}$$

The Theorem 14.1.15 can be stated correspondingly.

14.1.2 Motion of the Rigid Solid Subjected to Constraints

In what follows, after some general results, we study various particular cases of constraints in which parts of a rigid solid have given motions; we consider thus several general classes of constraints to which the rigid solid can be subjected, establishing the corresponding equations of motion.

14.1.2.1 General Results

If a rigid solid is subjected to constraints which may be holonomic or non-holonomic, scleronomic or rheonomous, bilateral or unilateral, ideal or with friction, then we must introduce in the system of equations of motion – using the axiom of liberation of constraints – also the torsor $\{\overline{\mathbf{R}}, \overline{\mathbf{M}}_O\}$ of the system of constraint forces at the pole O of the non-inertial frame of reference $\mathscr{R}$, rigidly linked to the solid. We replace thus the equations (14.1.51), (14.1.51') by the equations

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega}\times(\mathbf{v}'_O + \boldsymbol{\omega}\times\boldsymbol{\rho}) + \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}] = \mathbf{R} + \overline{\mathbf{R}}\,, \tag{14.1.60}$$

$$M\boldsymbol{\rho}\times(\dot{\mathbf{v}}'_O + \boldsymbol{\omega}\times\mathbf{v}'_O) + \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O + \overline{\mathbf{M}}_O\,, \tag{14.1.60'}$$

with respect to the frame $\mathscr{R}$. If $O \equiv C$, these equations become

$$M\mathbf{a}'_C = M(\dot{\mathbf{v}}'_C + \boldsymbol{\omega}\times\mathbf{v}'_C) = \mathbf{R} + \overline{\mathbf{R}}\,, \tag{14.1.61}$$

$$\mathbf{I}_C\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_C\boldsymbol{\omega}) = \mathbf{M}_C + \overline{\mathbf{M}}_C\,. \tag{14.1.62}$$

With respect to the central principal axes of inertia, we have

$$Ma'_{Ci} = R_i + \overline{R}_i\,,\quad i = 1,2,3\,, \tag{14.1.61'}$$

$$\begin{aligned} I_1^{(C)}\dot{\omega}_1 + \left(I_3^{(C)} - I_2^{(C)}\right)\omega_2\omega_3 &= M_{C1} + \overline{M}_{C1}\,,\\ I_2^{(C)}\dot{\omega}_2 + \left(I_1^{(C)} - I_3^{(C)}\right)\omega_3\omega_1 &= M_{C2} + \overline{M}_{C2}\,,\\ I_3^{(C)}\dot{\omega}_3 + \left(I_2^{(C)} - I_1^{(C)}\right)\omega_1\omega_2 &= M_{C3} + \overline{M}_{C3}\,. \end{aligned} \tag{14.1.62'}$$

We add to these equations initial conditions of Cauchy type too, e.g., of the form (14.1.56).

As a matter of fact, the equations (14.1.60), (14.1.61) correspond to the equations (14.1.38), (14.1.39), written in the form

$$\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} = \mathbf{R} + \overline{\mathbf{R}}\,, \tag{14.1.63}$$

$$M\mathbf{a}'_C = \mathbf{R} + \overline{\mathbf{R}}\,. \tag{14.1.63'}$$

We can thus state:

Theorem 14.1.16 (*theorem of momentum*) *The derivative with respect to time of the momentum of a rigid solid subjected to constraints, in an inertial frame of reference, is* *equal to the resultant of the given and constraint external forces which act upon this solid.*

Theorem 14.1.16' (*theorem of motion of the mass centre*) *The centre of mass of a rigid solid subjected to constraints moves, with respect to an inertial frame of reference, as a free particle at which would be concentrated the whole mass of the solid and which would be acted upon by the resultant of the given and constraint external forces.*

As well, the equations (14.1.60'), (14.1.62) correspond to the equations (14.1.40), (14.1.41'), written in the form

$$\boldsymbol{\rho} \times (M\mathbf{a}'_O) + \frac{\mathrm{d}\mathbf{K}^O}{\mathrm{d}t} = \mathbf{M}_O + \overline{\mathbf{M}}_O, \tag{14.1.64}$$

$$\frac{\mathrm{d}\mathbf{K}^C}{\mathrm{d}t} = \mathbf{M}_C + \overline{\mathbf{M}}_C. \tag{14.1.64'}$$

We thus state:

Theorem 14.1.17 (*theorem of moment of momentum*) *The derivative with respect to time of the pseudomoment of momentum of a rigid solid subjected to constraints, with respect to an arbitrary pole O, rigidly linked to the solid, in an inertial frame of reference, is equal to the resultant moment of the given and constraint external forces which act upon this solid, with respect to the same pole, from which one subtracts the dynamic moment, in the inertial frame, of the considered pole, translated at the centre of mass of the rigid solid, at which one assumes that would be concentrated the whole mass of the solid, with respect to the pole O.*

Theorem 14.1.17' *The derivative with respect to time of the pseudomoment of momentum of a rigid solid subjected to constraints, with respect to the mass centre, in an inertial frame of reference, is equal to the resultant moment of the given and constraint external forces which act upon this solid, with respect to the same centre.*

In what concerns the work, the formula (14.1.37) takes the form

$$\mathrm{d}W' + \mathrm{d}W'_R = (\mathbf{R} + \overline{\mathbf{R}}) \cdot \mathrm{d}\mathbf{r}'_O + (\mathbf{M}_O + \overline{\mathbf{M}}_O) \cdot \boldsymbol{\omega}\mathrm{d}t \tag{14.1.65}$$

and the formula (14.1.37') becomes

$$P' + P'_R = (\mathbf{R} + \overline{\mathbf{R}}) \cdot \mathbf{v}'_O + (\mathbf{M}_O + \overline{\mathbf{M}}_O) \cdot \boldsymbol{\omega}, \tag{14.1.65'}$$

so that we can state

Theorem 14.1.18 *The power of the given and constraint external forces which act upon a rigid solid subjected to constraints, with respect to a given frame of reference, is equal to the power of the torsor of the given and constraint external forces, at an arbitrary pole, rigidly linked to the solid, in the same frame.*

The formula (14.1.12) becomes

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = (\mathbf{R} + \overline{\mathbf{R}}) \cdot \mathbf{v}'_O + (\mathbf{M}_O + \overline{\mathbf{M}}_O) \cdot \boldsymbol{\omega}, \tag{14.1.66}$$

so that we state

Theorem 14.1.19 (*theorem of kinetic energy*) *The derivative with respect to time of the kinetic energy of a rigid solid subjected to constraints, in an inertial frame of reference, is equal to the power of the torsor of the given and constraint external forces, at an arbitrary pole, rigidly linked to the solid, in the same frame.*

Obviously, in this case too, the theorem of kinetic energy is a linear consequence of the theorems of momentum and moment of momentum.

In general, the rigid solid may be subjected to p holonomic constraints of the form (3.2.8), specified by the relations

$$f_l\left(\mathbf{r}'_O, \psi, \theta, \varphi; t\right) = f_l\left(x'_{O1}, x'_{O2}, x'_{O3}, \psi, \theta, \varphi; t\right) = 0\,, \quad l = 1,2,\ldots,p\,, \tag{14.1.67}$$

and to m non-holonomic constraints of the form (3.2.13), given by the relations

$$\boldsymbol{\alpha}_k \cdot \mathbf{v}'_O + \boldsymbol{\beta}_k \cdot \boldsymbol{\omega} + \gamma_k = \alpha_{kj} v'_{Oj} + \beta_{kj}\omega_j + \gamma_k = 0\,, \quad k = 1,2,\ldots,m\,. \tag{14.1.67'}$$

We have assumed that these constraints are bilateral; the unilateral constraints can be analogously introduced. Obviously, we can express the components ω_j as functions of the angular velocities $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$ in the constraint relations (14.1.67'). We notice that the relations (14.1.67) can be written in the form

$$\frac{\mathrm{d}f_l}{\mathrm{d}t} = \operatorname{grad} f_l \cdot \mathbf{v}'_O + \frac{\partial f_l}{\partial \psi}\dot{\psi} + \frac{\partial f_l}{\partial \theta}\dot{\theta} + \frac{\partial f_l}{\partial \varphi}\dot{\varphi} + \dot{f}_l = 0\,, \quad l = 1,2,\ldots,p \tag{14.1.67''}$$

too, wherefrom

$$\frac{\partial}{\partial v'_{Oj}}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right) = \frac{\partial f_l}{\partial x'_{Oj}}\,, \quad j = 1,2,3\,,$$

$$\frac{\partial}{\partial \dot{\psi}}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right) = \frac{\partial f_l}{\partial \psi}\,, \quad \frac{\partial}{\partial \dot{\theta}}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right) = \frac{\partial f_l}{\partial \theta}\,, \quad \frac{\partial}{\partial \dot{\varphi}}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right) = \frac{\partial f_l}{\partial \varphi}\,.$$

Taking into account (14.1.15), we can also write

$$\operatorname{grad} f_l \cdot \mathbf{v}'_O + \left[\left(\frac{\partial f_l}{\partial \psi} - \frac{\partial f_l}{\partial \varphi}\cos\theta\right)\frac{\sin\varphi}{\sin\theta} + \frac{\partial f_l}{\partial \theta}\cos\varphi\right]\omega_1$$
$$+\left[\left(\frac{\partial f_l}{\partial \psi} - \frac{\partial f_l}{\partial \varphi}\cos\theta\right)\frac{\cos\varphi}{\sin\theta} - \frac{\partial f_l}{\partial \theta}\sin\varphi\right]\omega_2 + \frac{\partial f_l}{\partial \varphi}\omega_3 + \dot{f}_l = 0\,, \quad l = 1,2,\ldots,p\,,$$

as well as

$$\frac{\partial \dot{\psi}}{\partial \omega_1} = \frac{\sin\varphi}{\sin\theta}\,, \quad \frac{\partial \dot{\psi}}{\partial \omega_2} = \frac{\cos\varphi}{\sin\theta}\,, \quad \frac{\partial \dot{\psi}}{\partial \omega_3} = 0\,,$$

$$\frac{\partial \dot{\theta}}{\partial \omega_1} = \cos\varphi\,, \quad \frac{\partial \dot{\theta}}{\partial \omega_2} = -\sin\varphi\,, \quad \frac{\partial \dot{\theta}}{\partial \omega_3} = 0\,,$$

$$\frac{\partial \dot{\varphi}}{\partial \omega_1} = -\sin\varphi \cot\theta\,, \quad \frac{\partial \dot{\varphi}}{\partial \omega_2} = -\cos\varphi \cot\theta\,, \quad \frac{\partial \dot{\varphi}}{\partial \omega_3} = 1\,.$$

Finally, we get

$$\begin{aligned} &\frac{\partial}{\partial v'_{Oj}}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right)v'_{Oj} + \frac{\partial}{\partial \omega_j}\left(\frac{\mathrm{d}f_l}{\mathrm{d}t}\right)\omega_j + \dot{f}_l \\ &= \mathrm{grad}_{\mathbf{v}'_O}\frac{\mathrm{d}f_l}{\mathrm{d}t}\cdot\mathbf{v}'_O + \mathrm{grad}_{\boldsymbol{\omega}}\frac{\mathrm{d}f_l}{\mathrm{d}t}\cdot\boldsymbol{\omega} + \dot{f}_l = 0\,, \quad l = 1,2,\ldots,p\,. \end{aligned} \tag{14.1.67'''}$$

These results have a general character and allow to determine the constraint forces in any particular case.

We notice that one must have $p + m < 6$, otherwise the rigid solid remains at rest; we assume, obviously, that the constraint relations are independent. On the other hand, the constraint forces involve q unknown scalars; if $q = p + m$, then the problem of the rigid solid motion is determined. In general, the equations of motion and the constraint relations are not independent, so that *the kinetostatic problem* can be separated in two problems only in particular cases: *the kinetic problem* (determination of the kinematic parameters $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ of the motion) and *the static problem* (determination of the torsor $\{\overline{\mathbf{R}}, \overline{\mathbf{M}}_O\}$ of the constraint forces).

We mention that the Theorems 14.1.14 and 14.1.15 concerning the virtual work are completed in the form:

Theorem 14.1.20 (*theorem of virtual work*) *The necessary and sufficient condition of equilibrium of a rigid solid subjected to ideal constraints and acted upon by a system of given external forces consists in equating to zero the virtual work of these forces for any system of virtual displacements of the respective solid.*

Theorem 14.1.21 (*theorem of virtual work; d'Alembert-Lagrange*) *The motion of a rigid solid subjected to ideal constraints takes place so that the virtual work of the lost forces of d'Alembert, which act upon this solid, vanishes for any system of virtual displacements of the respective solid.*

14.1.2.2 Rigid Solid a Point of Which has an Imposed Motion

Let us suppose that the point O rigidly linked to the solid has a known motion, given by the equation

$$\mathbf{r}'_O = \mathbf{r}'_O(t)\,, \tag{14.1.68}$$

with respect to the inertial frame of reference $\mathscr{R}'$; obviously, in this case, we know the velocity $\mathbf{v}'_O = \mathbf{v}'_O(t)$ and the acceleration $\mathbf{a}'_O = \mathbf{a}'_O(t)$ too. Eliminating the constraint by introducing the constraint force $\overline{\mathbf{R}}$ and using the system of equations (14.1.60), (14.1.60'), we can write the equations of motion of the rigid solid in the form

$$M[\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})] = \mathbf{R} - M\mathbf{a}'_O + \overline{\mathbf{R}}\,, \tag{14.1.69}$$

$$\mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O - M\boldsymbol{\rho} \times \mathbf{a}'_O . \tag{14.1.69'}$$

The equation (14.1.69') specifies the rotation about the point $\boldsymbol{O}$ (the vector), then the equation (14.1.69) determines the constraint force $\overline{\mathbf{R}}$; if $\mathbf{M}_O = \mathbf{M}_O(\psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t)$, then we associate to the vector equation (14.1.69') the system of scalar equations (14.1.53") too. We have thus introduced three relations of finite constraints of the form (14.1.63) ($p = 3$, $m = 0$) and three scalar components of the constraint force ($q = 3$).

In particular, if the pole O is fixed, then we have $\mathbf{r}'_O = \mathbf{v}'_O = \mathbf{a}'_O = \mathbf{0}$, obtaining the equations of motion of the rigid solid with a fixed point. If we introduce in these equations also the influence of the complementary force $-\iiint_V \mu(\mathbf{r})\mathbf{a}'_O \mathrm{d}V$ at the centre of mass C, then we find again the system of equations (14.1.69), (14.1.69') (one passes from the action of this force at C to the corresponding torsor at O).

If the point O is constrained to stay on a given perfectly smooth surface S (in general, movable), then the constraint relation (14.1.67) ($p = 1$, $m = 0$) is written in the form

$$f(\mathbf{r}'_O; t) = f(x'_{O1}, x'_{O2}, x'_{O3}; t) = 0 . \tag{14.1.70}$$

The liberation of constraints axiom leads to a constraint force $\overline{\mathbf{R}} = \lambda \operatorname{grad} f$, λ non-determinate scalar, normal at O to the surface S (the moment $\mathbf{M}'_O = \mathbf{0}$); the equations of motion (14.1.60), (14.1.60') of the rigid solid take the form

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}] = \mathbf{R} + \lambda \operatorname{grad} f , \tag{14.1.71}$$

$$M\boldsymbol{\rho} \times (\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times \mathbf{v}'_O) + \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O . \tag{14.1.71'}$$

To determine the vector unknowns $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ and the unknown scalar λ ($q = 1$; 7 scalar unknowns) we have the vector equations (14.1.71), (14.1.71') and the scalar equation (14.1.70) (7 scalar equations).

As well, if the point O stays on a perfectly smooth curve C (in general, movable), then the constraint relations (14.1.67) ($p = 2$, $m = 0$) become

$$f_l(\mathbf{r}'_O; t) = f_l(x'_{O1}, x'_{O2}, x'_{O3}; t) = 0, \quad l = 1,2 . \tag{14.1.72}$$

As in the previous case, we can write the equations (14.1.60), (14.1.60') in the form

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}] = \mathbf{R} + \lambda_1 \operatorname{grad} f_1 + \lambda_2 \operatorname{grad} f_2 , \tag{14.1.73}$$

$$M\boldsymbol{\rho} \times (\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times \mathbf{v}'_O) + \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O . \tag{14.1.73'}$$

The vector equations (14.1.73), (14.1.73') and the scalar equations (14.1.68) (8 scalar equations) determine thus the vector unknowns $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ and the scalar unknowns λ_1 and λ_2 ($q = 2$; 8 scalar unknowns).

14.1.2.3 Rigid Solid Two Points of Which have Imposed Motions

Let us suppose that the points O and O_1 of the rigid solid have known motions, given by the equations

$$\mathbf{r}'_O = \mathbf{r}'_O(t), \quad \mathbf{r}'_{O_1} = \mathbf{r}'_{O_1}(t), \tag{14.1.74}$$

with respect to the inertial frame of reference $\mathscr{R}'$; because the points O and O_1 are at an invariable mutual distance (equal to l), the equations (14.1.74) are not independent, and the relation

$$\left[\mathbf{r}'_{O_1}(t) - \mathbf{r}'_O(t)\right]^2 = l^2 \tag{14.1.74'}$$

takes place. Hence, there remain five independent finite constraint relations ($p = 5$, $m = 0$). As a matter of fact, the motion of any point O_2 on the straight line $\Delta \equiv OO_1$ is determined by the position vector $\mathbf{r}'_{O_2} = \mathbf{r}'_O + \lambda\left(\mathbf{r}'_{O_1} - \mathbf{r}'_O\right)$, λ scalar; the case thus considered is identical to that in which a straight line Δ rigidly linked to the solid (e.g., an axis of the frame of reference $\mathscr{R}$) describes a given motion, one of the points of this axis (e.g., the pole O) having a given motion too. Using the axiom of liberation of constraints, we introduce the constraint forces $\mathbf{R}'$ and $\mathbf{R}_1$ at the points O and O_1, respectively; the torsor of these forces at the point O is $\left\{\mathbf{R}' + \mathbf{R}_1, \left(\mathbf{r}'_{O_1} - \mathbf{r}'_O\right) \times \mathbf{R}_1\right\}$. The equations of motion (14.1.60), (14.1.60') are written in the form (the first equation (14.1.74) allows to calculate the acceleration $\mathbf{a}'_O = \mathbf{a}'_O(t)$ too)

$$M\left[\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\right] = \mathbf{R} - M\mathbf{a}'_O + \mathbf{R}' + \mathbf{R}_1, \tag{14.1.75}$$

$$\mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O - M\boldsymbol{\rho} \times \mathbf{a}'_O + \left(\mathbf{r}'_{O_1} - \mathbf{r}'_O\right) \times \mathbf{R}_1, \tag{14.1.76}$$

with respect to the non-inertial frame $\mathscr{R}$.

For the three vector unknowns $\boldsymbol{\omega}$, $\mathbf{R}'$ and $\mathbf{R}_1$ of the problem ($q = 6$; 9 scalar unknowns) we can use the vector equations (14.1.75), (14.1.76) and the second equation (14.1.74), together with the relation (14.1.74') (8 scalar equations). We can write the constraint relation also in the differential form (the velocity of the point O_1 expressed with respect to the velocity of the point O)

$$\mathbf{v}'_{O_1} = \mathbf{v}'_O + \boldsymbol{\omega} \times \left(\mathbf{r}'_{O_1} - \mathbf{r}'_O\right); \tag{14.1.74''}$$

a scalar product by $\mathbf{r}'_{O_1} - \mathbf{r}'_O$ leads to an identity, so that there result only two scalar relations. Hence, the number of unknowns is with a unity greater than the number of equations, the corresponding system being once indeterminate (as in the case of the static equilibrium of the rigid solid with a fixed axis).

Choosing the axis OO_1 as axis Ox_3, we can write the equations (14.1.75), (14.1.76) in components, in the form

$$M\left[\dot{\omega}_2\rho_3 - \dot{\omega}_3\rho_2 + \omega_1(\omega_1\rho_1 + \omega_2\rho_2 + \omega_3\rho_3) - \omega^2\rho_1\right] = R_1 - Ma'_{O_1} + R'_1 + R_{11},$$
$$M\left[\dot{\omega}_3\rho_1 - \dot{\omega}_1\rho_3 + \omega_2(\omega_1\rho_1 + \omega_2\rho_2 + \omega_3\rho_3) - \omega^2\rho_2\right] = R_2 - Ma'_{O_2} + R'_2 + R_{12},$$
$$M\left[\dot{\omega}_1\rho_2 - \dot{\omega}_2\rho_1 + \omega_3(\omega_1\rho_1 + \omega_2\rho_2 + \omega_3\rho_3) - \omega^2\rho_3\right] = R_3 - Ma'_{O_3} + R'_3 + R_{13},$$
$$\text{(14.1.75')}$$

$$I_{11}\dot{\omega}_1 + I_{12}\dot{\omega}_2 + I_{31}\dot{\omega}_3 + (I_{33} - I_{22})\omega_2\omega_3 + I_{23}\left(\omega_2^2 - \omega_3^2\right) + \omega_1(I_{31}\omega_2 - I_{12}\omega_3)$$
$$= M_{O1} - M(\rho_2 a'_{O3} - \rho_3 a'_{O2}) - lR_{12},$$
$$I_{22}\dot{\omega}_2 + I_{23}\dot{\omega}_3 + I_{12}\dot{\omega}_1 + (I_{11} - I_{33})\omega_3\omega_1 + I_{31}\left(\omega_3^2 - \omega_1^2\right) + \omega_2(I_{12}\omega_3 - I_{23}\omega_1)$$
$$= M_{O2} - M(\rho_3 a'_{O1} - \rho_1 a'_{O3}) - lR_{11},$$
$$I_{33}\dot{\omega}_3 + I_{31}\dot{\omega}_1 + I_{23}\dot{\omega}_2 + (I_{22} - I_{11})\omega_1\omega_2 + I_{12}\left(\omega_1^2 - \omega_2^2\right) + \omega_3(I_{23}\omega_1 - I_{31}\omega_2)$$
$$= M_{O3} - M(\rho_1 a'_{O2} - \rho_2 a'_{O1}).$$
$$\text{(14.1.76')}$$

As well, the condition (14.1.74") leads to

$$v'_{O_1 1} = v'_{O1} + l\omega_2, \quad v'_{O_1 2} = v'_{O2} - l\omega_1, \quad v'_{O_1 3} = v'_{O3}. \qquad \text{(14.1.74''')}$$

The first two equations (14.1.74'''), together with the last equation (14.1.76'), determine entirely the vector $\boldsymbol{\omega}$. The last column of the matrix (3.2.11''') allows to specify Euler's angles $\psi = \psi(t)$ and $\theta = \theta(t)$ in the motion of the Δ-line (if we know the direction cosines of this line with respect to the frame of reference $\overline{\mathscr{R}}$); in this case, the relations (5.2.35) allow to express the components of the vector $\boldsymbol{\omega}$ as function of the angle of proper rotation φ. Replacing in the last equation (14.1.76'), this one becomes a differential equation of second order in φ; by integration, we get $\varphi = \varphi(t)$ and then $\omega_i = \omega_i(t)$, $i = 1,2,3$. The first two equations (14.1.76') determine the constraint forces R_{11} and R_{12}, while the first two equations (14.1.75') give the constraint forces R'_1 and R'_2; the third equation (14.1.75') specifies the sum $R'_3 + R_{13}$ of the last unknown components, the character of the non-determination being thus put into evidence. As in the statical case (see Chap. 4, Subsec. 2.1.3), the non-determination is due to the modelling as a rigid adopted for the solid body.

If the axis Δ is a principal axis of inertia, the axes Ox_1 and Ox_2 having the same property, we have $I_{23} = I_{31} = I_{12} = 0$, so that the system of equations (14.1.76') takes the simplified form

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_{O1} - M(\rho_2 a'_{O3} - \rho_3 a'_{O2}) - lR_{12}, \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= M_{O2} - M(\rho_3 a'_{O1} - \rho_1 a'_{O3}) + lR_{11}, \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= M_{O3} - M(\rho_1 a'_{O2} - \rho_2 a'_{O1}).
\end{aligned} \tag{14.1.76''}
$$

In particular, if the points O and O_1 are fixed, then we have $v'_{O_1 i} = v'_{Oi} = a'_{Oi} = 0$, $i = 1,2,3$, while from the conditions (14.1.74''') it results $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$; we obtain thus the motion of the rigid solid with a fixed axis.

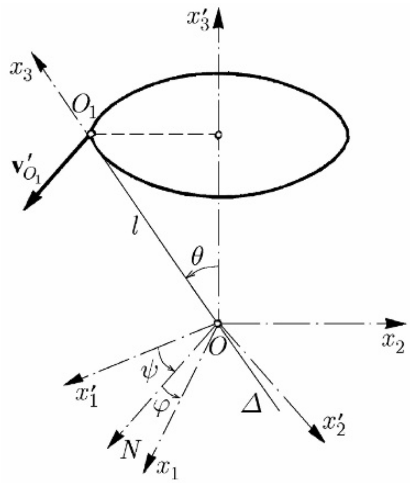

Fig. 14.5 Motion of a rigid solid a point O of which is fixed and an axis Δ of which passes through this point

Let us suppose, in particular, that the point O is fixed and that the straight line Δ describes a circular cone, having a motion of uniform rotation about the cone axis. To simplify the computation, we take $O' \equiv O$, the Ox'_3-axis being the axis of the cone. The point O_1 describes a circle in a plane normal to the fixed axis Ox'_3. Introducing Euler's angles ψ, θ, φ, the angular velocity of the point O_1 will be $\dot{\psi}$, so that the velocity $\mathbf{v}'_{O1}$ will be directed along the line of nodes ON, having the magnitude $v'_{O1} = \dot{\psi} l \sin\theta$ (Fig. 14.5); in components, there result $v'_{O_1 1} = \dot{\psi} l \sin\theta\cos\varphi$, $v'_{O_1 2} = -\dot{\psi} l \sin\theta \sin\varphi$. If we make $v'_{O1} = v'_{O2} = 0$ in (14.1.74''') or if we make $\dot{\theta} = 0$ (because $\theta = \text{const}$) in the relations (5.2.35), then we obtain

$$
\omega_1 = \dot{\psi}\sin\theta\sin\varphi, \quad \omega_2 = \dot{\psi}\sin\theta\cos\varphi, \quad \omega_3 = \dot{\varphi} + \dot{\psi}\cos\theta.
$$

We assume also that the axes Ox_i, $i = 1,2,3$, are principal axes of inertia; in this case, replacing in the third equation (14.1.76'') (we notice that $\dot{\psi} = \text{const}$, the rotation being uniform, and that $a'_{O1} = a'_{O2} = 0$), one obtains the equation of motion

$$\ddot{\Phi} + k \sin \Phi = 2 \frac{M_{O3}}{I_3}, \tag{14.1.77}$$

with the notations

$$\Phi = 2\varphi, \quad k = \frac{I_2 - I_1}{I_3} \dot{\psi}^2 \sin^2 \theta. \tag{14.1.77'}$$

If $M_{O3} = 0$ (e.g., if the rigid solid is subjected to the action of the own weight, the axis being a central principal axis of inertia), then the motion of rotation of the rigid solid is determined only by the angle of proper rotation $\varphi = \varphi(t)$, which is given by an equation of the simple pendulum equation type.

More general, we can assume that a frictionless sliding of the rigid solid along the Δ-axis takes place; the equations of this axis with respect to the inertial frame of reference $\mathscr{R}'$ can be written, e.g., in the form (we put in evidence the point O on the Δ-axis)

$$x'_{O1} = \alpha_1(t) x'_{O3} + \beta_1(t), \quad x'_{O2} = \alpha_2(t) x'_{O3} + \beta_2(t), \tag{14.1.78}$$

intervening thus two finite constraint relations; as above, knowing the motion of the straight line Δ, we can consider Euler's angles ψ and θ as known, so that only the angle of proper rotation φ remains to be determined. We notice that, due to the frictionless sliding, the constraint forces $\mathbf{R}'$ and $\mathbf{R}_1$ are normal to the Δ-axis. We have to determine eight scalar unknowns $R'_1, R'_2, R_{11}, R_{12}, x'_{Oi}$, $i = 1,2,3$ and φ, for which we have eight equations (the equations (14.1.75'), (14.1.76'), where we make $R'_3 = R_{13} = 0$, and (14.1.78)), the problem being thus determined.

If in the particular problem considered above we have $\overrightarrow{OO_1} = x'_{O_1 3}(t)\mathbf{i}_3$, then it results $\mathbf{v}'_{O_1}(t) = \boldsymbol{\psi} x'_{O_1 3}(t) \sin\theta$, wherefrom

$$v'_{O_1 1} = \dot{\psi} x'_{O_1 3} \sin\theta \cos\varphi, \quad v'_{O_1 2} = -\dot{\psi} x'_{O_1 3} \sin\theta \sin\varphi, \quad v'_{O_1 3} = \dot{x}'_{O_1 3}.$$

Observing that $\mathbf{a}'_{O_1} = \dot{\mathbf{v}}'_{O_1} + \boldsymbol{\omega} \times \mathbf{v}'_{O_1}$, where $\dot{\mathbf{v}}'_{O_1}$ is the derivative with respect to the movable frame of reference, using the results previously obtained for the vector $\boldsymbol{\omega}$ and taking into account that $\dot{\psi} = \text{const}$, we get

$$\begin{aligned} a'_{O_1 1} &= 2\dot{\psi}\dot{x}'_{O_1 3} \sin\theta\cos\varphi + \dot{\psi}^2 x'_{O_1 3} \sin\theta\cos\theta\sin\varphi, \\ a'_{O_1 2} &= -2\dot{\psi}\dot{x}'_{O_1 3} \sin\theta\sin\varphi + \dot{\psi}^2 x'_{O_1 3} \sin\theta\cos\theta\cos\varphi, \\ a'_{O_1 3} &= \ddot{x}_{O_1 3} - \dot{\psi}^2 x'_{O_1 3} \sin^2\theta. \end{aligned}$$

In the system of equations (14.1.75'), (14.1.76') we take the movable frame at the point O_1 (maintaining thus the notations in Fig. 14.5) and we introduce the results obtained above; the third equation (14.1.75') and the third equation (14.1.76') form a system of two second order differential equations with the unknowns $x'_{O_1 3} = x'_{O_1 3}(t)$ and $\varphi = \varphi(t)$. After integrating this system of equations, one obtains easily the other unknowns of the problem. Obviously, in this case, as in the preceding one, certain initial conditions of Cauchy type must be fulfilled too.

14.1.2.4 Other Cases of Motion of the Rigid Solid Subjected to Certain Constraint Conditions

Let us consider firstly the case of a rigid solid for which the point O is constrained to stay on a given perfectly smooth curve C (in general, movable) of equation (14.1.72), another point O_1 of which staying on a given perfectly smooth surface S (in general, movable) of equation (14.1.70), where we replace the point O by the point O_1 ($p = 3$, $m = 0$). Using the axiom of liberation of constraints, the equations of motion of this solid have the form

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}] = \mathbf{R} + \lambda \operatorname{grad} f + \lambda_1 \operatorname{grad} f_1 + \lambda_2 \operatorname{grad} f_2 , \tag{14.1.79}$$

$$M\boldsymbol{\rho} \times (\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times \mathbf{v}'_O) + \mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O + \lambda \left(\mathbf{r}'_{O_1} - \mathbf{r}'_O \right) \times \operatorname{grad} f . \tag{14.1.79'}$$

To determine the vector unknowns $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ and the scalar unknowns λ, λ_1 and λ_2 ($q = 3$, 9 scalar unknowns) we have the vector equations (14.1.79), (14.1.79') and the scalar equations (14.1.70), (14.1.72) (9 scalar equations).

Let be also the case of a rigid solid for which the points O_i, $i = 1,2,3,4$, are constrained to move on a perfectly smooth surfaces (in general, movable) S_i, $i = 1,2,3,4$, respectively, of equations ($p = 4$, $m = 0$)

$$f_i \left(\mathbf{r}'_{O_i} ; t \right) = f_i \left(x'_{O_i 1}, x'_{O_i 2}, x'_{O_i 3} ; t \right) = 0 , \quad i = 1,2,3,4 . \tag{14.1.80}$$

The rigid solid remains thus with two degrees of freedom. Proceeding as in the preceding case, we can write the equations (14.1.60), (14.1.60') in the form

$$M[\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times (\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}] = \mathbf{R} + \sum_{i=1}^{4} \lambda_i \operatorname{grad} f_i , \tag{14.1.81}$$

$$M\boldsymbol{\rho} \times (\dot{\mathbf{v}}'_O + \boldsymbol{\omega} \times \mathbf{v}'_O) + \mathbf{I}_O \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O \boldsymbol{\omega}) = \mathbf{M}_O + \sum_{i=1}^{4} \lambda_i \left(\mathbf{r}'_{O_i} - \mathbf{r}'_O \right) \times \operatorname{grad} f_i , \tag{14.1.81'}$$

The vector equations (14.1.81), (14.1.81') and the scalar equations (14.1.80) (ten scalar equations) determine thus the vector unknowns $\mathbf{v}'_O$ and $\boldsymbol{\omega}$ and the scalar unknowns λ_i, $i = 1,2,3,4$ ($q = 4$; ten scalar unknowns).

14.2. Motion of a Rigid Solid about a Fixed Axis. Plane-parallel Motion of the Rigid Solid

The general results previously obtained will be used, in what follows, to some important cases of motion of the rigid solid; we study thus the motion of the rigid solid about a fixed axis, as well as its plane-parallel motion.

14.2.1 Motion of the Rigid Solid about a Fixed Axis

An important particular case of that presented in Sect. 1.2.3 is that one in which the axis is a fixed one (the two points O and O_1 of the rigid solid are fixed). E.g., we mention the rigid pendulum. We will consider thus the pendulums of Borda, Kater and Bessel, the annular pendulum of Voinaroski, the inclined pendulum of Mach and the Weber-Gauss pendulum of torsion.

14.2.1.1 General Results

Choosing the straight line $\Delta \equiv OO_1$ as Ox_3-axis ($\left|\overrightarrow{OO_1}\right| = l$) and noting that this axis is fixed, it is convenient to take $O \equiv O'$ and $Ox_3 \equiv O'x'_3$; in this case $\omega_1 = \omega_2 = 0$, while $\omega_3 = \omega$, the angular velocity vector $\boldsymbol{\omega}$ being situated along the $O'x'_3$-axis (Fig. 14.6). All the points of the rigid solid describe circular trajectories in

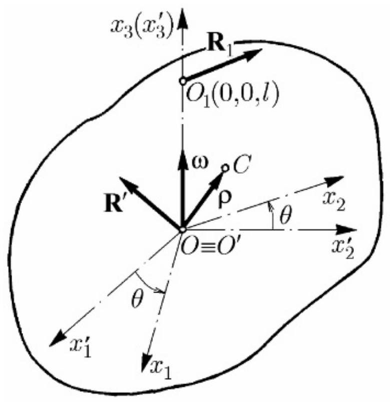

Fig. 14.6 Motion of a rigid solid about a fixed axis

planes normal to the $O'x'_3$-axis, the centres of which are on that axis (the fixed and the movable axoids are degenerated, coinciding with the axis). The motion has only one degree of freedom, to which corresponds the angle $\theta = \theta(t)$ (classical notation for the angle of proper rotation in this motion, different of that corresponding to Euler's angles); we have, obviously, $\omega = \dot{\theta}$. Taking into account the conditions $\mathbf{v}'_{O_1} = \mathbf{v}'_O = \mathbf{0}$, the relations (14.1.74''') are identically verified. If we put also $\mathbf{a}'_O = \mathbf{0}$, then the equations of motion (14.1.75'), (14.1.76') take the form

$$\begin{aligned} -M\left(\dot{\omega}\rho_2 + \omega^2\rho_1\right) &= R_1 + R_1' + R_{11}, \\ M\left(\dot{\omega}\rho_1 + \omega^2\rho_2\right) &= R_2 + R_2' + R_{12}, \\ 0 &= R_3 + R_3' + R_{13}, \end{aligned} \tag{14.2.1}$$

$$\begin{aligned} I_{31}\dot{\omega} - I_{23}\omega^2 &= M_{O1} - lR_{12}, \\ I_{23}\dot{\omega} + I_{31}\omega^2 &= M_{O2} + lR_{11}, \\ I_{33}\dot{\omega} &= M_{O3}, \end{aligned} \tag{14.2.1'}$$

where $\mathbf{R}'$ and $\mathbf{R}_1$ are the constraint forces at the points O' and O_1, respectively.

The third equation (14.2.1') can be written in the form

$$I_{33}\ddot{\theta} = M_{O3} \tag{14.2.2}$$

too. If $M_{O3} = M_{O3}\left(\theta, \dot{\theta}; t\right)$ and if initial conditions of Cauchy type

$$\theta(t_0) = \theta_0, \quad \dot{\theta}(t_0) = \omega(t_0) = \dot{\theta}_0 = \omega_0 \tag{14.2.2'}$$

are given too for the differential equation of second order (14.2.2), then we can determine univocally the angle of proper rotation $\theta = \theta(t)$, using the theorem of existence and uniqueness. If $M_{O3} = 0$, then the motion of rotation is uniform, and if we have $\dot{\theta}_0 = 0$ ($M_{O3}(\theta_0, 0; t) = 0$) too, then the rigid solid is at rest with respect to the fixed frame of reference. Observing that

$$T' = \frac{1}{2} I_{33}\omega^2$$

and

$$\mathrm{d}W' + \mathrm{d}W_R' = \left(\mathbf{M}_O + \overline{\mathbf{M}}_O\right) \cdot \boldsymbol{\omega}\mathrm{d}t = M_{O3}\omega\mathrm{d}t,$$

and applying the theorem of kinetic energy in the form (14.1.66), we find again the equation of motion (14.2.2). We mention that the pseudomoment of momentum is given by

$$\mathbf{K}^O = I_{33}\boldsymbol{\omega} + I_{31}\omega\mathbf{i}_1 + I_{23}\omega\mathbf{i}_2;$$

we have $\mathbf{K}^O = I_{33}\boldsymbol{\omega}$, so that the pseudomoment of momentum is directed along the fixed axis if and only if that axis is a principal axis of inertia.

The first two equations (14.2.1') and then the first two equations (14.2.1) allow to compute the constraint forces along the axes Ox_1 and Ox_2; we obtain

$$\begin{aligned} R_1' &= \frac{1}{l} M_{O2} - R_1 - \left[M\rho_2 + \frac{1}{l} I_{23} \right] \dot{\omega} - \left[M\rho_1 + \frac{1}{l} I_{31} \right] \omega^2, \\ R_2' &= -\frac{1}{l} M_{O1} - R_2 + \left[M\rho_1 + \frac{1}{l} I_{31} \right] \dot{\omega} - \left[M\rho_2 + \frac{1}{l} I_{23} \right] \omega^2, \end{aligned} \tag{14.2.3}$$

$$\begin{aligned} R_{11} &= -\frac{1}{l} \left[M_{O2} - \left(I_{23} \dot{\omega} + I_{31} \omega^2 \right) \right], \\ R_{12} &= \frac{1}{l} \left[M_{O1} - \left(I_{31} \dot{\omega} - I_{23} \omega^2 \right) \right]. \end{aligned} \tag{14.2.3'}$$

The third equation (14.2.1) specifies only the sum $R_3' + R_{13} = -R_3$ of the other two components of the constraint forces, the problem being thus indeterminate from this point of view (as in the static case), due to the model of rigid solid adopted for the solid body.

We notice that the constraint forces depend on the angular velocity $\boldsymbol{\omega}$ and on the angular acceleration $\dot{\boldsymbol{\omega}}$; for great values of these quantities (the square of the angular velocity, ω^2, or its non-uniformity given by $\dot{\boldsymbol{\omega}}$), e.g., for the rotation velocities of the propellers and of the turbines, the constraint forces increase very much; the stress in the axle increases also and it is possible to reach the state of fracture. As it was noticed by L. Euler, this dependence disappears if $I_{23} = I_{31} = 0$ and $\rho_1 = \rho_2 = 0$, hence if the axis of rotation is a central principal axis of inertia; it results

$$R_1' = \frac{1}{l} M_{O2} - R_1, \quad R_2' = -\frac{1}{l} M_{O1} - R_2, \tag{14.2.4}$$

$$R_{11} = -\frac{1}{l} M_{O2}, \quad R_{12} = \frac{1}{l} M_{O1}, \tag{14.2.4'}$$

corresponding to the formulae (4.2.7") obtained in the static case (the dynamic constraint forces coincide with the static ones). In this case, the constraint forces depend only on the given external forces, which are equilibrating them from the statical point of view, and are not influenced by the rotation of the rigid solid.

Putting the condition that the constraint force at the point O_1 be zero ($\mathbf{R}_1 = \mathbf{0}$) for any rotation of the rigid solid, we obtain $I_{23} = I_{31} = 0$, $M_{O1} = M_{O2} = 0$; we get thus the constraint forces

$$R_1' = -R_1 - M\left(\rho_2 \dot{\omega} + \rho_1 \omega^2 \right), \quad R_2' = -R_2 + M\left(\rho_1 \dot{\omega} - \rho_2 \omega^2 \right), \quad R_3' = -R_3, \tag{14.2.5}$$

the problem becoming statically determinate. Hence, *for any rotation of the rigid solid, it is sufficient only one point of support of the axis of rotation* (the fixed point O) *if and only if that one is a principal axis of inertia, the given external forces having a resultant moment with respect to the point of support, which is directed along this axis*. Such an axis is obtained, e.g., in the case in which the rigid solid is acted upon by a unique given force, situated in a plane normal to a principal axis of inertia, non-intersecting this axis, the trace of which on the plane being a fixed point. We obtain an analogous result if the support of the given force passes through a fixed point, the axis of rotation (which is a principal axis of inertia) passing through the same point; in this case, we have $M_{O3} = 0$, hence $\mathbf{M}_O = \mathbf{0}$ too, being led to a motion of uniform rotation (corresponding to the equation (14.2.2)), so that

$$R_1' = -R_1 - M\rho_1\omega^2 , \quad R_2' = -R_2 - M\rho_2\omega^2 , \quad R_3' = -R_3 . \tag{14.2.5'}$$

We can state

Theorem 14.2.1 *If a rigid solid with a fixed point is subjected to the action of a system of given forces equivalent to a resultant the support of which passes through this point and if this solid has an initial uniform motion of rotation about a principal axis of inertia which passes, as well, through the very same fixed point, then the rigid solid continuous to have this motion indefinitely.*

In this case, the axis of rotation is called permanent axis of rotation; sometimes, this denomination (justified by the Theorem 14.1.21) is used also in the more general case $M_{O3} \neq 0$.

The supplementary condition which imposes the vanishing of the constraint force at the point O too ($\mathbf{R}' = \mathbf{0}$), for any rotation of the rigid solid, leads to $\rho_1 = \rho_2 = 0$, $R_1 = R_2 = R_3 = 0$. *If and only if the axis of rotation is a central principal axis of inertia, the rigid solid being acted upon only by a couple contained in a plane normal to this axis, then no point of support is necessary for that axis*; in this case, the axis of rotation is *a free axis of rotation*, and the fixed axle is not acted by a given force. If $M_{O3} = 0$, hence if $\mathbf{M}_O = \mathbf{0}$ too, then we can state

Theorem 14.2.2 *If a free rigid solid is not acted upon by any given force and has a uniform motion of rotation about a central principal axis of inertia, then it will continue to have this motion indefinitely, the axis remaining fixed.*

In this case, the free axis of rotation is called *spontaneous axis of rotation*; sometimes, this denomination (justified by the Theorem 14.2.2) is used also in the more general case in which $M_{O3} \neq 0$. This result can be verified experimentally in a space laboratory, in conditions of imponderability. The Theorem 14.2.2 can be considered as a completion for the rigid solid of Newton's "principle of inertia", enounced for a particle (eventually for the mass centre of the rigid solid).

14.2.1.2 Physical Pendulum. Huygens's Theorems

We call *physical* (*rigid, compound*) *pendulum* a rigid solid which is rotating about a horizontal fixed axis, being subjected only to the action of its own weight. We take the $O'x_2'x_3'$-plane as horizontal plane, the fixed axis as $O'x_3'$-axis, the $O'x_1'$-axis being along the descendent vertical. Without any loss of generality, the Ox_1-axis will be taken so as to pass through the centre of mass C, at which acts the own weight $\mathbf{G}$; the position of the rigid solid will be thus specified by the angle of proper rotation

$\theta = \theta(t)$ (Fig. 14.7). Denoting $\left|\overrightarrow{OC}\right| = l$ and observing that $\mathbf{G} = M\mathbf{g}$, we get $M_{O3} = -Mgl\sin\theta$; we can thus write the equation of motion (14.2.2) in the form

$$\ddot{\theta} + \frac{g}{l'}\sin\theta = 0\,, \quad l' = \frac{I_{33}}{Ml}\,. \tag{14.2.6}$$

We find again the equation (7.1.38') of a simple pendulum of equivalent length l', called *the synchronous simple pendulum* of the considered physical pendulum; imposing the same initial conditions, the motions of the two pendulums will be specified by the same function $\theta = \theta(t)$, so that one can use all the results obtained in Chap. 7, Subsec. 1.3.1. In case of great oscillations, we use the formula (7.1.43") which, with a good approximation, leads to the period

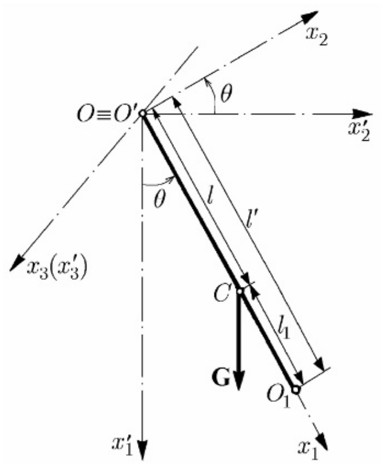

Fig. 14.7 Physical pendulum

$$T = 2\pi\sqrt{\frac{l'}{g}}\left(1 + \frac{\alpha^2}{16}\right) = 2\pi\sqrt{\frac{I_{33}}{Mgl}}\left(1 + \frac{\alpha^2}{16}\right), \tag{14.2.7}$$

where $\alpha = \theta_{\max}$; in case of isochronous (small) oscillations, we can write

$$T = 2\pi\sqrt{\frac{l'}{g}} = 2\pi\sqrt{\frac{I_{33}}{Mgl}}\,. \tag{14.2.7'}$$

The Huygens-Steiner theorem, expressed by the formula (3.1.113), allows to write $I_{33} = I_C + Ml^2$, where I_C is the moment of inertia of the physical pendulum with respect to an axis parallel to the horizontal $O'x_3'$-axis, passing through the mass centre C. If we denote by T_s the period of the simple pendulum of length l, then it results

$$T = T_s\sqrt{1 + \frac{I_C}{Ml^2}} = T_s\sqrt{1 + \left(\frac{i_C}{l}\right)^2}\,, \tag{14.2.7''}$$

where $i_C = \sqrt{I_C / M}$ is the central radius of gyration, defined by the relation (3.1.30), corresponding to the same axis which passes through C. Thus, we can write

$$l' = l + l_1 , \quad l_1 = \frac{I_C}{Ml} = \frac{i_C^2}{l} . \tag{14.2.8}$$

A point O_1 on the axis Ox_1 is defined so that $\overline{OO_1} = l' > l$, the centre of mass C being contained between the point O, called *centre of suspension*, and the point O_1, called *centre of oscillation* (Fig. 14.7). A particle situated at the point O_1 has the same motion as the synchronous simple pendulum. The horizontal axes parallel to the axis $O'x_3'$, which pass through the points O and O_1 are called *axis of suspension* and *axis of oscillation*, respectively. If we maintain the direction of the horizontal axis $O'x_3'$ (hence, the quantities I_C and i_C), as well as the distance from the centre of mass C to the centre of suspension O, the length l' of the synchronous simple pendulum remains invariant; we can thus state that *all the generatrices of a circular cylinder the axis of which is horizontal and passes through the mass centre lead to physical pendulums which, in the same initial conditions, have identical motions*. If we maintain constant only the axis of suspension, the length l' depends on the distance l. The graphic of the function $l' = l'(l)$ is a branch of a hyperbola contained in the second octant of the plane (Fig. 14.8); we notice that for $l = i_C$ we get $l'_{\min} = 2i_C = 2l$.

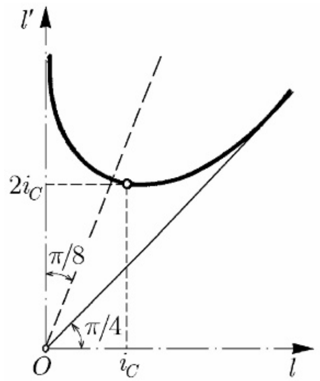

Fig. 14.8 Diagram of l' vs l in case of a physical pendulum

Assuming that the axis of suspension is a principal axis of inertia (e.g., in case of a geometrical and mechanical symmetry with respect to the Ox_1x_2-plane, we have $I_{23} = I_{31} = 0$) and observing that $M_{O1} = M_{O2} = 0$, it results that this one is a permanent axis of rotation too (in a large sense, with $M_{O3} \neq 0$); hence, it is sufficient only one point of support for the axis, having only one constraint force at $O' \equiv O$, the components of which are given by (14.2.5) in the form

$$R_1' = -M\left(g\cos\theta + l\omega^2\right), \quad R_2' = -M\left(g\cos\theta + l\dot{\omega}\right), \quad R_3' = 0, \tag{14.2.9}$$

and which is situated in the Ox_1x_2-plane. Starting from the equation (14.2.6), which we multiply by $\dot{\theta} = \omega$, we obtain

$$\frac{\omega^2}{2} = \frac{g}{l'}\cos\theta + C, \quad C = \frac{\omega_0^2}{2} - \frac{g}{l'}\cos\theta_0, \tag{14.2.9'}$$

where we take into account the initial conditions $\theta(t_0) = \theta_0$, $\omega(t_0) = \omega_0$; finally, we can write

$$R_1' = -Mg\frac{l' + 2l}{l'}\cos\theta - 2CMl, \quad R_2' = Mg\frac{l_1}{l'}\sin\theta, \quad R_3' = 0. \tag{14.2.9''}$$

Hence, the constraint forces are periodical functions of time with the period T; the component R_2' does not depend on the initial conditions, being directed in the sense of the motion (R_2' has the same sign as $\sin\theta$, hence as θ), but the component R_1' depends on these conditions, being always directed in the opposite sense to that in which is the mass centre ($R_1' < 0$).

If the angular velocity $\omega = \dot{\theta}$ maintains its sign, then the motion is circular and the pendulum is rotating in the same sense. From (14.2.9') we notice that, to have such a motion, it is necessary and sufficient that $C > g/l'$, wherefrom

$$\omega_0 > 2\sqrt{\frac{g}{l}}\cos\frac{\theta_0}{2}; \tag{14.2.10}$$

the initial angular velocity (considered to be positive) must be greater than a given value, depending on the initial conditions (angle θ_0) too. Such a situation is encountered, e.g., in case of a fly wheel, the axle of which (assumed to be horizontal) does not pass exactly through the mass centre, but at a short distance to that one; the phenomenon of resonance (the frequency of the perturbing constraint force is close to the frequency of the proper vibration), which leads to a critical rotation speed, must be avoid.

From the equation (14.2.6) it results that the physical pendulum is at rest with respect to the inertial frame of reference $\mathscr{R}'$ if $\sin\theta_0 = 0$, where θ_0 corresponds to the initial position (at the moment $t = t_0$); it results that $\theta_0 = 0$ or $\theta_0 = \pi$, the centre of mass C being situated on the vertical of the centre of suspension. Obviously, only the position $\theta_0 = 0$ is a stable position of equilibrium; in this case $\theta(t) = 0$ and $\dot{\theta}(t) = \omega(t) = \omega_0 = 0$.

We have seen that the centre of oscillation is at a distance $l' > l$ (we can have an equality only if $i_C = 0$, hence only if the rigid solid is reduced to a particle or to a material segment of a line, parallel to the axis of suspension) of the suspension centre;

Huygens showed that all the points of the axis of oscillation move as they would be mathematical pendulums connected to the axis of suspension by inextensible threads of length l'.

The second relation (14.2.8) can be written also in the form

$$ll_1 = \frac{I_C}{M} = i_C^2. \tag{14.2.11}$$

This allows to state

Theorem 14.2.3 (*Huygens*) *The suspension and oscillation axes of a physical pendulum are reciprocal.*

This property with an involutive character shows that if one takes the axis of oscillation as axis of suspension, then the axis of suspension becomes an axis of oscillation.

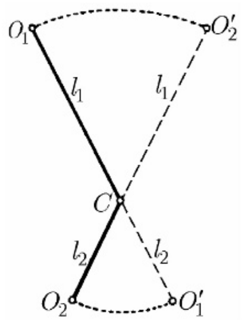

Fig. 14.9. Huygens's theorem

The relation (14.2.8) can be written in the form of an equation of second degree

$$l^2 - l'l + i_C^2 = 0, \tag{14.2.12}$$

of roots (for given l' and i_C correspond two values for l)

$$l_{1,2} = \frac{1}{2}\left(l' \pm \sqrt{l'^2 - 4i_C^2}\right) = \frac{l'}{2}\left[1 \pm \sqrt{1 - \left(\frac{2i_C}{l'}\right)^2}\right]. \tag{14.2.12'}$$

We can thus state

Theorem 14.2.4 (*Huygens*) *The locus of the straight lines of given direction, which – taken as axes of suspension – allow to a physical pendulum to oscillate about them with a given period is formed by two circular cylinders* (*the common axis of which passes through the mass centre and has the given direction*).

The period T being given, it results that the length l' of the synchronous simple pendulum is, as well, given. Assuming that $l' > 2i_C$, the formula (14.2.12') gives the lengths l_1 and l_2 of the radii of the two cylinders; if $l' = 2i_C = l'_{\min}$ (corresponding to Fig. 14.8), then the two cylinders coincide ($l_1 = l_2 = l'_{\min}/2 = i_C$).

Observing that $l_1 l_2 = i_C^2$, it results $l_1 \geq i_C$ and $l_2 \leq i_C$ (or conversely); as well, we have $l_1 + l_2 = l'$. If the centre of suspension O_i corresponds to the centre of oscillation O_i', $i = 1,2$, and if we assume that these centres are coplanar with the centre of mass C, so that the points O_1 and O_2' and the points O_2 and O_1', respectively, be on the same arcs of circle (Fig. 14.9), the Theorem 14.2.4 leads to

Theorem 14.2.4' (*Huygens*) *If two parallel axes of suspension of a physical pendulum lead to the same length for the corresponding synchronous simple pendulums, then this length is equal to the sum of the distances from the mass centre to the two axes.*

This theorem is justified for the relation of order $l_2 \leq i_C \leq l_1$. In the particular case in which the two axes of suspension are coplanar with the mass centre, we find again the Theorem 14.2.3.

14.2.1.3 Experimental Determination of Moments of Inertia of Rigid Solids

The theory of the physical pendulum allows an experimental determination of the moment of inertia of a rigid solid with respect to a given axis, which pierces this solid. To do this, one takes the respective axis as axis of suspension and one measures the isochronous oscillation period of the rigid solid, in its behaviour as a physical pendulum; knowing the own weight of the rigid solid (the mass and the gravity acceleration) and the distance from the centre of mass to the considered axis, the formula (14.2.7') gives the possibility to compute the axial moment of inertia in the form

$$I_{33} = \frac{T^2}{4\pi^2} Mgl = \left(\frac{T}{2\pi}\right)^2 Mgl . \tag{14.2.13}$$

Because the quantities M and I cannot be measured with a sufficient precision, one resorts to experimental artificial means. A rigid mass $\overline{M}$ uniformly distributed around the axis of suspension is added, so that its centre of mass be on this axis, the corresponding moment of inertia being $\overline{I}_{33}$. Obviously, we have

$$I_{33} + \overline{I}_{33} = \frac{\overline{T}^2}{4\pi^2} (M + \overline{M}) g\overline{l} .$$

Taking into account the relation of static moments $(M + \overline{M})\overline{l} = Ml$ and the relation (14.2.13), we obtain

$$I_{33} = \frac{T^2}{\overline{T}^2 - T^2} \overline{I}_{33} = \frac{1}{(\overline{T}/T)^2 - 1} \overline{I}_{33} . \tag{14.2.14}$$

Knowing the moment of inertia $\overline{I}_{33}$ and measuring the periods T and $\overline{T}$, we get the moment of inertia I_{33} with a sufficient good precision.

14.2.1.4 Experimental Determination of the Gravity Acceleration. The Borda, the Kater and the Bessel Pendulums

The physical pendulum and the formula (14.2.7') of its isochronous oscillations allow, as well, to determine experimentally the gravity acceleration; we get thus

$$g = \frac{4\pi^2}{T^2} l' = \frac{4\pi^2}{T^2} \frac{I_{33}}{Ml} = \frac{4\pi^2}{T^2} \left[1 + \left(\frac{i_C}{l} \right)^2 \right] l \,. \tag{14.2.15}$$

We assume that in this formula are known or measurable all the quantities, so that it is possible to obtain g. In 1792, Borda used a physical pendulum formed by a homogeneous sphere of platinum, suspended by a thin metallic thread, of negligible mass with respect to the mass of the sphere; the mass centre C of the physical pendulum is thus practically situated at the centre of the sphere of radius R and mass M. The formula (3.1.27) leads to the central radius of gyration $i_C = \sqrt{2/5}R$. If l is the distance from the centre of suspension to the centre of the sphere and if we determine experimentally the period T of the isochronous oscillations, then we obtain the acceleration g of gravity. We notice that, to have a result with the best precision, one must take into account the influence of the medium (temperature correction, reduction to vacuum, resistance of the air etc.), the influence of the suspension (the curvature of the supporting blade edge, the displacement of the support, the friction on the axle etc.) and the influence of the experimental measurements (the measure of the distance $l = \overline{OC}$ and the measure of the period T of the isochronous oscillations).

The period of oscillation of the pendulum is obtained by one of the known experimental methods (e.g., *the method of simultaneousness* or *the method of registering*); the use of the formula (14.2.15) corresponds to an *absolute determination* of the gravity constant g. A rigorous determination being particularly difficult, as we have seen, one obtains such results only in *gravimetrical stations of reference* (such stations are, e.g., these of Potsdam and Helsinki). In other stations one obtains *relative determinations*, starting from the determinations T_0 and g_0 in a station of reference, using an identical physical pendulum, in the same conditions (to have the same length l' of the synchronous simple pendulum),

$$g = \left(\frac{T_0}{T} \right)^2 g_0 \,. \tag{14.2.16}$$

The gravity acceleration g is thus obtained by a simple measurement of the period T.

If in the Theorem 14.2.4' the plane of the axes of suspension contains the centre of mass C too, so that the latter one be situated between the points O_1 and O_2 (in this case $O_2 \equiv O_1'$), then the length of the synchronous simple pendulum is equal to the distance between the two axes; the respective pendulum is a *reversible pendulum*, allowing the determination of the centre of oscillation and of the length $l' = l_1 + l_2$. Such a pendulum was built for the first time in 1818 by H. Kater from a bar of bronze, along which glide two masses m_1 and m_2, with the aid of a micrometric screw, and which has two centres of suspension O_1 and O_2, with two blades; thus, by the displacement of the mass centre C, one can make one of the centres of suspension to become centre of oscillation for the other one, and reciprocally. In fact, one obtains

$$T_1^2 = \frac{4\pi^2}{g}\left(l_1 + \frac{i_C^2}{l_1} \right), \quad T_2^2 = \frac{4\pi^2}{g}\left(l_2 + \frac{i_C^2}{l_2} \right),$$

where i_C must be the same in the two expressions, till an experimental error; eliminating the central radius of gyration, it results

$$T^2 = \frac{4\pi^2}{g} l' = \frac{4\pi^2}{g}(l_1 + l_2) = \frac{T_1^2 l_1 - T_2^2 l_2}{l_1 - l_2}, \quad l_1 \neq l_2, \tag{14.2.17}$$

wherefrom

$$T = T_1 \sqrt{1 + \frac{l_2}{l_1 - l_2}\left[1 - \left(\frac{T_2}{T_1} \right)^2 \right]} = T_1 + T_1 \frac{l_2}{2(l_1 - l_2)}\left[1 - \left(\frac{T_2}{T_1} \right)^2 \right] + \ldots \tag{14.2.17'}$$

Hence, the passing from a centre of suspension to another one leads to a term of correction applied to the period noticed in case of the first axis of suspension. From (14.2.17) we obtain also

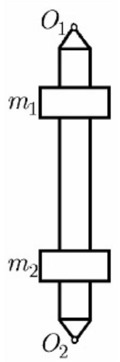

Fig. 14.10 Bessel's pendulum

$$g = 4\pi^2 \frac{l_1^2 - l_2^2}{T_1^2 l_1 - T_2^2 l_2}. \tag{14.2.17''}$$

This formula is much used in geodesy to determine experimentally the gravity acceleration by two measurements of periods corresponding to two axes of suspension.

Making $T_1 - T_2 \to 0$ and taking into account the Theorem 14.2.4', we find again the formula (14.2.15).

If the masses m_1 and m_2 have a different form, then one obtains experimental errors due to a different resistance of the air. To eliminate these errors, Bessel considered a physical pendulum for which the masses m_1 and m_2 have the same form and the same dimension, but different masses (a hollow cylinder and a full one) (Fig. 14.10).

14.2.1.5 Voinaroski's Annular Pendulum

One has used physical pendulums of various forms to determine the gravity acceleration. Such a *pendulum* is an *annular* one, imagined in 1939 by L. Teodoriu and R. Voinaroski; it is a homogeneous right cylinder of steel, having a height h and a circular annulus section, of radii R_i and R_e, suspended at a centre O on the internal face by a blade, along the respective generatrix (Fig. 14. 11). The moment of inertia with respect to a central principal axis of inertia along the direction of the generatrix is $I_C = \pi \mu h \left(R_e^4 - R_i^4 \right) / 2$, where μ is the density; the mass of the pendulum is, obviously, given by $M = \pi \mu h \left(R_e^2 - R_i^2 \right)$, so that $I_C = M \left(R_e^2 + R_i^2 \right) / 2$ and $i_C^2 = \left(R_e^2 + R_i^2 \right) / 2$. We obtain thus

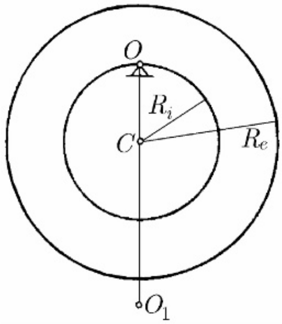

Fig. 14.11 Voinaroski's Annular Pendulum

$$l' = R_i + \frac{R_e^2 + R_i^2}{2R_i} = \frac{1}{2}\left(3R_i + \frac{R_e^2}{R_i} \right). \tag{14.2.18}$$

Observing that $\mathrm{d}l' / \mathrm{d}R_i = (1/2)\left[3 - (R_e / R_i)^2 \right]$, we obtain $l'_{\min} = R_e \sqrt{3} = 3R_i$ for $R_i = R_e / \sqrt{3}$; in this case, it results (Fig. 14.11)

$$g = \frac{4\pi^2 \sqrt{3}}{T^2} R_e = \frac{12\pi^2}{T^2} R_i . \tag{14.2.19}$$

The construction of the annular pendulum so that l' have a minimal value (l' is determined in a range zone) leads to small experimental errors due to various technical deficiencies. R. Voinaroski realized this pendulum in the Laboratory of Mechanics of the University of Bucharest in the above mentioned conditions, obtaining $g_{\mathrm{Buc}} = 9.806\ \mathrm{m/s^2}$ (with three exact decimals).

14.2.1.6 The Mach Inclined Pendulum

The inclined pendulum imagined by E. Mach is a physical pendulum, the suspension axis of which is inclined by an angle α, $0 < \alpha < \pi/2$, with respect to the horizontal plane (Fig. 14.12). The mass centre C oscillates in a plane Π normal to the axis Δ, the trace of which on the plane being just the suspension centre O. The equilibrium position of the pendulum is that for which the moment of the own weight $M\mathbf{g}$ with respect to the Δ-axis is equal to zero, the meridian plane determined by the axis Δ and the centre C being vertical for this position; during the motion, the meridian plane oscillates about this position of equilibrium. We decompose the force $M\mathbf{g}$ in a component $Mg\sin\alpha$, parallel to the axis Δ, and a component $Mg\cos\alpha$, normal to this axis, contained in the plane Π; observing that the first component gives a null moment with respect to the Δ-axis, we obtain the equation of motion of Mach's pendulum in the form

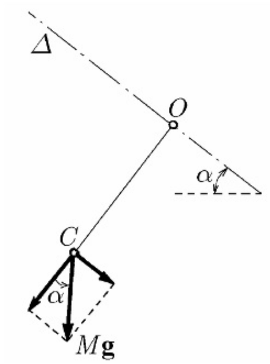

Fig. 14.12 Mach's inclined pendulum

$$\ddot{\theta} + \frac{g}{l'}\sin\theta = 0\,, \quad l' = \frac{I_\Delta}{Ml}\cos\alpha\,. \tag{14.2.20}$$

Hence, we find again the previous results, where we replace the distance l by $l\cos\alpha$. In case of isochronous oscillations, we find the period

$$T_\alpha = 2\pi\sqrt{\frac{I_\Delta}{Mgl}\sec\alpha}\,. \tag{14.2.21}$$

For $\alpha = 0$ we find again the period (14.2.7'), so that

$$\cos^2 \alpha = \frac{T_0^2}{T_\alpha^2}, \tag{14.2.22}$$

and we have an experimental verification of the above obtained results.

If $\alpha = \pi / 2$, then the Δ-axis is vertical and the equation of motion becomes $\ddot{\theta} = 0$; it results a uniform rotation if $\omega_0 = \dot{\theta}(t_0) \neq 0$. If $\omega_0 = 0$, then any position $\theta = \theta(t_0) = \theta_0$ is a position of equilibrium. A door with a vertical axis illustrates this assertion.

14.2.1.7 The Weber-Gauss Torsion Pendulum

Let be a homogeneous rigid solid $\mathscr{S}$ with axial symmetry, suspended at a point O' by an inextensible thread along the symmetry axis (the descendent vertical of the point O'), which is taken as $O'x_3'$-axis (Fig.14.13,a). This solid is rotated by an angle θ_0 (till an initial position), being then let free. We obtain thus *the Weber-Gauss torsion pendulum*. The thread plays the rôle of a constraint and acts as a couple of torsion of moment $\mathrm{M} = -\nu^2 \theta \mathbf{i}_3$ (we take $Ox_3 \equiv O'x_3'$, $O \equiv O'$) upon the rigid body, $\theta = \theta(t)$ being the angle of rotation about the stable position of equilibrium. The equation of motion with respect to the axis of rotation will be

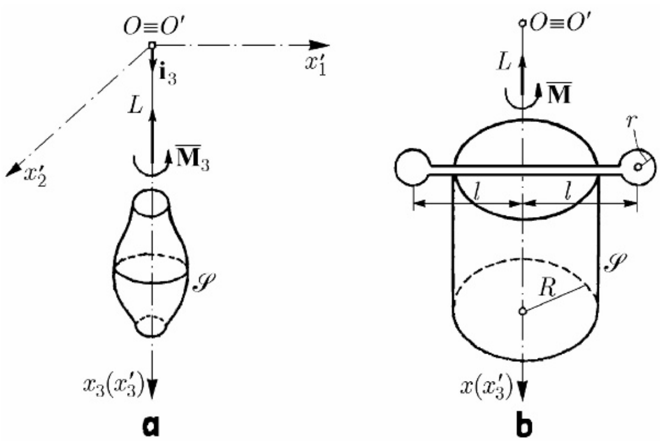

Fig. 14.13 The Weber-Gauss torsion pendulum

$$I_{33}\ddot{\theta} = -\nu^2 \theta, \tag{14.2.23}$$

where we assume that $\nu^2 = k \mu_f d^4 / L$, d and L being the diameter of the cross section of the thread and its length, respectively, μ_f is the linear density of it, while $k > 0$ is a constant of the nature of an angular acceleration. We obtain thus

$$\theta(t) = \theta_0 \cos \frac{\nu}{\sqrt{I_{33}}} t, \quad T = \frac{2\pi}{\nu} \sqrt{I_{33}}, \quad \nu > 0, \tag{14.2.23'}$$

so that

$$T = \frac{2\pi}{\nu}\sqrt{\frac{MR^2}{2}} = \frac{\pi}{\nu}\sqrt{2M}R \tag{14.2.24}$$

for isochronous oscillations.

If the rigid solid $\mathscr{S}$ is a circular cylinder of radius R and mass M, then we have $I_{33} = MR^2/2$. We can verify the mathematical model considered for the torsion of threads adding to the rigid solid two equal homogeneous spheres, each of radius r and mass m, connected by a thin rod of length $2l$ and negligible mass with respect to m (Fig. 14.13,b). Denoting by I'_{33} the axial moment of inertia of the new obtained mechanical system and using the Huygens-Steiner theorem (formula (3.1.113)), we obtain

$$T' = \frac{2\pi}{\nu}\sqrt{I'_{33}} = \frac{2\pi}{\nu}\sqrt{\frac{MR^2}{2} + 2ml^2 + \frac{4}{5}mr^2}\,. \tag{14.2.24'}$$

Assuming that the periods T and T' are obtained experimentally, one determines the constant ν of torsion of the thread. Practically, the values thus obtained differ very little; if the difference is not great, then one can assume that the determination is sufficiently good. Eliminating ν between the relations (14.2.24) and (14.2.24'), one obtains the moment of inertia I_{33} by experimental determinations; this result can be of interest for any rigid solid of revolution $\mathscr{S}$.

14.2.2 Plane-parallel Motion of the Rigid Solid

The plane-parallel motion of the rigid solid is a particular case of motion which is frequently encountered in practice. In this order of ideas, after some applications, we make considerations concerning the dynamics of the three-dimensional motion of an airplane and then the plane-parallel motion of it. As well, we present the motion with sliding and rolling friction on an inclined or on a horizontal plane.

14.2.2.1 General results

As we have seen in Chap. 5, Subsec. 2.3.4, a rigid solid $\mathscr{S}$ has a *plane-parallel motion* if three non-collinear points of it are contained during the motion in a fixed plane, hence if a plane section (the $O\overline{x}_1\overline{x}_2$-plane or the Ox_1x_2-plane) of the rigid solid slides on a fixed plane (the $O'x'_1x'_2$-plane); the axes $O'x'_3$ and $O\overline{x}_3$ (or Ox_3) are normal to these planes. Without particularizing the motion, we can take the mass centre as pole of the non-inertial frame of reference ($O \equiv C$) (Fig. 14.14). In this case, the rigid solid $\mathscr{S}$ has only three degrees of freedom and we can choose as parameters which specify the motion the co-ordinates ρ'_1 and ρ'_2 ($\rho'_3 = 0$) of the mass centre C and the angle θ made by the axis Cx_1 with the axis $C\overline{x}_1$. The fixed and the movable axoids are two cylinders the traces of which on the fixed plane are the fixed and movable centrods, respectively, tangent one to the other at the instantaneous centre of rotation (see Chap. 5, Sect. 2.3.4).

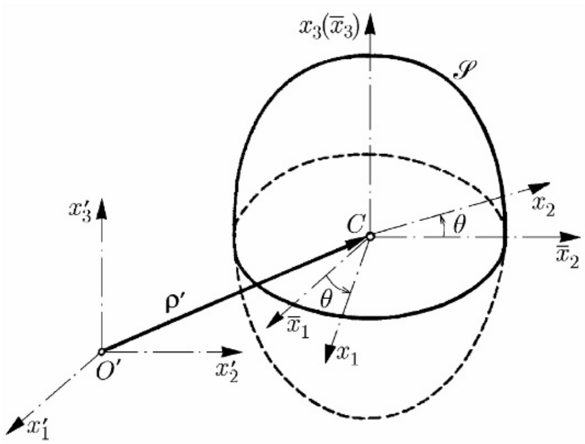

Fig. 14.14 Plane-parallel motion of the rigid solid

Starting from the equations (14.1.48), (14.1.52), we can write the equations of motion along the axes of the frame of reference $\mathscr{R}$ in the form

$$\begin{aligned} M(\dot{v}'_{C1} - \omega v'_{C2}) &= R_1 + \overline{R}_1\,, \\ M(\dot{v}'_{C1} + \omega v'_{C2}) &= R_2 + \overline{R}_2\,, \\ 0 &= R_3 + \overline{R}_3\,, \end{aligned} \tag{14.2.25}$$

$$\begin{aligned} I_{31}\dot{\omega} - I_{23}\omega^2 &= M_{C1} + \overline{M}_{C1}\,, \\ I_{23}\dot{\omega} + I_{31}\omega^2 &= M_{C2} + \overline{M}_{C2}\,, \\ I_{33}\dot{\omega} &= M_{C3} + \overline{M}_{C3}\,, \end{aligned} \tag{14.2.25'}$$

where we noticed that $v'_{C3} = 0$, while $\boldsymbol{\omega} = \omega \mathbf{i}_3$; here $\{\mathbf{R}, \mathbf{M}_C\}$ is the torsor of the given forces, while $\{\overline{\mathbf{R}}, \overline{\mathbf{M}}_C\}$ is the torsor of the constraint forces. Projecting the equation (14.1.47) on the axes of the inertial frame of reference $\mathscr{R}'$, we obtain the scalar equations (which replace the equations (14.2.25))

$$M\frac{\mathrm{d}^2\rho_1'}{\mathrm{d}t^2} = R_1 + \overline{R}_1\,,\quad M\frac{\mathrm{d}^2\rho_2'}{\mathrm{d}t^2} = R_2 + \overline{R}_2\,,\quad 0 = R_3 + \overline{R}_3\,, \tag{14.2.26}$$

where the notations correspond to these co-ordinate axes.

To have a plane-parallel motion of the rigid solid, it is sufficient that three non-collinear points of it (e.g., the points C, $P_1(h_1, 0, 0)$ and $P_2(0, h_2, 0)$, $h_1, h_2 > 0$) be during the motion in the fixed plane; assuming that the motion of these points is frictionless, the corresponding constraint forces are of the form $\mathbf{R}_C = R_{C3}\mathbf{i}_3$, $\mathbf{R}_1 = R_{13}\mathbf{i}_3$, $\mathbf{R}_2 = R_{23}\mathbf{i}_3$, which leads to $\overline{R}_1 = \overline{R}_2 = 0$, $\overline{R}_3 = R_{C3} + R_{13} + R_{23}$, $\overline{M}_{C1} = h_2 R_{23}$, $\overline{M}_{C2} = -h_1 R_{13}$, $\overline{M}_{C3} = 0$. Associating adequate initial conditions of Cauchy type, we notice that the third equation (14.2.25') determines the angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$; then the first two equations (14.2.25) give the components $v'_{Cj} = v'_{Cj}(t)$, $j = 1,2$, of the velocity of the mass centre C with respect to the inertial frame of reference. In this case too, we can adopt correspondingly the theorem of existence and uniqueness. The first two equations (14.2.25') allow to calculate the components R_{23} and R_{13} of the constraint forces, while the third equation (14.2.25) specifies the constraint force R_{C3}. Thus, the plane-parallel motion of the rigid solid is statically determinate; in the case in which the number of the points of the solid which coincide all the time with the fixed plane is greater than three, then the problem becomes statically indeterminate (due to the model of rigid solid adopted). If the motion of the rigid solid is a plane-parallel one, the constraint forces vanishing, one must have $R_3 = 0$; hence, the resultant of the given forces must be parallel to the fixed plane. The moment of the given forces is directed along the normal to the fixed plane ($M_{C1} = M_{C2} = 0$) if and only if the Cx_3-axis is a central principal axis of inertia, hence if $I_{23} = I_{31} = 0$. In this case, the pseudomoment of momentum $\mathbf{K}^C$ has the components $K_1^C = \text{const}$, $K_2^C = \text{const}$; because at the initial moment we have $K_1^C = K_2^C = 0$, it results that during the motion we have $\mathbf{K}^C = K_3^C \mathbf{i}_3$.

14.2.2.2 The Plane-Parallel Motion of a Rigid Straight Bar

Le be a rigid straight bar AB of length $2l$, which is moving in the fixed plane $O'x'_1x'_2$ (in a modelling of unidimensional solid). Assuming that the bar is homogeneous, of linear density μ, the mass centre will be at its middle; in the same fixed plane, we consider also the non-inertial frames of reference $C\overline{x}_1\overline{x}_2$ (the axes of which are parallel to the axes of the inertial frame $O'x'_1x'_2$) and Cx_1x_2, the axis Cx_1 being along the bar. The position of the bar will be specified by the co-ordinates ρ'_1 and ρ'_2 of the centre of mass and by the angle θ, made by the Cx_1-axis with the $C\overline{x}_1$-axis. Because of the symmetry, the Cx_1-axis is a central principal axis of inertia, any normal axis at C to that one having the same property; thus, we have $I_{23} = I_{31} = 0$. On the other hand, $I_{33} = \int_{-l}^{l} \mu x_1^2 \mathrm{d}x_1$, so that $I_{33} = 2\mu l^3/3 = Ml^2/3$, where $M = 2\mu l$ is the mass of the bar.

We assume that the bar is in a vertical plane and is subjected to the action of its own weight $\mathbf{G} = M\mathbf{g}$; its centre of mass C is connected to the fixed point O' by an inextensible thread (in which arises the tension T) (Fig. 14.15a). It results $\mathbf{M}_C = \mathbf{0}$ so that $\dot{\omega} = 0$, wherefrom $\theta = \omega_0 t + \theta_0$, θ_0 and ω_0 corresponding to the position and to the angular velocity at the initial moment $t = 0$, respectively. The equations of motion of the mass centre in the non-parallel frame of reference $\mathscr{R}$ have the form

$$\begin{aligned} M(\dot{v}'_{C1} - \omega_0 v'_{C2}) &= Mg\cos\theta - T\cos(\theta - \varphi), \\ M(\dot{v}'_{C2} + \omega_0 v'_{C1}) &= -Mg\sin\theta - T\sin(\theta - \varphi), \end{aligned} \tag{14.2.27}$$

but it is difficult to study them directly; it is more convenient to use the equations (14.2.26), which lead to

$$M\frac{\mathrm{d}^2\rho'_1}{\mathrm{d}t^2} = Mg - T\cos\varphi, \quad M\frac{\mathrm{d}^2\rho'_2}{\mathrm{d}t^2} = -T\sin\varphi. \tag{14.2.27'}$$

Observing that $\rho'_1 = h\cos\varphi$, $\rho'_2 = h\sin\varphi$, $h = \overline{O'C}$ and eliminating the tension T, we obtain the equation of the mathematical pendulum

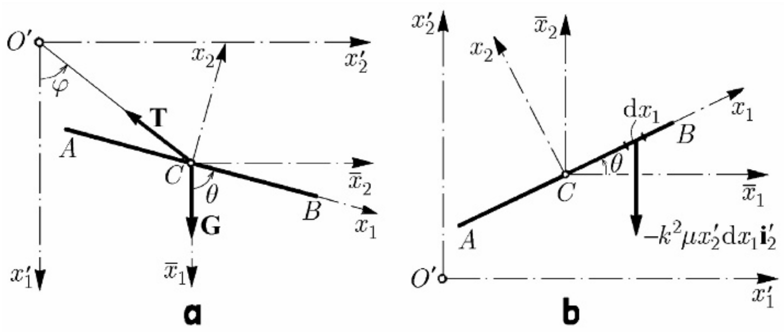

Fig. 14.15 Plane-parallel motion of a rigid straight bar

$$h\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + g\sin\varphi = 0,$$

the tension being given by

$$T = M\left[g\cos\varphi + h\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right].$$

Hence, the centre of mass C oscillates as a simple pendulum about the fixed point O', the bar AB rotating uniformly around it in the considered vertical plane.

Let us suppose now that the straight bar slides frictionless on a horizontal fixed plane $O'x'_1x'_2$, being attracted by the fixed axis $O'x'_1$ in direct proportion to the mass and to the distance of each point of it to this axis (Fig. 14.15b). In this case (we use the formula (3.1.9') of static moments)

$$R_1 = 0, \quad R_2 = -\int_{-l}^{l} k^2 \mu x'_2 \mathrm{d}x_1 = -k^2 M\rho'_2, \quad k = \mathrm{const},$$

the equations of motion (14.2.26) leading to $\rho_1' = at + b$, $\rho_2' = A\sin(kt + \alpha)$; eliminating the time t, we obtain

$$\rho_2' = A\sin\left[\frac{k}{a}(\rho_2' - b) + \alpha\right], \tag{14.2.28}$$

the trajectory of the mass centre C being a sinusoid in the fixed plane. As well (the static moment with respect to an axis which passes through C vanishes)

$$\begin{aligned} M_{C3} &= -\int_{-l}^{l} k^2 \mu \bar{x}_1 x_2' \mathrm{d}x_1 = -k^2 \int_{-l}^{l} \mu \bar{x}_1 (\rho_2' + \bar{x}_2) \mathrm{d}x_1 = -k^2 I_{\bar{x}_1 \bar{x}_2} \\ &= -k^2 \int_{-l}^{l} \mu \sin\theta \cos\theta x_1^2 \mathrm{d}x_1 = -k^2 I_{33} \sin\theta \cos\theta \,. \end{aligned}$$

The angle $\theta = \theta(t)$ verifies the equation

$$\ddot{\theta} + k^2 \sin\theta \cos\theta = 0\,. \tag{14.2.29}$$

We find thus again the equation of the mathematical pendulum for the argument 2θ; the oscillations defined by the equation (14.2.29) have been denoted by W. Thomson and P.G. Tait, in 1861, quadrantal oscillations. Multiplying by $2\dot{\theta}$ and integrating, we get

$$\dot{\theta}^2 = \omega_0^2 - k^2 \sin\theta\,, \tag{14.2.29'}$$

where ω_0 is the angular velocity for $\theta = 0$; if $\omega < k$, then the bar oscillates about the axis $C\bar{x}_1$, while if $\omega > k$, then the bar has a circular motion (it rotates in the same sense). If $\omega = k$, we obtain

$$\dot{\theta} = \omega\cos\theta\,, \quad t = \frac{1}{\omega}\ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right), \tag{14.2.30}$$

where $\theta = 0$ for $t = 0$; in this case, the bar tends to the $C\bar{x}_2$-axis in an infinite time ($t \to \infty$ for $\theta \to \pi/2$).

14.2.2.3 The Plane-Parallel Motion of a Rigid Double Circular Cone and of a Rigid Sphere on an Inclined Plane

Let be an inclined plane formed by two axes Δ' and Δ'', concurrent at O', equally inclined on a horizontal plane and situated over it, and let be $O'x_1'$ the bisectrix of the angle formed by these axes. We put on them a rigid double cone, formed by two equal homogeneous circular cones, the bases of which are joined, so that their common plane coincides with the vertical plane Π and passes through the bisectrix $O'x_1'$, which makes the angle α with the horizontal. We assume that this solid is placed, in the initial position, at the lowest part of the angle formed by the axes Δ' and Δ'', in the proximity of the point O'; then, it is left to a free rolling, without sliding, on these axes, under the action of its own weight Mg. In the fixed plane Π we specify the inertial frame of reference $O'x_1'x_2'$. The axes Δ' and Δ'' are projected on the plane Π along the $O'x_1'$-axis, while the points at which the double cone lays on them are projected at the point I; as well, the vertices of the cones are projected at the point C, the centre of mass of the double cone, the whole mechanical system being symmetric with respect to the plane Π. The planes which pass through the axes Δ' and Δ'', respectively, and are tangent to the two cones, make constant angles with the horizontal plane, being thus fixed. Let Δ be their intersection straight line, which is fixed too and is contained in the plane Π, making an angle β with the horizontal. The base circle of the two cones is tangent to the axis Δ, while its centre describes the axis $\overline{\Delta}$, parallel to the axis Δ. We assume that the axis Δ is situated below the horizontal, at the initial moment the mass centre lying over the $O'x_1'$-axis (Fig. 14.16). This plane-parallel motion, in which *the rigid solid, without initial velocity, seems to ascend on the inclined plane* (its centre of mass moves downwards), has been studied in the XIXth century by Résal, Fleury, Mannheim and Saint-Germain.

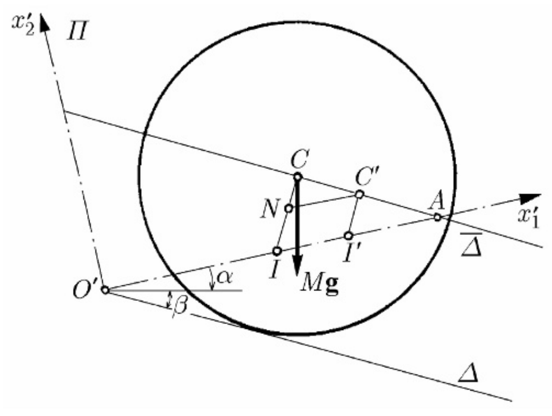

Fig. 14.16 Plane-parallel motion of a rigid double cone

In the plane of symmetry Π, the point I is the instantaneous centre of rotation and lies on the $O'x_1'$-axis; hence, the straight line IC is normal to the trajectory $\overline{\Delta}$ of the point C and we have $\overline{CC'} = \mathrm{d}s = r\mathrm{d}\theta$, $r = \overline{IC}$ and $v_C' = \mathrm{d}s/\mathrm{d}t = r\mathrm{d}\theta/\mathrm{d}t = r\dot{\theta}$, where $\dot{\theta} = \omega$ is the angular velocity, while θ is the angle by which the double cone is rotating, starting from an initial position. At the moment $t + \mathrm{d}t$, $\overline{IC} = r$ becomes $\overline{I'C'} = r - \mathrm{d}r$, $I'C' \parallel IC$, while the parallel through C' to $O'x_1'$ pierces IC at N; we have

$$\overline{CN} = -\mathrm{d}r = \overline{CC'} + \tan(\alpha + \beta),$$

hence

$$\mathrm{d}\overline{\rho}_1 = r\mathrm{d}\theta = -\cot(\alpha + \beta)\mathrm{d}r. \tag{14.2.31}$$

The theorem of kinetic energy is written in the form

$$\mathrm{d}\left[\frac{1}{2}Mr^2\dot{\theta}^2 + \frac{1}{2}Mi_C^2\dot{\theta}^2\right] = -Mg\sin\beta\cot(\alpha + \beta)\mathrm{d}r,$$

where we have used the theorem of Koenig, i_C being the gyration radius with respect to the symmetry axis of the double cone, which is a central principal axis of inertia ($I_C = Mi_C^2$); the elementary work is given by $M\mathbf{g} \cdot \mathrm{d}\overline{\boldsymbol{\rho}}$. By integration,

$$(i_C^2 + r^2)\dot{\theta}^2 = k^2(r_0 - r)\cot^2(\alpha + \beta), \quad k^2 = 2g\sin\beta\tan(\alpha + \beta), \tag{14.2.32}$$

where r_0 corresponds to the initial moment $t = 0$, for which we have $\dot{\theta} = 0$ too. Taking into account (14.2.31) and integrating, the time will be given by the elliptic integral

$$t = -\frac{1}{k}\int_{r_0}^{r}\frac{1}{r}\sqrt{\frac{i_C^2 + r^2}{r_0 - r}}\mathrm{d}r, \tag{14.2.32'}$$

where we have taken the sign minus because it is obvious, from a geometrical point of view, that r diminishes starting from r_0. For $r \to 0$ it results $t \to \infty$, hence the mass centre C tends to the limit position $A \equiv O'x_1' \cap \overline{\Delta}$, without reaching it (or in an infinite time). Starting from (14.2.31) we can also write

$$r = r_0\mathrm{e}^{-\theta\tan(\alpha+\beta)}, \quad s - s_0 = (r_0 - r)\cot(\alpha + \beta), \tag{14.2.33}$$

where we have assumed that $\theta = 0$ for $r = r_0$, while s is the abscissa along the $\overline{\Delta}$-axis. Thus, the relation (14.2.32) may lead to $\overline{\rho}_1 = \overline{\rho}_1(t)$, obtaining the law of motion of the mass centre along the $\overline{\Delta}$-axis; the same relation shows that its velocity is given by

$$v_C' = kr\sqrt{\frac{r_0 - r}{i_C^2 + r^2}}\cot(\alpha + \beta). \tag{14.2.32''}$$

This velocity vanishes at the initial position $r = r_0$ and at the final position $r = 0$; it has a maximum in the interval of time in which the motion takes place. From a kinematical point of view, the motion of the base circle of the cones is obtained rolling the logarithmic spiral given by the first equation (14.2.33) on the $O'x_1'$-axis.

Mannheim considers in 1859 a similar problem for a homogeneous sphere of radius R which, as well, seems to ascend on the inclined plane in the same conditions (the centre of mass moves downwards). We keep the previous notations. If we denote by N the point at which the axis Δ', e.g., is tangent to the sphere, we have $\overline{O'C}^2 = R^2 + \overline{O'N}^2 = \rho_1'^2 + \rho_2'^2$, where we have specified the mass centre C (the centre of the sphere) with respect to the non-inertial frame of reference $O'x_1'x_2'$; observing that $\overline{O'N} = \rho_1' \cos\varphi$, where 2φ is the angle formed by the axes Δ' and Δ'', it results

$$\frac{\rho_1'^2}{R^2}\sin^2\varphi + \frac{\rho_2'^2}{R^2} = 1\,.$$

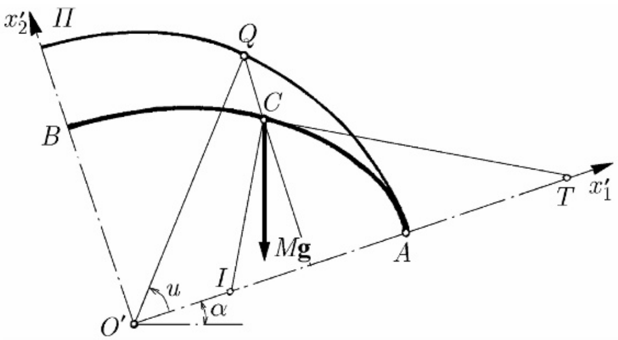

Fig. 14.17 Plane-parallel motion of a rigid sphere

Hence, the centre of mass C describes an arc of ellipse of semiaxes $a = \overline{O'A} = R/\sin\varphi$ and $b = \overline{O'B} = R$ (Fig. 14.17); the normal at C to the ellipse pierces the $O'x_1'$-axis at the instantaneous centre of rotation I in the vertical plane Π. The normal from C to $O'x_1'$ pierces the circle of radius a and centre O' at Q; introducing the eccentric anomaly u (see Chap. 9, Sect. 2.1.3, Fig. 9.8 too), the parametric equations of the ellipse are $\rho_1' = a\cos u$, $\rho_2' = b\sin u$. The tangent at C to the ellipse (the equation of which is obtained by halving) pierces the $O'x_1'$-axis at T so that $\overline{O'T} = a^2/\rho_1'$; in the right triangle ICT we get

$$r^2 = \overline{IC}^2 = \left(\frac{b}{a}\right)^2 \left(a^2\sin^2 u + b^2\cos^2 u\right).$$

We can thus write

$$\mathrm{d}s^2 = \mathrm{d}\rho_1'^2 + \mathrm{d}\rho_2'^2 = \left(a^2\sin^2 u + b^2\cos^2 u\right)\mathrm{d}u^2 = \left(\frac{a}{b}\right)^2 r^2\mathrm{d}u^2$$

for the element of arc of ellipse. But $\mathrm{d}s = r\mathrm{d}\theta$, so that $\mathrm{d}\theta = (a/b)\mathrm{d}u$; the velocity of the mass centre will be $v'_C = r\dot{\theta} = r(a/b)\dot{u}$. Applying the theorem of kinetic energy, as in the preceding case, we get the equation

$$\left[\left(\frac{a}{b}\right)^2 i_C^2 + a^2 \sin^2 u + b^2 \cos^2 u\right]\dot{u}$$
$$= 2g\left[a\sin\alpha\left(\cos u_0 - \cos u\right) + b\cos\alpha\left(\sin u_0 - \sin u\right)\right], \qquad (14.2.34)$$

which allows to compute the time t as function of the anomaly u by a quadrature. From a kinematical point of view, the motion of the sphere is obtained by rolling an epicycloid on the $O'x'_1$-axis.

14.2.2.4 Dynamics of Motion of the Airplane

We present firstly the general equations of motion of the airplane; this one will be modelled as a free rigid mechanical system with a plane of symmetry, subjected to the action of given forces: own weight (dead and useful load), forces of aerodynamical resistance and driving forces. The inertial frame of reference $\mathscr{R}'$, called *geodesic frame*, has the origin O' at the initial position of the mass centre C; the $O'x'_1$-axis is taken along the direction of the initial velocity of the centre C and the $O'x'_3$-axis along the descendent vertical. The non-inertial frame $\mathscr{R}$ will be rigidly linked to the airplane, having the pole at C; the Ox_1-axis is along the longitudinal axis of the airplane, its sense coinciding with the sense of advance, while the Ox_3-axis is contained in the symmetry plane.

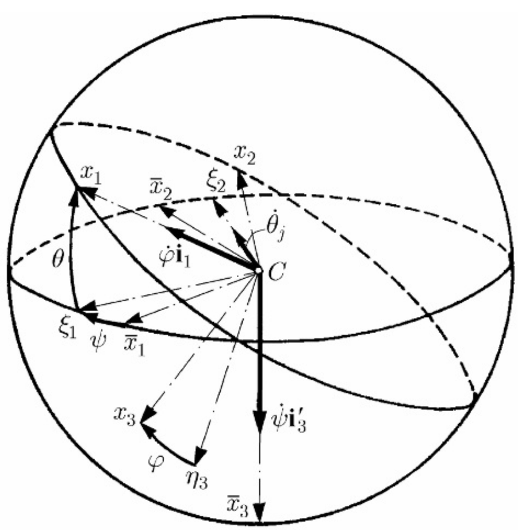

Fig. 14.18 Dynamics of the motion of an airplane

To pass from the initial position of the airplane to its actual position, hence from the frame $\mathscr{R}'$ to the frame $\mathscr{R}$, one performs a translation, which brings the pole O' at the pole C so as to coincide with the frame $\overline{\mathscr{R}}$ (the axes of which are parallel to those of the frame $\mathscr{R}'$) and a rotation, composed of three successive rotations (Fig. 14.18); i) a rotation $\mathbf{\Psi}$ of angle ψ (*angle of gyration*, which puts in evidence the rotation with respect to the initial direction), about the $C\overline{x}_3$-axis, which carries the $C\overline{x}_1$-axis in the vertical plane which contains $C\overline{x}_3$ (let be $C\xi_1\xi_2\overline{x}_3$ the frame in this position); ii) a rotation $\mathbf{\Theta}$ of angle θ (*angle of pitching*, which is the angle of inclination of the longitudinal axis of the airplane), about the $C\xi_2$-axis, which carries the $C\xi_1$-axis in the position Cx_1 (let $Cx_1\xi_2\eta_3$ be the corresponding position of the frame); iii) a rotation $\mathbf{\Phi}$ of angle φ (*the angle of rolling*, which indicates the inclination on a wing), about the axis Cx_1, which carries $C\xi_2$ in the position Cx_2 and $C\eta_3$ in Cx_3. These three rotations are, in fact, specified by angles of Eulerian type (ψ, θ and φ), adapted to the problem of motion of the airplane. As in Chap. 3, Subsec. 2.2.3, we introduce the column matrices (3.2.11), the unit vectors $\mathbf{i}_j$, $j = 1,2,3$, of the frame $\mathscr{R}$ being expressed with respect to the unit vectors $\mathbf{i}'_k$, $k = 1,2,3$, of the frame $\mathscr{R}'$ by means of the matric product (3.2.11"). Noting that

$$\mathbf{\Psi} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}, \tag{14.2.35}$$

we introduce the matrix $\boldsymbol{\beta}$ of components $\beta_{jk} = \mathbf{i}_j \cdot \mathbf{i}'_k$, $j,k = 1,2,3$, which specifies the direction cosines of the axes of the frame $\mathscr{R}$ with respect to the axes of the frame $\overline{\mathscr{R}}$ in the form

$$\boldsymbol{\beta} = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ -\sin\psi\cos\varphi + \cos\psi\sin\theta\sin\varphi & \cos\psi\cos\varphi + \sin\psi\sin\theta\sin\varphi & \cos\theta\sin\varphi \\ \sin\psi\sin\varphi + \cos\psi\sin\theta\cos\varphi & -\cos\psi\sin\varphi + \sin\psi\sin\theta\cos\varphi & \cos\theta\cos\varphi \end{bmatrix}. \tag{14.2.35'}$$

Observing that $\boldsymbol{\omega} = \dot{\psi}\mathbf{i}'_3 + \dot{\theta}\mathbf{j} + \dot{\varphi}\mathbf{i}_1$, $\mathbf{j} = \operatorname{vers}\overrightarrow{C\xi_2}$, we obtain the components of the rotation angular velocity vector in the frame $\mathscr{R}$ in the form

$$\begin{aligned} \omega_1 &= \dot{\varphi} - \dot{\psi}\sin\theta\,, \\ \omega_2 &= \dot{\psi}\cos\theta\sin\varphi + \dot{\theta}\cos\varphi\,, \\ \omega_3 &= \dot{\psi}\cos\theta\cos\varphi - \dot{\theta}\sin\varphi\,, \end{aligned} \tag{14.2.36}$$

wherefrom

$$\begin{aligned}\dot{\psi} &= (\omega_2 \sin\varphi + \omega_3 \cos\varphi)\sec\theta\,,\\ \dot{\theta} &= \omega_2 \cos\varphi - \omega_3 \sin\varphi\,,\\ \dot{\varphi} &= \omega_1 + (\omega_2 \sin\varphi + \omega_3 \cos\varphi)\tan\theta\,.\end{aligned} \tag{14.2.36'}$$

The angles ψ, θ and φ are measured by the gyrocompass, the angular velocity being then given by (14.2.36). With the aircraft instruments one can also measure the components v'_{Ci} of the velocity $\mathbf{v}'_C$ in the frame $\mathscr{R}'$, along the axes of the frame $\mathscr{R}$, the formula $V'_{Cj} = \beta_{ij} v'_{Ci}$, $j = 1,2,3$, allowing then to pass to the axes of the frame $\mathscr{R}'$; integrating the system of equations $\mathrm{d}\rho'_j / \mathrm{d}t = V'_{Cj}$, $j = 1,2,3$, we determine the motion of the mass centre C in the latter frame of reference.

The equations (14.1.48), (14.1.52) take the form ($I_{23} = I_{12} = 0$, because the plane Cx_1x_3 is a plane of symmetry)

$$\begin{aligned}M(\dot{v}'_{C1} + \omega_2 v'_{C3} - \omega_3 v'_{C2}) &= -Mg\sin\theta + R_1\,,\\ M(\dot{v}'_{C2} + \omega_3 v'_{C1} - \omega_1 v'_{C3}) &= Mg\cos\theta\sin\varphi + R_2\,,\\ M(\dot{v}'_{C3} + \omega_1 v'_{C2} - \omega_2 v'_{C1}) &= Mg\cos\theta\cos\varphi + R_3\,,\end{aligned} \tag{14.2.37}$$

$$\begin{aligned}I_{11}\dot{\omega}_1 + I_{31}\dot{\omega}_3 + (I_{33} - I_{22})\omega_2\omega_3 + I_{31}\omega_1\omega_2 &= M_{C1}\,,\\ I_{22}\dot{\omega}_2 + (I_{11} - I_{33})\omega_3\omega_1 + I_{31}\left(\omega_3^2 - \omega_1^2\right) &= M_{C2}\,,\\ I_{33}\dot{\omega}_3 + I_{31}\dot{\omega}_1 + (I_{22} - I_{11})\omega_1\omega_2 - I_{31}\omega_2\omega_3 &= M_{C3}\,,\end{aligned} \tag{14.2.37'}$$

where $\mathbf{G} = M\mathbf{g}$ is the weight of the airplane, while $\mathbf{R} = \mathbf{R}(\boldsymbol{\omega}, \mathbf{v}'_C)$ and $\mathbf{M}_C = \mathbf{M}_C(\boldsymbol{\omega}, \mathbf{v}'_C)$ are the components of the torsor of the aerodynamic actions at C. We notice that components of a pseudomoment of momentum $\widetilde{\mathbf{K}}^O$ can intervene too, leading to supplementary terms of the form $\dot{\widetilde{K}}_i^O + \in_{ijk} \omega_j \widetilde{K}_k^O$, $i = 1,2,3$, in the left member of the equations (14.2.37'), due to the rotation of some parts of the airplane (e.g., of a wing of it); thus, in the right member of the same equations appear supplementary moments, called *moments of gyration*. In any case, in a *leeway* (*drift angle*) *navigation* we have $\widetilde{\mathbf{K}}^O = \mathbf{0}$.

The airplane can change its direction of advance modifying the angle between the fixed and the movable surfaces by which its wings are fitted out; the movable surfaces (called *ailerons*) corresponding to the frontal wings are coupled so that if one of them is moving upwards the other one is moving downwards, acting thus upon the rolling axis Cx_2. *The horizontal empennage* (the movable part of which is called *depth rudder*) and *the vertical empennage* (the movable part of which is called *direction*) are at the back part of the airplane, helping – as well – to change its direction. We denote by ψ' the angle between the horizontal empennage and the depth rudder, by θ' the angle between the vertical empennage and the direction and by φ' the angle between the wing and the aileron; one can establish *the driving* (*manoeuvre*) *equations*

$$\begin{gathered}
I_1 \ddot{\psi}' + M e_1 (\dot{v}'_{C1} + \omega_2 v'_{C3} - \omega_3 v'_{C2}) + J_1 (\omega_3 \omega_1 - \dot{\omega}_2) = R'_1 + R_{d1}, \\
I_2 \ddot{\theta}' - M e_2 (\dot{v}'_{C2} + \omega_3 v'_{C1} - \omega_1 v'_{C3}) - J_2 (\omega_1 \omega_2 + \dot{\omega}_3) \\
-J (\omega_2 \omega_3 - \dot{\omega}_1) = R'_2 + R_{d2}, \\
I_3 \ddot{\varphi}' + 2 J_3 (\omega_2 \omega_3 + \dot{\omega}_1) = R'_3 + R_{d3},
\end{gathered} \tag{14.2.38}$$

where I_k, J_k, $k = 1,2,3$, and J are axial and centrifugal moments of inertia, respectively, of some movable parts (e.g., I_1 is the moment of inertia of the depth rudder with respect to its axis of rotation), M_1 and M_2, e_1 and e_2 are the masses and the eccentricities of the depth rudder and of the direction, respectively, R'_k are the aerodynamic forces which appear due to these rotations, while R_{dk}, $k = 1,2,3$, are the corresponding driving actions. To determine the 12 unknown functions $v'_{Ck} = v'_{Ck}(t)$, $\omega_k = \omega_k(t)$, $k = 1,2,3$, $\psi = \psi(t)$, $\theta = \theta(t)$, $\varphi = \varphi(t)$, $\psi' = \psi'(t)$, $\theta' = \theta'(t)$ and $\varphi' = \varphi'(t)$ we have thus at our disposal the system of 12 differential equations (14.2.36')-(14.2.38), corresponding to a given command. In particular problems, these equations can be simplified (e.g., in case of *dynamic equilibrated commands* we have $J_1 = J_2 = J_3 = J = 0$, $e_1 = e_2 = 0$, while in *problems of stability with free wings* we put $R_{d1} = R_{d2} = R_{d3} = 0$).

Besides the aerodynamic and manoeuvre loads, one can take into consideration *the storm loads* (of aerodynamical nature too), *the loads which arise at take-off and landing*, various types of loads with a local character etc.

14.2.2.5 Plane-Parallel Motion of the Airplane

In case of the plane-parallel motion, the problem in the preceding subsection is considerably simplified; we put thus in evidence the motion in the symmetry plane of the airplane, which is a vertical plane. Unlike the general case, we report the motion to the inertial frame of reference $O'x'_1x'_2$ (the $O'x'_1$-axis being horizontal) and to a non-inertial frame $C\overline{x}_1\overline{x}_2$, with the axes parallel to those of the inertial one. At the mass centre C, which moves with the velocity $\mathbf{v}'_C$, which makes the angle α with the longitudinal axis Δ of the airplane, acts *the own weight* $\mathbf{G} = M\mathbf{g}$, *the propelling force* $\mathbf{F}$ and the torsor $\{\mathbf{R}, \mathbf{M}_C\}$ of *the aerodynamic forces* exerted upon the aircraft's surface. It is convenient to decompose the resultant of the aerodynamic forces in the form $\mathbf{R} = \mathbf{W} + \mathbf{N}$, where $\mathbf{W}$ (*the so-called resistance*) is along the velocity $\mathbf{v}'_C$, while $\mathbf{N}$ (*force of uplift*) is normal to $\mathbf{W}$ (Fig. 14.19a). The magnitudes of these components are obtained by studies of aerodynamical nature, in the form

$$W = C_W(\alpha) A \frac{\gamma}{2g} v_C'^2, \quad N = C_N(\alpha) A \frac{\gamma}{2g} v_C'^2, \tag{14.2.39}$$

where A is the area of the lifting surface, γ is the unit weight of the air, while the coefficients of resistance $C_W(\alpha)$ and $C_N(\alpha)$, characteristic for each airplane, are two non-dimensional functions of angle α; we assume that for $\alpha = 0$ we have $C_N(0) = 0$, hence $N = 0$, the velocity $\mathbf{v}'_C$ being along the longitudinal axis of the aircraft.

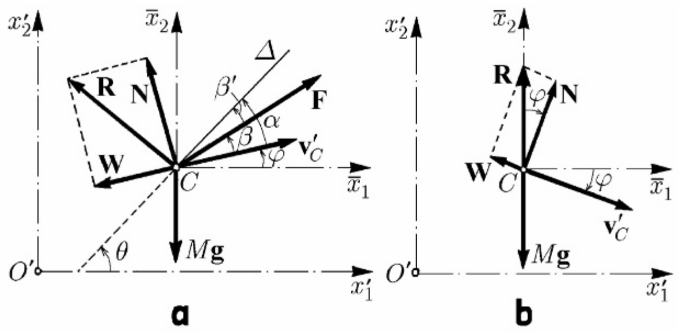

Fig. 14.19 Plane-parallel motion of an airplane

The motion of the aircraft's centre of mass will be specified by the equation

$$M \frac{\mathrm{d}^2 \boldsymbol{\rho}'}{\mathrm{d}t^2} = M\mathbf{g} + \mathbf{F} + \mathbf{N} + \mathbf{W}, \tag{14.2.40}$$

written with respect to the inertial frame of reference $\mathscr{R}'$, and its rotation by the equation

$$I_C \ddot{\theta} = M_C, \tag{14.2.40'}$$

where $\theta = \theta(t)$ is the angle made by the longitudinal axis Δ with the $O'x'_1$-axis, while I_C and M_C correspond to the Cx_3-axis, normal to the considered vertical plane. The magnitude of the moment $\mathbf{M}_C$ is obtained, by aerodynamical research too, in the form

$$M_C = -m(\alpha, \delta) v^2 - n v'_C \dot{\theta}, \tag{14.2.39'}$$

where the function $m(\alpha, \delta)$, of the nature of a mass, depends on α and on the angle δ made by *the altitude rudder* with its normal position, while the coefficient n, of the nature of a product of a mass by a length, is due to the motion of the rudder surfaces (the damping action of the moment $\mathbf{M}_C$); here too, m and n are characteristics of each aircraft. We obtain thus three scalar equations (14.2.40), (14.2.40') for the unknown functions $\rho'_1 = \rho'_1(t)$, $\rho'_2 = \rho'_2(t)$ and $\theta = \theta(t)$.

If the airplane advances with *switched off motor* ($\mathbf{F} = \mathbf{0}$) and constant velocity $\mathbf{v}'_C = \overrightarrow{\mathrm{const}}$ (case of a *gliding flight*), then the equation (14.2.40) leads to

$$M\mathbf{g} + \mathbf{N} + \mathbf{W} = \mathbf{0}\,. \tag{14.2.41}$$

Hence, the resultant $\mathbf{R}$ of the aerodynamic forces equilibrates the weight of the aircraft ($\mathbf{R} = -M\mathbf{g}$). The soaring angle φ, made by the velocity $\mathbf{v}'_C$ with the horizontal axis $C\bar{x}_1$ (Fig. 14.19b), is given by $\tan\varphi = -W/N = -C_W / C_N$, where we took into account (14.2.39); analogously, starting from the relation $G^2 = M^2 g^2 = N^2 + W^2$, the magnitude of the soaring velocity $\mathbf{v}'_C$ is given by

$$v_C'^2 = \frac{2Mg^2}{\gamma A\sqrt{C_W^2 + C_N^2}}\,. \tag{14.2.41'}$$

In case of a normal flight (*horizontal flight with constant velocity*) we have $\varphi = 0$ and $v'_C = \text{const}$; the equation (14.2.40) takes the form

$$M\mathbf{g} + \mathbf{F} + \mathbf{N} + \mathbf{W} = \mathbf{0}\,. \tag{14.2.42}$$

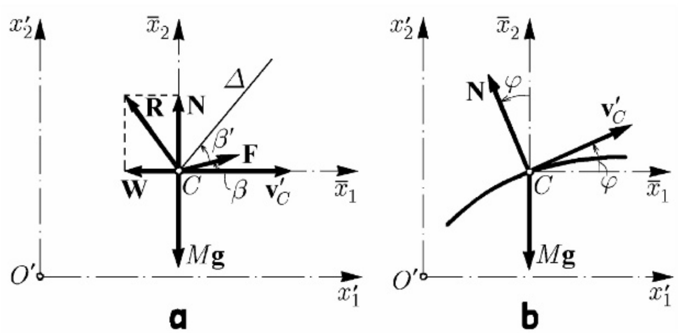

Fig. 14.20 Case of a horizontal flight with constant velocity

In projection on the axes $C\bar{x}_1$ and $C\bar{x}_2$, we get (Fig. 14.20a)

$$-W + F\cos\beta = 0\,, \quad N - G + F\sin\beta = 0\,, \tag{14.2.42'}$$

where β is the angle made by the force $\mathbf{F}$ with the velocity $\mathbf{v}'_C$. We obtain thus

$$\tan\beta = \frac{G - N}{W}\,, \quad F = \sqrt{(G-N)^2 + W^2}\,. \tag{14.2.42''}$$

We can easily calculate the angle $\alpha = \beta + \beta'$, because the angle β' between the force $\mathbf{F}$ and the longitudinal axis Δ is known. Observing that $\theta = \text{const}$, from the equations (14.2.39'), (14.2.40') it results $m(\alpha, \delta) = 0$, determining thus δ, that is the necessary deviation angle of the altitude rudder.

Lanchester studied in 1909 the motion of an aircraft with switched of motor ($\mathbf{F} = \mathbf{0}$) and with a nearly vanishing resultant ($\mathbf{W} \cong \mathbf{0}$), the velocity $\mathbf{v}'_C$ of which makes a constant angle with the longitudinal axis ($\alpha = \mathrm{const}$). It results the equation of motion of the mass centre

$$M \frac{\mathrm{d}^2 \boldsymbol{\rho}'}{\mathrm{d}t^2} = M\mathbf{g} + \mathbf{N} . \tag{14.2.43}$$

Projecting on the direction of the velocity $\mathbf{v}'_C$ and on a direction normal to it, we obtain (using the expressions (5.1.14) of the velocity in polar co-ordinates (Fig. 14.20b)

$$\frac{\mathrm{d}v'_C}{\mathrm{d}t} = -g \sin\varphi , \quad v'_C \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -g\cos\varphi + Kv^2 , \tag{14.2.43'}$$

where $K = C_N \gamma A / 2Mg$ is a coefficient of the nature of the inverse of a length. We have thus $v'_C \mathrm{d}v'_C / \mathrm{d}s = -g \sin\varphi$ for an element of arc along the trajectory of the centre *C*; observing that $\mathrm{d}x'_2 = \mathrm{d}s \sin\varphi$, it results

$$v'^2_C = -2gx'_2 + \frac{2h}{M} , \tag{14.2.44}$$

where h is the energy constant (corresponding to the first integral of the kinetic energy). Observing that $v'_C \mathrm{d}\varphi / \mathrm{d}t = v'^2_C \sin\varphi \mathrm{d}\varphi / \mathrm{d}x'_2$ and taking into account (14.2.44), we can write the second equation (14.2.43') in the form

$$\frac{\mathrm{d}\cos\varphi}{\mathrm{d}x'_2} - \frac{Mg\cos\varphi}{2(h - Mgx'_2)} + K = 0 .$$

The change of variable $x = x'_1$, $y = h / Mg - x'_2$ (one passes to a left-handed inertial frame of reference Oxy, with the Oy-axis along the ascendent vertical and with the Ox-axis at a level at which, in conformity to the formula (14.2.44), the velocity of the mass centre vanishes) leads to

$$\frac{\mathrm{d}\cos\varphi}{\mathrm{d}y} + \frac{\cos\varphi}{2y} = K ,$$

wherefrom

$$\cos\varphi = \frac{2a}{\sqrt{y}} + by , \quad b = \frac{2}{3} K , \quad a, b = \mathrm{const} . \tag{14.2.45}$$

Observing that

$$\cos\varphi = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}x}{\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)^{1/2}} = \frac{1}{\left[1 + (\mathrm{d}y/\mathrm{d}x)^2\right]^{1/2}},$$

we obtain the differential equation of the trajectory of the mass centre in the form

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{2a}{\sqrt{y}} + by\,.$$

Differentiating with respect to the variable x, we obtain

$$\frac{\mathrm{d}^2 y/\mathrm{d}x^2}{\left[1 + (\mathrm{d}y/\mathrm{d}x)^2\right]^{3/2}} = \frac{1}{R} = \frac{a}{y\sqrt{y}} - b\,, \tag{14.2.45'}$$

where R is the curvature radius of the trajectory at the point C. Using the relations (14.2.45), (14.2.45'), one obtains the trajectory by a graphical integration; the curve thus obtained is called a *figoid*, the corresponding motion being a *figoidal motion*.

14.2.2.6 Rolling on an Inclined Plane

Let be a homogeneous circular cylinder of radius R and weight $\mathbf{G} = M\mathbf{g}$, situated on a plane inclined with the angle α with respect to the horizontal; considering the plane-parallel motion of the cylinder, it is sufficient to make a study in a vertical plane, which passes through the mass centre of the cylinder (Fig. 14.21). Assuming at the beginning that sliding friction does not exist, the circular disc of radius R (to which is reduced the cylinder in our study) glides downwards in a uniform accelerated motion, under the action of the own weight G and of the constraint force **N**. Writing the equations of motion of the mass centre in the inertial frame of reference $O'x_1'x_2'$, we find the magnitude of the normal constraint force

$$N = G\cos\alpha = Mg\cos\alpha\,, \tag{14.2.46}$$

as well as $M\mathrm{d}v_C'/\mathrm{d}t = G\sin\alpha$, wherefrom $\mathrm{d}v_C'/\mathrm{d}t = g\sin\alpha$. Hence, the centre $\boldsymbol{C}$ moves frictionless as a heavy particle of mass M on a plane inclined by the angle α with respect to the horizontal; its trajectory is, obviously, the $C\bar{x}_1$-axis, the disc having a motion of translation, while the component of $\boldsymbol{\rho}'$ along the $O'x_1'$-axis is given by $\rho_1' = \left(gt^2/2\right)\sin\alpha + \rho_1'^0$, where $\rho_1'^0$ corresponds to the initial moment $t = 0$.

If a tangential constraint force T, applied at the point I, intervenes too, the inclined plane being rough, we obtain the equations of motion (I_C is the moment of inertia with respect to an axis normal at C to the fixed plane in which the motion takes place; we use a left-handed frame of reference)

$$M\frac{\mathrm{d}v_C'}{\mathrm{d}t} = Mg\sin\alpha - T\,, \quad I_C\dot{\omega} = TR\,, \tag{14.2.47}$$

the ideal constraint force being given by the relation (14.2.46) too; here $\boldsymbol{\omega}$ is the angular velocity by which the disc is rolling without sliding on the inclined plane and we have $v_C' = R\omega$, I being the instantaneous centre of rotation. We notice also that $\rho_1' = \rho_1'^0 + R(\theta - \theta_0)$, where θ_0 corresponds to the initial moment $t = 0$, being the angle made by the Cx_1-axis with the $C\overline{x}_1$-axis. Hence, $T = \left(I_C / R^2\right)\mathrm{d}v_C' / \mathrm{d}t$; replacing in the first equation (14.2.47), it results

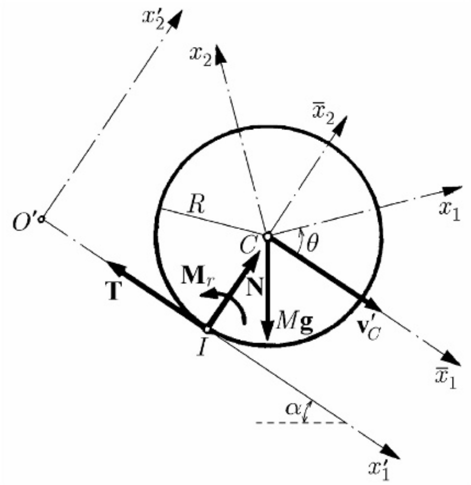

Fig. 14.21 Rolling on an inclined plane

$$\frac{\mathrm{d}v_C'}{\mathrm{d}t} = \frac{g\sin\alpha}{1+\lambda}\,, \quad \lambda = \frac{m}{M}\,, \quad m = \frac{I_C}{R^2} \tag{14.2.48}$$

for the rolling without sliding, where m is *the peripheral mass* of the disc (the mass which, situated at the distance R from the centre C, leads to the axial moment of inertia I_C). Observing that $I_C = MR^2/2$, it results $\lambda = 1/2$. If the disc is reduced to a peripheral circle of mass M (corresponds to a hollow cylinder, with a very thin wall), then we have $\lambda = 1$. We can thus state that a homogeneous full cylinder is rolling without sliding on the inclined plane with an acceleration greater (hence, quicker) than that of the hollow cylinder, assuming that both cylinders are rolling without initial velocity.

Studying, analogously, the rolling without sliding of a homogeneous sphere in a vertical plane, which contains its centre of mass, we notice that $I_C = 2MR^2/5$; we find thus $\lambda = 2/5$, the corresponding acceleration being greater than the acceleration of the two cylinders. Observing that $\rho_1' = [g/2(1+\lambda)]t^2\sin\alpha + \rho_1'^0$ and denoting by $t_{\rm cf}$, $t_{\rm ch}$, $t_{\rm s}$ and $t_{\rm p}$ the times in which a full and a hollow cylinder, a sphere and a particle, respectively, are rolling without sliding on the same inclined plane, without initial velocity, travelling through the same space, we obtain

$$t_{\rm p} = \frac{t_{\rm s}}{\sqrt{7/5}} = \frac{t_{\rm cf}}{\sqrt{3/2}} = \frac{t_{\rm ch}}{\sqrt{2}}. \tag{14.2.49}$$

The tangential constraint force is given by

$$T = \frac{\lambda}{1+\lambda} Mg \sin\alpha . \tag{14.2.50}$$

The above results hold also for a rigid solid with an axis of geometric and mechanical symmetry and with a plane of geometric and mechanical symmetry, normal to that axis (including thus also the case of a non-homogeneous rigid solid).

To have a rolling without sliding, it is necessary that $T \leq fN$, where $f = \tan\varphi$ is the coefficient of sliding friction, while φ is the angle of sliding friction between the cylinder and the inclined plane; hence, the condition of rolling without sliding of the cylinder is

$$\tan\alpha \leq \frac{1+\lambda}{\lambda} f = \frac{1+\lambda}{\lambda}\tan\varphi \tag{14.2.51}$$

or

$$\alpha \leq \varphi', \quad \tan\varphi' = \frac{1+\lambda}{\lambda}\tan\varphi, \quad \varphi' > \varphi. \tag{14.2.51'}$$

Corresponding to the results in Chap. 4, Subsecs 1.1.8 and 2.1.6 too, a particle would slide on the inclined plane if $\varphi < \alpha$; if $\alpha > \varphi'$, the cylinder *is rolling with sliding*.

Experimentally, one sees firstly that, for a sufficiently small angle α, the cylinder remains at rest, due to the apparition of a moment of rolling friction M_r, which equates to zero the couple TR; observing that, in this case, $v_C' = 0$, the first equation (14.2.47) gives $T = Mg\sin\alpha$, so that the respective couple will be $MgR\sin\alpha = NR\tan\alpha$. For an angle $\alpha > \alpha_0$, $M_r = NR\tan\alpha_0$, the cylinder recommences to roll; introducing the coefficient of rolling friction $s = R\tan\alpha_0$, it results $M_r = sN$, corresponding to the considerations in Chap. 3, Subsec. 2.2.12 and Chap. 4, Subsec. 2.1.4. The experiments show that, in general, the influence of the rolling friction is smaller than the influence of the sliding friction and we have $\alpha_0 < \varphi$, hence $f < s/R$.

If the rolling friction appears too, then the second equation (14.2.47) is completed in the form

$$I_C\dot{\omega} = TR - sN, \tag{14.2.47'}$$

so that, analogously, we obtain

$$\frac{\mathrm{d}v_C'}{\mathrm{d}t} = \frac{g}{1+\lambda}\left(\sin\alpha - \frac{s}{R}\cos\alpha\right) = \frac{g\sin(\alpha-\alpha_0)}{(1+\lambda)\cos\alpha_0}. \tag{14.2.48'}$$

Comparing the relations (14.2.48) and (14.2.48') and observing that $\sin(\alpha-\alpha_0) < \sin\alpha\cos\alpha_0$, it results that *the intervention of the rolling friction diminishes the acceleration of the mass centre of the disc*. The tangential constraint force will be given, in the same way, by

$$T = m\frac{\mathrm{d}v_C'}{\mathrm{d}t} + \frac{s}{R}N = \frac{Mg}{1+\lambda}\left(\lambda\sin\alpha + \frac{s}{R}\cos\alpha\right). \tag{14.2.50'}$$

The condition of rolling without sliding ($T \le fN$) becomes

$$\tan\alpha \le f + \frac{1}{\lambda}\left(f - \frac{s}{R}\right) = \tan\varphi + \frac{1}{\lambda}(\tan\varphi - \tan\alpha_0) \tag{14.2.52}$$

or

$$\alpha \le \varphi'', \quad \tan\varphi'' = \tan\varphi + \frac{1}{\lambda}(\tan\varphi - \tan\alpha_0), \tag{14.2.52'}$$

where we suppose that $\varphi > \alpha_0$. Finally, for $\alpha \le \alpha_0$ the disc is at rest, for $\alpha_0 < \alpha \le \varphi''$ the disc is rolling without sliding, while for $\alpha > \varphi''$ the rolling takes place with sliding, assuming a null velocity at the initial moment.

The above results can be put in connection also with the study of the equilibrium problems of the drawn and of the motive wheels (see Chap. 4, Subsec. 2.1.4).

14.2.2.7 Rolling on a Horizontal Plane

If the cylinder considered at the preceding subsection lies on a horizontal plane (the case $\alpha = 0$), then the problem has a different character; indeed, if upon the cylinder (a disc in a vertical cross section) acts only its own weight $\mathbf{G} = M\mathbf{g}$ as a given force, the initial velocity being equal to zero, this one remains at rest. To roll on the horizontal plane, the disc must be acted upon also by other given forces (which will be considered constant in time), of torsor $\{\mathbf{F}, \mathbf{M}_C\}$ at the mass centre C (Fig. 14.22); as well, the initial velocity can be non-zero. In the case of rolling without sliding, the contact point I of the disc with the horizontal plane is the instantaneous centre of rotation, while the velocity of the mass centre C (which moves along a horizontal) is given by

$$\mathbf{v}_C' = R\boldsymbol{\omega}, \tag{14.2.53}$$

where $\boldsymbol{\omega}$ is the rotation angular velocity about the centre C; the relation takes place at the initial moment $t = 0$ too, in the form $\mathbf{v}_C'^0 = R\boldsymbol{\omega}_0$. If the point I slides along the

$O'x_1'$-axis with the velocity $\mathbf{v}_I'$ (the disc has a motion of translation too), then – by composition of velocities – we can write

$$\mathbf{v}_C' = \mathbf{v}_I' + R\boldsymbol{\omega}. \tag{14.2.53'}$$

At the initial moment we have, obviously, $\mathbf{v}_C'^0 = \mathbf{v}_I'^0 + R\boldsymbol{\omega}_0$. In general, arises a force of sliding friction **T**, opposite to the motion (to the velocity $\mathbf{v}_I'$) and a moment of rolling friction of magnitude $M_r = sN$, $s = R\tan\alpha_0$, which, as well, is opposite to the motion or to the tendency of motion.

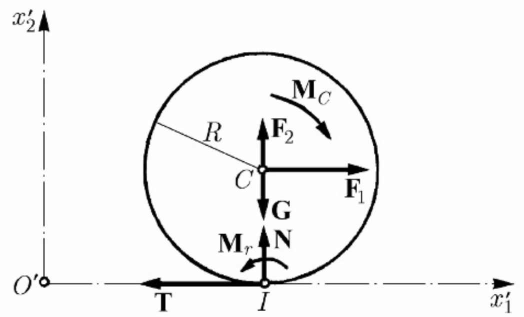

Fig. 14.22 Rolling on a horizontal plane

From the theorem of motion of the mass centre, it results

$$M\frac{\mathrm{d}v_C'}{\mathrm{d}t} = -T + F_1, \quad N - G + F_2 = 0, \tag{14.2.54}$$

while the theorem of moment of momentum with respect to the same centre gives (we use a left-handed frame of reference)

$$I_C\dot{\omega} = M_C - M_r + TR = \mathscr{M} + TR, \quad \mathscr{M} = M_C - M_r. \tag{14.2.54'}$$

We find the normal constraint force

$$N = G - F_2, \tag{14.2.55}$$

where we assume that $G > F_2$ (otherwise, the cylinder would be detached from the plane). Eliminating the constraint force T, it results

$$I_C\dot{\omega} + MR\frac{\mathrm{d}v_C'}{\mathrm{d}t} = \mathscr{M} + F_1R. \tag{14.2.56}$$

In case of rolling without sliding (*pure rolling*) we can assume that at a moment t (as well, at the initial moment $t = 0$) takes place the relation (14.2.53). The equation (14.2.56) becomes

$$(I_C + MR^2)\dot{\omega} = (1+\lambda)MR^2\dot{\omega} = \mathcal{M} + F_1 R\,, \quad \lambda = \frac{m}{M}\,, \quad m = \frac{I_C}{R^2}\,,$$

whence, by integration,

$$\omega = \omega_0 + \frac{\mathcal{M} + F_1 R}{(1+\lambda)MR^2}t\,, \quad v'_C = v'^0_C + \frac{\mathcal{M} + F_1 R}{(1+\lambda)MR}t\,. \tag{14.2.57}$$

The force of sliding friction will be given by

$$T = \frac{\lambda F_1 R - \mathcal{M}}{(1+\lambda)R}\,. \tag{14.2.58}$$

Imposing the condition $|\mathbf{T}| \leq fN = f(G - F_2)$, we find $|\lambda F_1 R - \mathcal{M}| \leq (1+\lambda)f(G - F_2)R$, so that the condition of rolling without sliding becomes

$$\mathcal{M}' - \mathcal{M}'' \leq \mathcal{M} \leq \mathcal{M}' + \mathcal{M}''\,, \quad \mathcal{M}' = \lambda F_1 R\,, \quad \mathcal{M}'' = (1+\lambda)f(G - F_2)R\,. \tag{14.2.59}$$

If $\mathcal{M} + F_1 R > 0$, then the quantities ω and v'_C are always positive, while the angular velocity ω (hence, the velocity v'_C too) grows indefinitely, in direct proportion to time; in case of equality ($\mathcal{M} + F_1 R = 0$), the velocities ω and v'_C remain constant during the motion. If $\mathcal{M} + F_1 R < 0$, then the rolling takes place with an angular velocity which diminishes in direct proportion to time till the moment $\bar{t} = -(1+\lambda)MR^2\omega_0 / (\mathcal{M} + F_1 R)$, when $\omega = 0$ and $v'_C = 0$; for $t > \bar{t}$ one obtains a rolling in the opposite sense ($\omega < 0$), where $\mathcal{M} = M_C + M_r$ is taken (because the moment of rolling friction is opposed to the motion). If $-M_r \leq M_C + F_1 R \leq M_r$ and $\omega_0 = 0$, then the cylinder remains at rest.

In the case in which one of the relations

$$\mathcal{M} < \mathcal{M}' - \mathcal{M}''\,, \quad \mathcal{M} > \mathcal{M}' + \mathcal{M}'' \tag{14.2.60}$$

takes place, we must assume that $|\mathbf{T}| = fN = f(G - F_2)$, having to do with a rolling without sliding; in this case, the equations (14.2.54), (14.2.54') lead to

$$v'_C = v'^0_C + \frac{1}{M}(F_1 - T)t\,, \quad \omega = \omega_0 + \frac{1}{I_C}(\mathcal{M} + TR)\,. \tag{14.2.61}$$

We suppose that $v'_I = 0$ at the initial moment; but at a moment t takes place the relation (14.2.53'). We obtain thus the relation (we notice that we have a relation of the form (14.2.53) at the initial moment)

$$v'_I = \frac{1}{mR}(-\mathscr{M} + \mathscr{M}' \mp \mathscr{M}'')t\,, \tag{14.2.62}$$

where we have used the relations introduced above; the double sign corresponds to the sense of the force of sliding friction **T**, obtaining thus $v'_I > 0$ or $v'_I < 0$, as the sign – (for **T** in a sense opposite to the motion, as in Fig. 14.22) or the sign + is taken, respectively. Hence, if the first relation (14.2.60) is taken, then the motion involves a sliding of velocity $v'_I > 0$ on the $O'x'_1$-axis, while – in case of the second relation (14.2.60) – the sliding velocity is $v'_I < 0$.

If $v'^0_I \neq 0$, then the relation (14.2.53') takes place at the initial moment too. Because v'_I is a function continuous in time, it must have the same sign as v'^0_I, at least for small values of t. On the same way, we can write

$$v'_I = \begin{cases} v'^0_I + \dfrac{1}{mR}(-\mathscr{M} + \mathscr{M}' - \mathscr{M}'')t & \text{for } v'^0_I > 0, \\ v'^0_I + \dfrac{1}{mR}(-\mathscr{M} + \mathscr{M}' + \mathscr{M}'')t & \text{for } v'^0_I < 0. \end{cases} \tag{14.2.62'}$$

Firstly, let us suppose that $v'^0_I > 0$. If $\mathscr{M} \leq \mathscr{M}' - \mathscr{M}''$, then we have, $v'_I > 0$ for any time, while if $\mathscr{M} > \mathscr{M}' - \mathscr{M}''$, then $v'_I > 0$ at the initial moment and it is diminishing and then vanishing for $\bar{t} = mRv'^0_I / (\mathscr{M} - \mathscr{M}' + \mathscr{M}'')$; thus, we come back to the case considered before ($v'^0_I = 0$), taking the moment $t = \bar{t}$ as initial moment. Let be now $v'^0_I < 0$. If $\mathscr{M} \geq \mathscr{M}' + \mathscr{M}''$, then $v'_I < 0$ at any moment t, while if $\mathscr{M} < \mathscr{M}' + \mathscr{M}''$, then $v'_I < 0$ at the initial moment and then increases till vanishing for $\bar{t} = -mRv'^0_I / (\mathscr{M} + \mathscr{M}' - \mathscr{M}'')$; we return thus to the case $v'^0_I = 0$.

Finally, if $\mathscr{M}' - \mathscr{M}'' \leq \mathscr{M} \leq \mathscr{M}' + \mathscr{M}''$, then takes place firstly a rolling with sliding and then a rolling without sliding, while if $\mathscr{M} > \mathscr{M}' + \mathscr{M}''$ or $\mathscr{M} < \mathscr{M}' - \mathscr{M}''$, then a phenomenon of rolling with sliding takes place for any time. In the first of these cases, the sliding velocity v'_I vanishes at the moment $t = \bar{t}$, while the force of sliding friction has no more the absolute value fN but it is given by (14.2.58), which leads to discontinuities both for T and for the derivatives $\mathrm{d}v'_C / \mathrm{d}t$ and $\dot{\omega}$ in the equations (14.2.54), (14.2.54'); but the velocities v'_C and ω remain continuous functions. These results can be put in connection with the study corresponding to the drawn and motive wheel (see Chap. 4, Sect. 2.1.4).

In our study, we have assumed till now that the disc reaches the horizontal plane at only one point I. In this case, under the action of a moment $\mathbf{M}_C$ and of a tangential force **T**, we obtain the equations of motion (Fig. 14.23a)

$$M\frac{\mathrm{d}v'_C}{\mathrm{d}t} = -T\,, \quad I_C\dot{\omega} = M_C + TR\,. \tag{14.2.63}$$

Taking into account the relation $v_C' = R\omega$ corresponding to the phenomenon of rolling (without sliding) and eliminating the force of sliding friction T, we obtain

$$\left(I_C + MR^2\right)\dot{\omega} = (1+\lambda)MR^2\dot{\omega} = M_C\,, \tag{14.2.63'}$$

which shows that the disc begins to roll for any $M_C > 0$ (having $\dot{\omega} > 0$ too). In reality, the disc cannot remain perfectly rigid; it is deformed, the contact with the horizontal plane taking place on the segment $P'P''$ (Fig. 14.23b). Taking into account the sense of rolling of the disc, that one will effect – in fact – a motion of rotation about the point P'', being detached from the point P'; we can thus assume that the normal constraint force **N** is applied at the point P'', at a distance s from the support of the own weight **G**, the equation of motion being written in the form

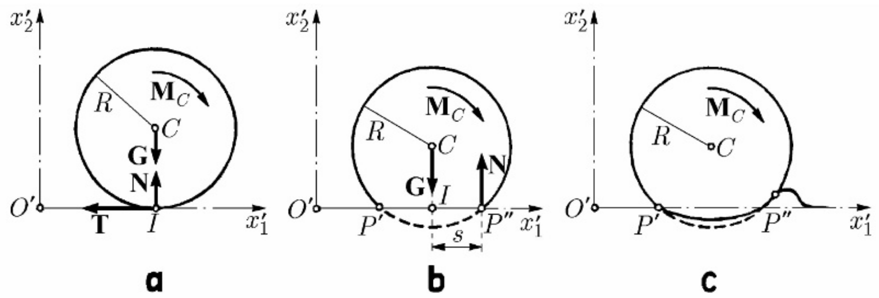

Fig. 14.23 Rolling of a rigid and of a non-rigid disc on a horizontal plane

$$I_C\dot{\omega} = M_C + TR - sN\,, \tag{14.2.64}$$

so that

$$\left(I_C + MR^2\right)\dot{\omega} = (1+\lambda)MR^2\dot{\omega} = M_C - sN = \mathscr{M}\,. \tag{14.2.64'}$$

Hence, for a phenomenon of rolling it is necessary that $M_C > sN = M_r$. We must mention that, in reality, the horizontal plane is also deformed, so that – locally – the contact zone is as in the Fig. 14.23c.

14.2.2.8 Departure and Stopping of a Powercraft

By a powercraft we mean a vehicle which is displaced by the rolling of its wheels, put in motion by a motor (of any nature), e.g., automobile, locomotive, street motorcar, motorcycle etc.

We, firstly, consider the departure of a powercraft (the starting, the departure from a state of rest) with respect to an inertial frame of reference. We point our attention on a motive wheel, assuming that it is acted upon by a driving torque of moment $\mathbf{M}_C$ and that a rolling moment $\mathbf{M}_r$ arises too; we assume also that $F_1 = F_2 = 0$ (we use the notations in the previous subsection). At the initial moment, when the wheel is at rest, we have $\omega_0 = 0$, $v_C'^0 = v_I'^0 = 0$. The formulae (14.2.59) lead to

$$\mathscr{M}' = 0, \quad \mathscr{M}'' = (1+\lambda) fGR. \tag{14.2.65}$$

To can put the wheel in motion, we must have $\mathscr{M} = M_C - M_r > 0$.

If $\mathscr{M} \leq \mathscr{M}''$, then the wheel begins to roll without sliding, and the motion can continue till infinity; the angular velocity ω grows in direct proportion to time. From the second formula (14.2.57) it results that in an interval of time given by $\bar{t} = (1+\lambda) MRV / \mathscr{M}$ the powercraft reaches the velocity V; thus, it results

$$\mathscr{M} = (1+\lambda)\frac{MRV}{\bar{t}}, \tag{14.2.66}$$

hence, the moment $\mathscr{M}$ of the driving couple necessary to reach the velocity V in an interval of time equal to $\bar{t}$. Taking into account (14.2.65), (14.2.66), the condition $\mathscr{M} \leq \mathscr{M}''$ takes the form

$$V \leq f\frac{G}{M}\bar{t}, \tag{14.2.67}$$

obtaining thus a limitation of the velocity V function of the time $\bar{t}$; we notice that G is the part of the whole weight of the powercraft which corresponds to the axle of the wheel, M being the mass of it.

The kinetic energy of the wheel at the moment $\bar{t}$ will be (we have $\overline{\omega} = V / R$, $\overline{\omega} = \omega(\bar{t})$)

$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_C\overline{\omega}^2 = \frac{1}{2R^2}\left(I_C + MR^2\right)V^2 = \frac{1}{2}(1+\lambda)MV^2,$$

while the kinetic energy consumed by rolling friction is calculated in the form (we use the first formula (14.2.57) and the formula (14.2.66))

$$T_r = \int_0^{\bar{t}} M_r \omega \mathrm{d}t = sG\frac{\mathscr{M}}{(1+\lambda)MR^2}\frac{\bar{t}^2}{2} = \frac{sG}{2}\frac{(1+\lambda)MV^2}{\mathscr{M}}.$$

Thus, the kinetic energy consumed by the powercraft to reach the velocity V is given by

$$\overline{T} = T + T_r = \frac{1}{2}(1+\lambda)\left(1 + \frac{sG}{\mathscr{M}}\right)MV^2. \tag{14.2.68}$$

If $\mathscr{M} > \mathscr{M}''$, then the rolling will be accompanied by sliding and the wheel will *slip*. Obviously, in this case the kinetic energy necessary to reach the velocity V in an interval of time $\bar{t}$ will be greater. To avoid this loss of kinetic energy it is necessary that the inequality $\mathscr{M} \leq \mathscr{M}''$ takes place; in this order of ideas, from the relations (14.2.66) and (14.2.67) we obtain

$$\mathscr{M} \leq (1 + \lambda) fGR \,. \tag{14.2.69}$$

Assuming that the moment M_r of the driving couple is realized by a force **F** applied tangentially at the peripheral of the wheel ($M_r = FR$) and observing that $M_r = sG = GR \tan \alpha_0$ we can also write ($\mathscr{M} = M_C - M_r$)

$$F \leq [\tan \alpha_0 + (1 + \lambda) f] G \,. \tag{14.2.70}$$

Neglecting the angle of rolling α_0 (e.g., in case of a locomotive), it results

$$F \leq (1 + \lambda) fG \,, \tag{14.2.70'}$$

a formula particularly useful in applications.

Analogously, one can make a study of the stopping of the powercraft by braking the wheels; in this case, we must have $M_C = -R\psi < 0$, where $\psi > 0$ is a coefficient of the nature of a length, which corresponds to the brake slipper friction on the peripheral of the wheel. We notice that $v'_C = V$, $\omega_0 = V/R$, $v'^0_I = 0$, $F_1 = 0$, hence $\mathscr{M}' = 0$.

If $\mathscr{M} > -\mathscr{M}''$, then we have a rolling without sliding; the time $\bar{t}$ necessary to make vanishing the velocity V by deceleration will be given by the second formula (14.2.57) in the form (we have $M_r = sN = sG = GR \tan \alpha_0$)

$$\bar{t} = -(1 + \lambda) \frac{MRV}{\mathscr{M}} = (1 + \lambda) \frac{MRV}{R\psi + M_r} = (1 + \lambda) \frac{MV}{\psi + G \tan \alpha_0} \,. \tag{14.2.71}$$

If $\mathscr{M} < -\mathscr{M}''$, then the powercraft will stop by the rolling with sliding of the wheels; the formulae (14.2.61) lead to (we have put $T = fN = fG$)

$$v'_C = V - \frac{1}{M} fGt \,, \quad \omega = \overline{\omega} + \frac{1}{I_C} (\mathscr{M} + fGR) t \,, \quad \overline{\omega} = \frac{V}{R} \,.$$

In this case,

$$\overline{t'} = -\frac{I_C \overline{\omega}}{\mathscr{M} + fGR} = -\frac{I_C V}{R(\mathscr{M} + fGR)} = \frac{\lambda MV}{\psi + G \tan \alpha_0} \,, \quad \overline{t''} = \frac{MV}{fG} \tag{14.2.72}$$

represents the interval of time after which the linear velocity v'_C and the angular velocity ω, respectively, vanish; it results

$$\overline{t''} - \overline{t'} = \frac{MV}{fG} \frac{\mathscr{M} + (1 + \lambda) fGR}{\mathscr{M} + fGR} \,. \tag{14.2.72'}$$

As in the case of the departure of the powercrafts, the moment M'' is given by (14.2.65); from $\mathscr{M} + \mathscr{M}'' < 0$ we obtain $\mathscr{M} + fGR < \mathscr{M} + (1+\lambda)fGR < 0$, so that $\overline{t''} > \overline{t'}$. Hence, after an interval of time $\overline{t'}$, the wheel is slipping (it does no more rotate) with the velocity $v_I' = v_C'$ till the moment $\overline{t''}$ when $v_I' = v_C' = 0$, and the powercrafts stops.

If we use simplified models (the modelling as particles), for the powercraft and for the wheel in the interval of time $\overline{t''}$ in which the wheel is slipping and if $G = Mg$ (corresponding to the powercraft), we can write ($\overline{t'} = 0$, assuming a null peripheral mass, hence $\lambda = 0$)

$$\overline{t''} = \frac{V}{fg}. \tag{14.2.73}$$

The distance l travelled through in this interval of time will, obviously, be given by

$$l = \frac{V^2}{2fg}, \tag{14.2.73'}$$

hence a formula of the Torricelli type. Taking $g = 9.806\ \mathrm{m/s^2}$ and assuming a

Table 14.1

f \ V	30	40	50	60	70	80	90	100
0.3	11.80	20.98	32.79	47.21	64.26	83.93	106.22	131.14
0.6	5.90	10.49	16.39	23.61	32.13	41.97	53.11	65.57

coefficient of friction $f = 0.3$ for a wet ground and a coefficient of friction $f = 0.6$ for a dry ground, we give in Table 14.1 *the braking distance* l in m for various values of the velocity V in $\mathrm{km/h}$.

Chapter 15

Dynamics of the Rigid Solid with a Fixed Point

The frictionless motion of a rigid solid about a point of it, fixed with respect to an inertial (fixed) frame of reference, is one of the basic problems in the motion of a rigid solid, important from a theoretical point of view (as an intermediary phase in the solution of other problems or as a problem in itself), as well as from the point of view of practical applications. This problem has been considered for the first time in 1749 by d'Alembert, but the final form of the equations of motion has been given by L. Euler in 1758. Subsequent researches are due to J.-L. Lagrange, L. Poinsot, S.-D. Poisson, C.G.J. Jacobi, Ch. Hermite and Sonya Kovalevsky. Further, many other studies have been made, which are continuing also now.

After a general study of the motion of a rigid solid with a fixed point, one considers the most important cases of integrability of the corresponding system of differential equations.

15.1 General Results. Euler-Poinsot Case

Using the general results which are presented, one considers the Euler-Poinsot case of integrability, interesting both in the study of the heavy rigid solid and in the study of other cases of loading; thus, various analytical and geometric aspects of the problem are developed.

15.1.1 General Results

In what follows, one makes firstly some preliminary considerations; general methods of computation are then presented, using the theory of the last multiplier, as well as the most important cases of integrability.

15.1.1.1 Kinematical Considerations

Fixing one of the points of the rigid solid $\mathscr{S}$, the number of degrees of freedom of it is reduced to three, corresponding to the three components of the rotation vector applied at the fixed point. Obviously, this point is chosen as pole O' of the inertial frame of reference $\mathscr{R}'$; as well, without any loss of generality, the pole O of the non-inertial frames $\overline{\mathscr{R}}$ and $\mathscr{R}$ can be situated at the same point $O \equiv O'$ (hence, $\mathbf{r}'_O = \mathbf{0}$), so that $\overline{\mathscr{R}} \equiv \mathscr{R}'$ (the frame $\overline{\mathscr{R}}$ being inertial too).

The position of the frame of reference $\mathscr{R}$ (hence, of the rigid solid) with respect to the frame $\mathscr{R}'$ is specified by Euler's angles: $\psi = \psi(t)$, $0 \leq \psi < 2\pi$, $\theta = \theta(t)$,

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$0 \leq \theta \leq \pi$, and $\varphi = \varphi(t)$, $0 \leq \varphi < 2\pi$ (Fig. 15.1); as a matter of fact, these angles can be considered as components of the rotation of the rigid solid about the fixed point. Indeed, be PQ a segment of a line which belongs to the rigid solid, situated in the position $P'Q'$ with respect to the frame $\mathscr{R}'$, at the initial moment at which the frame $\mathscr{R}$ coincides with the frame $\mathscr{R}'$; after the motion of the rigid solid about the fixed point, the frame $\mathscr{R}$ reaches the actual position, while the segment of a line PQ has another position with respect to the frame $\mathscr{R}'$, but the same position with respect to

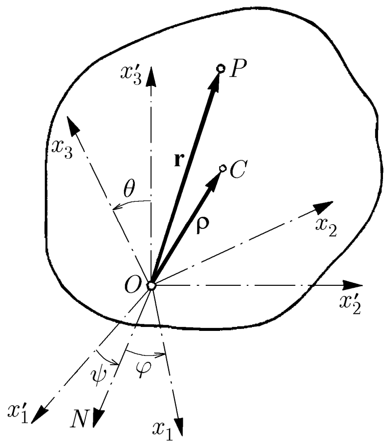


Fig. 15.1 The fixed and movable frames of reference. Euler's angles

the frame $\mathscr{R}$. Obviously, we have $\overline{P'Q'} = \overline{PQ}$ and $\overline{OP'} = \overline{OP}$, $\overline{OQ'} = \overline{OQ}$. The plane Π_P, normal to the segment of a line $P'P$ at its middle $\overline{P}$, passes through the point O, being a plane of symmetry for the isosceles triangle $P'OP$; the plane Π_Q, normal to the segment of a line $Q'Q$ at its middle $\overline{Q}$, has the same property. The intersection of the two planes is a line OR (Fig. 15.2). Because of symmetry reasons with respect to the plane Π_P, we have $\widehat{P'OR} = \widehat{POR}$; analogously, we can write $\widehat{Q'OR} = \widehat{QOR}$. But, by means of the considered motion, the point P' reaches the point P, while Q' reaches the point Q, the angles $\widehat{P'OR}$ and $\widehat{Q'OR}$ remaining invariant; the straight line OR remains fixed with respect to the frame $\mathscr{R}'$ during the motion, so that it is an axis of rotation. We find thus again Euler's Theorem 14.1.1, which can be applied in the case of a finite rotation, as well as in the case of an instantaneous one.

Hence, one passes from the frame of reference $\mathscr{R}'$ to the frame $\mathscr{R}$ by a rotation of angle χ about an axis of unit vector $\mathbf{u}$; this motion can be represented in various forms (see Sects. 14.1.1.1-14.1.1.3). Denoting $\overrightarrow{OP'} = \mathbf{r}'$ and $\overrightarrow{OP} = \mathbf{r}$ and using the results in Sect. 14.1.1.1, we can write (see Fig.14.3 too)

$$\mathbf{r} = \mathbf{r}'\cos\chi + (\mathbf{r}'\cdot\mathbf{u})(1-\cos\chi)\mathbf{u} + \mathbf{u}\times\mathbf{r}'\sin\chi\,. \tag{15.1.1}$$

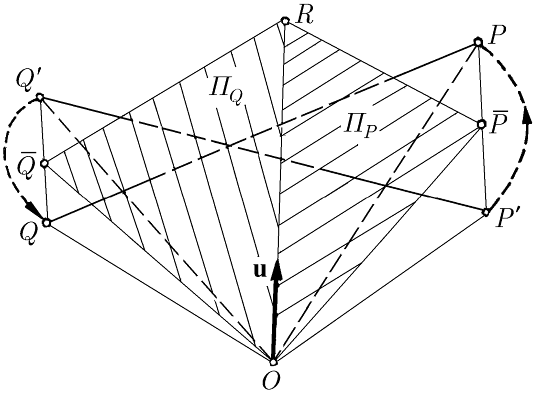


Fig. 15.2 Passing from the frame of reference $\mathscr{R}'$ to the frame $\mathscr{R}$

Projecting on the axes of the frame $\mathscr{R}'$, we obtain the transformation matrix $\boldsymbol{\alpha}$ of components ($\mathbf{r} = \boldsymbol{\alpha}\mathbf{r}'$, $x_i = \alpha_{ij}x'_j$, $\alpha_{ij} = \mathbf{i}_i\cdot\mathbf{i}'_j$, $i,j=1,2,3$)

$$\alpha_{ij} = \delta_{ij}\cos\chi + u_i u_j(1-\cos\chi) - \in_{ijk} u_k \sin\chi\,,\quad i,j=1,2,3\,. \tag{15.1.1'}$$

Calculating the trace of this tensor, the rotation angle χ will be given by ($u_k u_k = 1$)

$$\cos\chi = \frac{1}{2}(\alpha_{ll} - 1)\,, \tag{15.1.2}$$

corresponding to the considerations in Sect. 14.1.1.1. Noting that $\mathbf{i}_j \times \mathbf{i}_k = \in_{jkl} \mathbf{i}_l$, $j,k=1,2,3$, and projecting on the axes of the frame $\mathscr{R}'$, it results $\in_{ijk} \alpha_{mj}\alpha_{mk} = \in_{lmn} \alpha_{li}$, $i,m,n=1,2,3$. Taking into account the relation $\alpha_{ij}\alpha_{ik} = \delta_{jk}$, $j,k=1,2,3$, too, we get $\alpha_{jj}\alpha_{kk} - 2\alpha_{ll} = 3 - 2\alpha_{[jk]}\alpha_{[jk]}$, wherefrom $(\alpha_{ll}-1)^2/4 \le 1$; the formula (15.1.2) leads thus always to acceptable values for the angle χ. Considering the antisymmetric part of the tensor (15.1.1'), we obtain, analogously,

$$u_i = -\frac{\in_{ijk} \alpha_{jk}}{2\sin\chi}\,,\quad i=1,2,3\,. \tag{15.1.2'}$$

Hence, $-\mathbf{u}\sin\chi$ is the vector associated to the antisymmetric part of the tensor $\boldsymbol{\alpha}$. Knowing the matrix $\boldsymbol{\alpha}$, we can determine the rotation of angle χ about the axis of unit vector $\mathbf{u}$; as we have seen in Sect. 14.1.1.2, this representation is multiform. Even if we restrict ourselves to the interval $[0,2\pi]$, the relation (15.1.2) leads to the values χ and $2\pi-\chi$ (distinct values, excepting the particular case $\chi=\pi$); from (15.1.2') one obtains the components u_i and $-u_i$, respectively, in fact the same rotation otherwise represented.

Denoting

$$\lambda = u_1 \sin\frac{\chi}{2}, \quad \mu = u_2 \sin\frac{\chi}{2}, \quad \nu = u_3 \sin\frac{\chi}{2}, \quad \rho = \cos\frac{\chi}{2}, \tag{15.1.3}$$

we find again the representation (14.1.6) of the matrix $\boldsymbol{\alpha}$.

The rotation velocity of the movable frame of reference $\mathscr{R}$, rigidly linked to the solid, with respect to the fixed frame $\mathscr{R}'$, is characterized by the angular velocity vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$, which is expressed, by means of Euler's angles ψ, θ, φ, in the vector form (5.2.34) and, in components, in the form (5.2.35), with respect to the frame $\mathscr{R}$, or in the form (5.2.35'), with respect to the frame $\mathscr{R}'$; we mention also the inverse relations (14.1.15) and (14.1.15'), respectively (see Sect. 14.1.1.4 and Fig. 14.1 too). Introducing the functions $\alpha_i = \alpha_i(t)$, $i = 1,2,3$, which represent the direction cosines of the Ox_3'-axis with respect to the movable frame $\mathscr{R}$, we obtain the relations (5.2.36) and (14.1.16). The functions α_i and ω_i are linked by the differential relations (5.2.37') (see Chap. 5, Sect. 2.3.3). The accelerations distribution will be of the form (5.2.38), the fixed point being the only one of null acceleration. We remark also that the motion is reduced to a finite rotation if and only if the vectors $\dot{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}$ are collinear or if $\dot{\boldsymbol{\omega}} = \mathbf{0}$. Noting that $\omega^2 = \omega_i \omega_i$, we get

$$\omega^2 = \dot{\psi}^2 + \dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta\,. \tag{15.1.4}$$

The fixed and movable axoids are two tangent cones, with the vertices at the fixed point (*Poinsot's cones*); the motion of the rigid solid with a fixed point will be thus characterized by *the rolling without sliding of the polhodic* (movable) *cone* $\mathscr{C}_p$ *over the herpolhodic* (fixed) *cone* $\mathscr{C}_h$ (see Fig. 5.16 too). Let ω_i, $i = 1,2,3$, be the components of the vector $\boldsymbol{\omega}$ in the frame of reference $\mathscr{R}$; an arbitrary point P on the support of the vector $\boldsymbol{\omega}$ has the co-ordinates $x_i = \lambda\omega_i$, $i = 1,2,3$, λ scalar, with respect to this frame. Replacing in the relations (5.2.35) and eliminating the parameter λ and the time t between these three relations, we obtain the equation of the polhodic cone with respect to the non-inertial (movable) frame $\mathscr{R}$. Analogously, we denote by ω_i', $i = 1,2,3$, the components of the same vector $\boldsymbol{\omega}$ with respect to the frame $\mathscr{R}'$; the point P will have the co-ordinates $x_i' = \lambda\omega_i'$, $i = 1,2,3$. We replace then in the relations (5.2.35') and eliminate the parameter λ and the time t between these three relations; we get thus the equation of the herpolhodic cone with respect to the inertial (fixed) frame $\mathscr{R}'$.

The points of the rigid solid which are situated on a sphere of centre O form a spherical figure $\mathscr{F}$, of invariable form, movable on this sphere. The traces of the cones $\mathscr{C}_h$ and $\mathscr{C}_p$ of vertex O on this sphere are two curves: the curve C_h, fixed on the sphere, and the curve C_p, invariably linked to the spherical figure mentioned above; *the motion of the spherical figure* $\mathscr{F}$ *is obtained by rolling without sliding of the curve* C_p *over the curve* C_h.

15.1.1.2 Kinetical Considerations

In case of the rigid solid with a fixed point we have $\mathbf{v}'_O = \mathbf{0}$, so that the velocity of the mass centre C is given by (see the results in Sect. 14.1.1.5)

$$\mathbf{v}'_C = \boldsymbol{\omega} \times \boldsymbol{\rho}\,, \tag{15.1.5}$$

and the momentum of the rigid solid will be expressed in the form

$$\mathbf{H}' = M\boldsymbol{\omega} \times \boldsymbol{\rho}\,. \tag{15.1.6}$$

We notice that $|\boldsymbol{\rho}| = \rho = \mathrm{const}$, the centre C describing a curve situated on the sphere $(0, \rho)$; in this case, the magnitudes $|\mathbf{v}'_C|$ and $|\mathbf{H}'|$ are in direct proportion to ρ.

As well, we obtain the moment of momentum

$$\mathbf{K}'_{O'} = \mathbf{K}'_O = \mathbf{I}_O \boldsymbol{\omega}\,; \tag{15.1.7}$$

making $O \equiv C$, we notice that this formula is identical with the formula (14.1.26'), which takes place in the case of a free rigid solid. This result was to be expected, because we have seen that the general motion of a free rigid solid can be studied in two steps: (i) the motion of the mass centre C as a free particle at which is concentrated the whole mass M of the rigid solid; (ii) the motion (rotation) of the rigid solid about the centre C (considered as a fixed point). Hence, if we take $O \equiv C$, then our study is useful for the second step of the mentioned general problem too. In this case, $\boldsymbol{\rho} = \mathbf{0}$, hence $\mathbf{v}'_C = \mathbf{0}$ and $\mathbf{H}' = \mathbf{0}$, while the formula (15.1.7) is reduced to the formula (14.1.26').

The kinetic energy is given by

$$T' = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{K}'_O = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_O \boldsymbol{\omega})\,, \tag{15.1.8}$$

and we can use all the considerations in Sect. 14.1.1.6; if $O \equiv C$, then we can write

$$T' = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{K}'_C = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_C \boldsymbol{\omega})\,. \tag{15.1.8'}$$

Besides the torsor $\{\mathbf{R}, \mathbf{M}_O\}$ of the given forces, we introduce the constraint force $\overline{\mathbf{R}}$, applied at the fixed point O; we notice that $\overline{\mathbf{M}}_O = \mathbf{0}$ (Fig.15.3a). In this case, the elementary work of the given external forces is given by

$$\mathrm{d}W' = \mathbf{M}_O \cdot \boldsymbol{\omega}\mathrm{d}t\,, \tag{15.1.9}$$

while the elementary work of the external constraint forces vanishes; as well, the power of the given forces is of the form

$$P' = \mathbf{M}_O \cdot \boldsymbol{\omega}\,. \tag{15.1.9'}$$

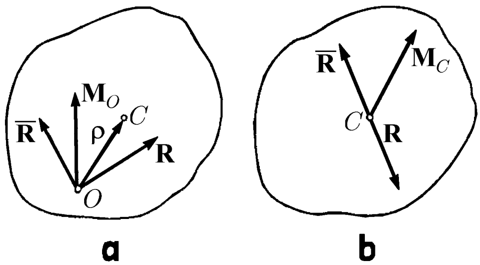


Fig. 15.3 The rigid solid with a fixed point acted upon by arbitrary external forces; the case O distinct from C (a) and the case $O \equiv C$ (b)

The vector equations of motion (14.1.60), (14.1.60') become (see Sect. 14.1.2.2 too)

$$M\mathbf{a}'_C = M\frac{\mathrm{d}\mathbf{v}'_C}{\mathrm{d}t} = M(\dot{\mathbf{v}}'_C + \boldsymbol{\omega}\times\mathbf{v}'_C) = M[\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})] = \mathbf{R} + \overline{\mathbf{R}}\,, \tag{15.1.10}$$

$$\mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{M}_O\,, \tag{15.1.11}$$

where $\dot{\mathbf{v}}'_C = \partial\mathbf{v}'_C/\partial t$ is the derivative of the velocity $\mathbf{v}'_C$ with respect to time, in the frame of reference $\mathscr{R}$. In components, along the axes of the non-inertial frame $\mathscr{R}$ we can write

$$Ma'_{Ci} = M\left(\in_{ijk}\ \dot{\omega}_j\rho_k + 2\omega_j\omega_{[i}\rho_{j]}\right) = R_i + \overline{R}_i\,,\quad i = 1,2,3\,, \tag{15.1.10'}$$

$$I_{kl}\left(\delta_{ik}\dot{\omega}_l + \in_{ijk}\ \omega_j\omega_l\right) = M_{Oi}\,,\quad i = 1,2,3\,; \tag{15.1.11'}$$

with respect to the principal axes of inertia, we obtain Euler's equations (taken again by Lagrange, Poisson, Poinsot, P. Saint-Guilhem, R.B. Hayward, J.C. Maxwell, G. Schmidt, P.V. Harlamov etc.)

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_{O1}\,,\\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= M_{O2}\,,\\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= M_{O3}\,, \end{aligned} \tag{15.1.11''}$$

where $I_1 \geq I_2 \geq I_3$ are the principal moments of inertia relative to the pole O.

Assuming that $O \equiv C$, the equation (15.1.11) becomes (Fig. 15.3,b)

$$\mathbf{R} + \overline{\mathbf{R}} = \mathbf{0}\,, \tag{15.1.10''}$$

as in the static case, while the equation (15.1.11) takes the form (14.1.48).

We can express the kinetic energy also in the form

$$T' = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \right) = \frac{1}{2} I_\Delta \omega^2 , \tag{15.1.8''}$$

where we have reported to the principal axes of inertia and to the instantaneous axis of rotation Δ, respectively (along the direction of the vector $\boldsymbol{\omega}$).

The theorem of kinetic energy (14.1.66) reads

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = \mathbf{M}_O \cdot \boldsymbol{\omega} , \tag{15.1.12}$$

and if the fixed point is just the mass centre, then we have

$$\frac{\mathrm{d}T'}{\mathrm{d}t} = \mathbf{M}_C \cdot \boldsymbol{\omega} . \tag{15.1.12'}$$

In components, along the principal axes of inertia, it results

$$\frac{1}{2}\left(I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 \right) = M_{O1}\omega_1 + M_{O2}\omega_2 + M_{O3}\omega_3 , \tag{15.1.12''}$$

while if $O \equiv C$, then we become the formula (14.1.49').

Sometimes it is useful that the non-inertial frame of reference $\mathscr{R}$ with the pole at O have a rotation $\boldsymbol{\Omega}$ with respect to the rigid solid (not being rigidly connected to it); obviously, in this case the equations of motion given by the general theorems have a more intricate form, as it was mentioned in Sect. 14.1.1.4.

Using Euler's angles and the relations (5.2.35), we can express the kinetic energy (15.1.8'') in the form

$$\begin{aligned} T' = \frac{1}{2}\Big[& I_1 \left(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \right)^2 + I_2 \left(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \right)^2 \\ & + I_3 \left(\dot{\psi}\cos\theta + \dot{\varphi} \right)^2 \Big] \end{aligned} \tag{15.1.13}$$

too. If the $O3$-axis is a kinetic axis of symmetry of the rigid solid (*the ellipsoid of inertia is of rotation*), then we have $I_1 = I_2 = J$ (important case in applications) and we obtain the remarkable formula

$$T' = \frac{1}{2}\left[J\left(\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2 \right) + I_3 \left(\dot{\psi}\cos\theta + \dot{\varphi} \right)^2 \right] . \tag{15.1.13'}$$

We notice that, in this case, $\partial T / \partial\psi = \partial T / \partial\varphi = 0$. If *the ellipsoid of inertia is a sphere* ($I_1 = I_2 = I_3 = I$), then we get

$$T' = \frac{1}{2} I \left(\dot{\psi}^2 + \dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta \right) = \frac{1}{2} I\omega^2 , \tag{15.1.13''}$$

in conformity to the formula (15.1.4).

Taking into account the results in Sect. 14.1.1.6 (e.g., the formula (14.1.31)), we can write

$$\mathrm{d}T' = \mathbf{K}'_O \cdot \mathrm{d}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{K}'_O \,. \tag{15.1.14}$$

Observing that the ellipsoid of inertia is the locus of the points P for which the position vector $\overrightarrow{OP}$ is given by (see the formulae (14.1.32) and (15.1.8"))

$$\overrightarrow{OP} = \frac{K}{\sqrt{2T'}}\boldsymbol{\omega}, \tag{15.1.15}$$

we obtain $\mathbf{K}'_O \cdot \mathrm{d}\overrightarrow{OP} = 0$; we can thus state that the moment of momentum $\mathbf{K}'_O$ is along the normal OQ from O to the plane Π ($Q \in \Pi$), tangent to the ellipsoid of inertia (see Fig. 3.9 too). The distance from the point O to the plane Π is given by

$$h = \left|\overrightarrow{OQ}\right| = \overrightarrow{OP} \cdot \operatorname{vers}\mathbf{K}'_O = \overrightarrow{OP} \cdot \frac{\mathbf{K}'_O}{K'_O} = K \frac{\boldsymbol{\omega} \cdot \mathbf{K}'_O}{K'_O \sqrt{2T'}} \,.$$

Taking into account (15.1.8), it results

$$h = \frac{K\sqrt{2T'}}{K'_O} \,. \tag{15.1.16}$$

The vector $\mathbf{J}_O$ *associated to the moment of inertia tensor* $\mathbf{I}_O$ is thus defined in the form

$$\mathbf{J}_O = \frac{\mathbf{K}'_O}{\sqrt{2T'}} = \frac{I_O}{\sqrt{I_\Delta}} \operatorname{vers}\boldsymbol{\omega} \tag{15.1.17}$$

and is situated along the same normal OQ. In conformity to the formula (14.1.34'), $J_O \overline{OQ} = K$; the locus of the extremity P' of the vector $\mathbf{J}_O$ (we take $K = R^2 \sqrt{M}$, so that P' be the inverse of the point Q with respect to a sphere of centre O and of radius R; Fig. 3.9) is the ellipsoid of gyration. We obtain thus a graphic method for the determination of the moment of momentum $\mathbf{K}'_O$.

15.1.1.3 General Methods of Computation. Case of the Heavy Rigid Solid

To determine the motion of the rigid solid with a fixed point O, subjected to the action of given forces of torsor $\{\mathbf{R}, \mathbf{M}_O\}$, as well as to the constraint force $\overline{\mathbf{R}}$ (Fig. 15.3,a), knowing its position at a given moment, we have at our disposal the vector equations of motion (15.1.10), (15.1.11) (or the equivalent equations) and the relations (5.2.35), hence two vector equations and three scalar ones for the two vector functions $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ and $\overline{\mathbf{R}} = \overline{\mathbf{R}}(t)$ and the three scalar functions $\psi = \psi(t)$, $\theta = \theta(t)$ and $\varphi = \varphi(t)$.

In general, $R_i = R_i(\psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t)$ and $M_{Oi} = M_{Oi}(\psi, \theta, \varphi, \omega_1, \omega_2, \omega_3; t)$, $i = 1,2,3$. The six unknown scalar functions (Euler's angles $\psi = \psi(t)$, $\theta = \theta(t)$, $\varphi = \varphi(t)$ and the components $\omega_i = \omega_i(t)$, $i = 1,2,3$, of the angular velocity vector of the rigid solid) are determined by the system of differential equations (15.1.11"), written in the normal form

$$\begin{aligned}
\dot{\omega}_1 &= \frac{1}{I_1}\widetilde{M}_{O1} = \frac{1}{I_1}[M_{O1} + (I_2 - I_3)\omega_2\omega_3],\\
\dot{\omega}_2 &= \frac{1}{I_2}\widetilde{M}_{O2} = \frac{1}{I_2}[M_{O2} + (I_3 - I_1)\omega_3\omega_1],\\
\dot{\omega}_3 &= \frac{1}{I_3}\widetilde{M}_{O3} = \frac{1}{I_3}[M_{O3} + (I_1 - I_2)\omega_1\omega_2],
\end{aligned} \tag{15.1.18}$$

and by the system of equations (14.1.53"); the initial conditions (at the moment $t = t_0$) of Cauchy type will be of the form

$$\psi(t_0) = \psi^0, \quad \theta(t_0) = \theta^0, \quad \varphi(t_0) = \varphi^0, \quad \omega_i(t_0) = \omega_i^0, \quad i = 1,2,3. \tag{15.1.19}$$

Starting from the Theorem 14.1.12, we can state a theorem of existence and uniqueness of the solution.

It is interesting to see that the formulation of the problem would be more intricate (equations (14.1.55)–(14.1.55")) by choosing the centre of mass C as pole of the movable frame of reference $\mathcal{R}$.

If we determine the position of the rigid solid and the angular velocity $\boldsymbol{\omega}$, then the equation (15.1.10) allows to express the constraint force in the form

$$\overline{\mathbf{R}} = -\mathbf{R} + M[\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]. \tag{15.1.20}$$

If $O \equiv C$, then the constraint force is

$$\overline{\mathbf{R}} = -\mathbf{R}, \tag{15.1.20'}$$

hence the same as in the static case (Fig. 15.3,b).

Introducing the direction cosines α_i, $i = 1,2,3$, of the Ox_3'-axis with respect to the axes of the frame of reference $\mathcal{R}$ (one can pass to Euler's angles by means of the relations (5.2.36)), the components of the torsor of the given forces are of the form $R_i = R_i(\alpha_1, \alpha_2, \alpha_3, \omega_1, \omega_2, \omega_3; t)$ and $M_{Oi} = M_{Oi}(\alpha_1, \alpha_2, \alpha_3, \omega_1, \omega_2, \omega_3; t)$; the six unknown functions $\alpha_i = \alpha_i(t)$ and $\omega_i = \omega_i(t)$, $i = 1,2,3$, are determined by Poisson's system of geometric equations (14.1.54), written in the normal form, and by the system of dynamic equations (15.1.18). Associating the initial conditions

$$\alpha_i(t_0) = \alpha_i^0, \quad i = 1,2,3, \tag{15.1.19'}$$

to the initial conditions (15.1.19) (with which they must be compatible), we can state, analogously, a theorem of existence and uniqueness.

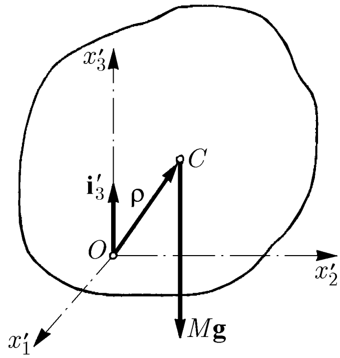


Fig. 15.4 The rigid solid with a fixed point acted upon by its own weight **G**

An important particular case is that in which the rigid solid is acted only by its own weight $\mathbf{G} = M\mathbf{g} = -Mg\mathbf{i}_3'$ at the centre of mass C (Fig. 15.4). The equations (15.1.11") take the form

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= Mg(\rho_3\alpha_2 - \rho_2\alpha_3), \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= Mg(\rho_1\alpha_3 - \rho_3\alpha_1), \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= Mg(\rho_2\alpha_1 - \rho_1\alpha_2). \end{aligned} \tag{15.1.21}$$

If the fixed point is just the mass centre, then Euler's equations (15.1.21) read

$$\begin{aligned} I_1^{(C)}\dot{\omega}_1 + \left(I_3^{(C)} - I_2^{(C)}\right)\omega_2\omega_3 &= 0, \\ I_2^{(C)}\dot{\omega}_2 + \left(I_1^{(C)} - I_3^{(C)}\right)\omega_3\omega_1 &= 0, \\ I_3^{(C)}\dot{\omega}_3 + \left(I_2^{(C)} - I_1^{(C)}\right)\omega_1\omega_2 &= 0 \end{aligned} \tag{15.1.21'}$$

and form a system of homogeneous equations.

The latter case can be seen as a second step in the study of the motion of the free rigid solid subjected only to the action of its own weight; the first step is represented by the study of the motion of the mass centre, the trajectory of which is a parabola (in particular, the local vertical; see Sect. 14.1.1.8 too).

15.1.1.4 Jacobi's Multiplier

To determine the first integrals of the systems of differential equations considered above, a particularly important rôle is played by Jacobi's multiplier. We will present the corresponding theory for a system of differential equations

$$\dot{x}_j = X_j(x_1, x_2, ..., x_{n-1}; t), \quad j = 1, 2, ..., n-1. \tag{15.1.22}$$

As it was shown in Chap. 6, Sect. 1.2.2 and in Sect. 11.1.1.6, if the functions X_j and $\partial X_j / \partial x_k$, $j, k = 1, 2, ..., n-1$, are defined and continuous in a neighbourhood $\mathscr{V}$ of the point $(t_0; x_1^0, x_2^0, ..., x_{n-1}^0)$, then there exists a neighbourhood $\overline{\mathscr{V}} \subset \mathscr{V}$ in which the solution of the system of differential equations (15.1.22) is obtained by means of the first integrals

$$f_j(x_1, x_2, ..., x_{n-1}; t) = C_j, \quad C_j = \text{const}, \quad j = 1, 2, ..., n-1. \tag{15.1.23}$$

We can set up at the most $n-1$ independent first integrals; we say that the first integrals $\{f_k, k = 1, 2, ..., n-1\}$ form a fundamental system of first integrals if they are of class C^1 with respect to all variables (x_k, $k = 1, 2, ..., n-1$, and t), the corresponding functional determinant being non-zero.

$$\det\left[\frac{\partial(f_1, f_2, ..., f_{n-1})}{\partial(x_1, x_2, ..., x_{n-1})}\right] \neq 0. \tag{15.1.23'}$$

The initial conditions of Cauchy type allow the determination of the constants C_j, $j = 1, 2, ..., n-1$, the solution

$$x_j = x_j(t; C_1, C_2, ..., C_{n-1}), \quad j = 1, 2, ..., n-1 \tag{15.1.22'}$$

thus obtained being unique.

In a study of the problem of first integrals it is convenient to denote $t = x_n$ (t has no more a privileged position), the system of differential equations (15.1.22) being written in the form ($X_n = 1$)

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = ... = \frac{\mathrm{d}x_n}{X_n}. \tag{15.1.24}$$

Thus, a function $f = f(x_1, x_2, ..., x_n)$ is a first integral of the system (15.1.22) if

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \mathrm{d}x_j = 0. \tag{15.1.25}$$

Taking into account (15.1.24), we can write this condition in the form

$$\mathscr{D}f = 0, \quad \mathscr{D} = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}. \tag{15.1.25'}$$

We notice that the system (15.1.24) is just the characteristic differential system associated to the equation with partial derivatives of first order (15.1.25'), written by means of the differential operator $\mathscr{D}$; hence, we may state

Theorem 15.1.1 *The relation* (15.1.25') *represents the necessary and sufficient condition so that the function f be a first integral of the system* (15.1.24).

We can write (the function f depends on the functions f_k, $k = 1,2,...,m$, $m \leq n-1$)

$$\mathrm{d}f = \sum_{j=1}^{n}\sum_{k=1}^{m}\frac{\partial f}{\partial f_k}\frac{\partial f_k}{\partial x_j}\mathrm{d}x_j = \sum_{k=1}^{m}\frac{\partial f}{\partial f_k}\sum_{j=1}^{n}\frac{\partial f_k}{\partial x_j}\mathrm{d}x_j\,,$$

as well as

$$\mathscr{D}f = \sum_{k=1}^{m}\frac{\partial f}{\partial f_k}\mathscr{D}f_k = 0\,, \tag{15.1.25''}$$

if $\mathscr{D}f_k = 0$, $k = 1,2,...,m$; it results

Theorem 15.1.2 *Any function f which depends on $m \leq n-1$ first integrals f_k, $k = 1,2,...,m$, of the system of differential equations* (15.1.24) *is a first integral of this system.*

If the rank of the matrix

$$\left[\frac{\partial(f, f_1, f_2, ..., f_m)}{\partial(x_1, x_2, ..., x_n)}\right]$$

is $m+1$, $m < n-1$, then the first integral f is independent of the first integrals f_k, $k = 1,2,...,m$. Let us suppose now that the relations $\mathscr{D}f_k = 0$ take place for the functions $X_k \neq 0$, $k = 1,2,...n-1$, the rank of the matrix

$$\left[\frac{\partial(f_1, f_2, ..., f_{n-1})}{\partial(x_1, x_2, ..., x_n)}\right]$$

being $n-1$; the $n-1$ first integrals f_k are, in this case, independent and form a *fundamental system of first integrals* $\{f_k, k = 1,2,...,n-1\}$. We have $\mathscr{D}f = 0$ and can state

Theorem 15.1.3 *A fundamental system of first integrals of the system* (15.1.24) *being given, any other first integral of this system is a function of the considered fundamental system.*

The corresponding functional determinant vanishes

$$D = \det\left[\frac{\partial(f, f_1, f_2, ..., f_{n-1})}{\partial(x_1, x_2, ..., x_n)}\right] = 0\,. \tag{15.1.26}$$

Developing Jacobi's determinant D after the first line, it results

$$D = \sum_{i=1}^{n} \Delta_i \frac{\partial f}{\partial x_i}, \quad \Delta_i = (-1)^{i+1} D_i, \quad i = 1,2,\ldots,n, \tag{15.1.26'}$$

where D_i is the minor of the element $\partial f / \partial x_i$ of the first line, while Δ_i is the corresponding algebraic complement. The relations (15.1.25') and (15.1.26), (15.1.26') are equivalent, both being the necessary and sufficient conditions so that the function f be a first integral of the differential system (15.1.24); hence, there exists a function M, called *Jacobi's multiplier*, so that

$$\Delta_i = M X_i, \quad i = 1,2,\ldots,n. \tag{15.1.27}$$

The relation

$$M \mathscr{D} f = D \tag{15.1.27'}$$

takes place too.

Together with Jacobi, we consider the determinant

$$U = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ u_{11} & u_{12} & \cdots & u_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{n-1,1} & u_{n-1,2} & \cdots & u_{n-1,n} \end{vmatrix}, \quad u_{ij} = \frac{\partial f_i}{\partial x_j}, \quad a_j = \text{const},$$

where $i = 1,2,\ldots,n-1$, $j = 1,2,\ldots,n$; using the above notations, we can write

$$U = \sum_{j=1}^{n} a_j \Delta_j,$$

wherefrom $\Delta_j = \partial U / \partial a_j$, $j = 1,2,\ldots,n$. Noting that Δ_j depends on x_j by means of the quantities u_{ik}, $i = 1,2,\ldots,n-1$, $k = 1,2,\ldots,n$, $k \neq j$, we have

$$\frac{\partial \Delta_j}{\partial x_j} = \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{\partial \Delta_j}{\partial u_{ik}} \frac{\partial u_{ik}}{\partial x_j} = \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{\partial \Delta_j}{\partial u_{ik}} \frac{\partial^2 f_i}{\partial x_j \partial x_k}, \quad k \neq j, \quad j = 1,2,\ldots,n,$$

so that

$$\sum_{j=1}^{n} \frac{\partial \Delta_j}{\partial x_j} = \sum_{j=1}^{n} \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{\partial \Delta_j}{\partial u_{ik}} \frac{\partial^2 f_i}{\partial x_j \partial x_k} = \sum_{i=1}^{n-1} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial \Delta_j}{\partial u_{ik}} \frac{\partial^2 f_i}{\partial x_j \partial x_k}$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \Delta_j}{\partial u_{ik}} + \frac{\partial \Delta_k}{\partial u_{ij}} \right) \frac{\partial^2 f_i}{\partial x_j \partial x_k} = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 U}{\partial u_{ik} \partial a_j} + \frac{\partial^2 U}{\partial u_{ij} \partial a_k} \right) \frac{\partial^2 f_i}{\partial x_j \partial x_k},$$

$$k \neq j ,$$

where we took the symmetric part of the parenthesis with respect to the indices j and k (the contribution of the corresponding antisymmetric part vanishes) and where we took into account the expression of the algebraic complement Δ_j. Let us consider now, in the determinant U, the minor of second order formed by the lines 1 and i and by the columns j and k, that is

$$\begin{vmatrix} a_j & a_k \\ u_{ij} & u_{ik} \end{vmatrix} = a_j u_{ik} - a_k u_{ij} .$$

The determinant considered above will be thus of the form

$$U = \left(a_j u_{ik} - a_k u_{ij} \right) U' + U'' , \quad j \neq k ,$$

where the algebraic complement U' of this minor does not contain any of the elements a_j, a_k, u_{ij}, u_{ik}, while U'' may depend at the most linearly on these elements. In this case,

$$\frac{\partial^2 U}{\partial u_{ik} \partial a_j} + \frac{\partial^2 U}{\partial u_{ij} \partial a_k} = U' - U'' = 0 ,$$

and one obtains *Jacobi's identity* in the form

$$\sum_{i=1}^{n} \frac{\partial \Delta_i}{\partial x_i} = 0 . \tag{15.1.28}$$

Taking into account (15.1.27), one gets the equation of Jacobi's multiplier in the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (M X_i) = 0 \tag{15.1.29}$$

or in the form

$$\sum_{i=1}^{n} \left[X_i \frac{\partial}{\partial x_i} (\ln M) + \frac{\partial X_i}{\partial x_i} \right] = 0 . \tag{15.1.29'}$$

Writing the latter condition for two multipliers M_1 and M_2 and subtracting the two relations thus obtained one from the other, we get

$$\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \left(\ln \frac{M_1}{M_2} \right) = \mathscr{D} \ln \frac{M_1}{M_2} = 0 .$$

Hence, $\ln(M_1 / M_2)$ is a first integral of the differential system (15.1.24); but a function of a first integral is a first integral too. We can thus state

Theorem 15.1.4 *If* M_1 *and* M_2 *are two multipliers of the differential system* (15.1.24), *then their ratio* M_1 / M_2 *is also a first integral of this system.*

Starting from the conditions (15.1.25') and (15.1.29), we may write

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(MfX_i) = 0$$

obtaining thus another form of the Theorem 15.1.4. We state

Theorem 15.1.4' *If M is a multiplier of the differential system* (15.1.24) *and f is a first integral of this system, then* Mf *is a multiplier of the respective system too.*

If

$$\sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = 0, \qquad (15.1.30)$$

hence if the divergence of the n-dimensional vector of components X_i, $i = 1,2,...,n$, vanishes, then the equation of Jacobi's multiplier is verified for $M = 1$ (any constant is a Jacobi multiplier). Taking into account the above result, we can state

Theorem 15.1.4'' *Any non-constant multiplier is a first integral of the differential system* (15.1.24) *which verifies the condition* (15.1.30).

15.1.1.5 Properties of Invariance. Theory of the Last Multiplier

Let be the change of variables

$$x_i = x_i(y_1, y_2, ... y_n), \quad i = 1,2,...,n, \qquad (15.1.31)$$

where x_i are functions of class C^1, the functional determinant being non-zero

$$J \equiv \det\left[\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}\right] \neq 0. \qquad (15.1.31')$$

Writing again the differential system (15.1.24) in the form (15.1.22), we have $(x_n = t, X_n = 1)$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{\partial y_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial y_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}t} = \sum_{j=1}^{n} X_j \frac{\partial y_i}{\partial x_j} = \mathscr{D} y_i, \quad i = 1,2,...,n.$$

Denoting $\mathscr{D} y_i = Y_i$, where Y_i, $i = 1,2,...,n$, are functions of y_j, $j = 1,2,...,n$, by means of the relations (15.1.31), the differential system (15.1.24) takes the form

$$\frac{\mathrm{d}y_1}{Y_1} = \frac{\mathrm{d}y_2}{Y_2} = ... = \frac{\mathrm{d}y_n}{Y_n}. \qquad (15.1.32)$$

We notice that

$$\mathscr{D}f = \sum_{j=1}^{n} X_j \frac{\partial f}{\partial x_j} = \sum_{j=1}^{n}\sum_{i=1}^{n} X_j \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial x_j} = \sum_{i=1}^{n} Y_i \frac{\partial f}{\partial y_i}.$$

Hence, the differential operator $\mathscr{D}$ is invariant to a change of variable (15.1.31). Let M_0 be a multiplier of the differential system (15.1.24), which satisfies the relation (15.1.27'), written in the form

$$M_0 \mathscr{D}f = D_x , \tag{15.1.33}$$

to put in evidence the differentiation with respect to the variables x_i, $i = 1,2,...,n$, in the functional determinant. We observe that the matric relation

$$\left[\frac{\partial(f, f_1, f_2, ..., f_{n-1})}{\partial(x_1, x_2, ..., x_n)}\right] = \left[\frac{\partial(f, f_1, f_2, ..., f_{n-1})}{\partial(y_1, y_2, ..., y_n)}\right]\left[\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}\right]$$

takes place, whence we have $D_x = D_y J$; taking into account (15.1.33), we can write

$$M_0 J^{-1} \mathscr{D}f = D_y , \quad J^{-1} = \frac{1}{J}, \tag{15.1.33'}$$

because the operator $\mathscr{D}$ remains constructively invariant. If f is a first integral of the differential system (15.1.24), then $M = M_0 f$ is a multiplier corresponding to this system; hence, we get $MJ^{-1} = M_0 J^{-1} f$. But $M_0 J^{-1}$ verifies the relation (15.1.33'), which is of the form (15.1.27'), being thus a multiplier of the differential system (15.1.32); on the other hand, f is a first integral for this differential system too. Hence, we can state

Theorem 15.1.5 *If M is a multiplier for the differential system* (15.1.24), *then* MJ^{-1} *is a multiplier for the differential system* (15.1.32).

By a direct calculation, one can also show that

$$\sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = J \sum_{i=1}^{n} \frac{\partial (Y_i J^{-1})}{\partial y_i}, \tag{15.1.34}$$

which justifies once more the above statement, if we take into account the equation (15.1.29) of the multiplier.

Let us suppose that the differential system (15.1.24) admits the independent first integrals f_i, $i = 1,2,...,k$, $k < n$, and let us make the change of variable

$$y_i = x_i , \quad i = 1,2,...,n-k , \quad y_{n-k+j} = f_j , \quad j = 1,2,...,k . \tag{15.1.35}$$

In this case, the considered differential system takes the form

$$\frac{\mathrm{d}y_1}{Y_1} = \frac{\mathrm{d}y_2}{Y_2} = \ldots = \frac{\mathrm{d}y_{n-k}}{Y_{n-k}}, \tag{15.1.36}$$

where the functions Y_i, $i = 1,2,\ldots,n-k$, are obtained from the corresponding functions X_i, where we take into account the transformation (15.1.35) (x_i is replaced by y_i for $i = 1,2,\ldots,n-k$; then, for $i = n-k+1$, $n-k+2,\ldots,n$, the second group of relations (15.1.35) allows to replace x_i as functions of $x_1 = y_1$, $x_2 = y_2,\ldots$, $x_{n-k} = y_{n-k}$ and of $y_{n-k+j} = c_{n-k+j}$, $j = 1,2,\ldots,k$, c_{n-k+j} being constants). In this case, the functional determinant (15.1.31') becomes

$$J = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_2}{\partial x_1} & \ldots & \dfrac{\partial f_k}{\partial x_1} \\ 0 & 1 & 0 & \ldots & 0 & \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} & \ldots & \dfrac{\partial f_k}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & \dfrac{\partial f_1}{\partial x_{n-k}} & \dfrac{\partial f_2}{\partial x_{n-k}} & \ldots & \dfrac{\partial f_k}{\partial x_{n-k}} \\ 0 & 0 & 0 & \ldots & 0 & \dfrac{\partial f_1}{\partial x_{n-k+1}} & \dfrac{\partial f_2}{\partial x_{n-k+1}} & \ldots & \dfrac{\partial f_k}{\partial x_{n-k+1}} \\ 0 & 0 & 0 & \ldots & 0 & \dfrac{\partial f_1}{\partial x_{n-k+2}} & \dfrac{\partial f_2}{\partial x_{n-k+2}} & \ldots & \dfrac{\partial f_k}{\partial x_{n-k+2}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \dfrac{\partial f_1}{\partial x_n} & \dfrac{\partial f_2}{\partial x_n} & \ldots & \dfrac{\partial f_k}{\partial x_n} \end{vmatrix}$$

$$= \det\left[\frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n-k+1}, x_{n-k+2}, \ldots, x_n)}\right] = [D]_k; \tag{15.1.37}$$

in the notation introduced above, the index k puts in evidence a section in the set $\{1,2,\ldots,n\}$. Taking into account the properties mentioned above, it results that $M[D]_k^{-1}$, $[D]_k^{-1} = 1/[D]_k$, is a multiplier for the differential system (15.1.36) if M is a multiplier for the differential system (15.1.24); this multiplier verifies the equation

$$\sum_{i=1}^{n-k} \frac{\partial}{\partial y_i}\left(M[D]_k^{-1} Y_i\right) = 0. \tag{15.1.38}$$

The knowledge of k independent first integrals reduces thus the number of differential equations which remain to be integrated with k units (we remain with a sequence of $n-k$ equal ratios), eliminating k variables. In particular, if f_1 is a first integral of the differential system (15.1.24) ($k = 1$), then we eliminate the variable x_n (we remain with a sequence of $n-1$ equal ratios); if M is a multiplier for the differential system (15.1.24), then $M(\partial f_1 / \partial x_n)^{-1}$ is a multiplier for the new differential system thus obtained.

A particularly important case is that in which $k = n-2$; if we would also know a first integral f_{n-1} independent on the first $n-2$ first integrals f_i, $i = 1,2,...,n-2$, then we could express $n-1$ variables as functions of the nth variable and of $n-1$ constants of integration, obtaining thus a general integral of the differential system (15.1.24). But this system is reduced to the system

$$\frac{\mathrm{d}y_1}{Y_1} = \frac{\mathrm{d}y_2}{Y_2}, \tag{15.1.39}$$

which is, in fact, an ordinary differential equation in the Pfaff form

$$Y_2 \mathrm{d}y_1 - Y_1 \mathrm{d}y_2 = 0. \tag{15.1.39'}$$

Multiplying by the integrant factor μ, we obtain an exact differential of the form $\mathrm{d}\varphi(y_1, y_2) = 0$, hence $\varphi(y_1, y_2) = \text{const}$, which represents the first integral which was still necessary to integrate the differential system; the equation which must be verified by the integrant factor is written in the form

$$\frac{\partial(\mu Y_1)}{\partial y_1} + \frac{\partial(\mu Y_2)}{\partial y_2} = 0. \tag{15.1.38'}$$

Comparing with the equation (15.1.38), we can state that the multiplier $M[D]_{n-2}^{-1}$, called the last multiplier, is an integrant factor for the differential equation (15.1.39'). We can state

Theorem 15.1.6 *If we know a multiplier M for the differential system* (15.1.24), *then* $n-2$ *first integrals are sufficient to integrate it.*

In particular (the system (15.1.24) has a multiplier equal to unity), it results

Theorem 15.1.6' *There are necessary* $n-2$ *first integrals to obtain the general integral of a differential system* (15.1.24), *which verifies the condition* (15.1.30).

15.1.1.6 Cases of Integrability

Let us assume that the rigid solid with a fixed point O is acted upon only by its own weight $\mathbf{G} = M\mathbf{g}$, applied at the mass centre C. In this case, the problem is governed by the dynamic equations (15.1.21) and by the geometric equations (14.1.54). In the particular case in which $O \equiv C$, the dynamic equations read

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0, \end{aligned} \tag{15.1.40}$$

no more containing the unknown functions $\alpha_i = \alpha_i(t)$, $i = 1,2,3$; starting from this system, we can determine firstly the unknown functions $\omega_i = \omega_i(t)$, passing then to the system (14.1.54) to obtain the position of the rigid solid.

Introducing the notations $x_i = \omega_i$, $x_{i+3} = \alpha_i$, $X_i = \in_{ijk} \left(K'_{Oj}\omega_k + Mg\alpha_j\rho_k \right) / I_i$ (without summation with respect to i), $X_{i+3} = \in_{ijk} \alpha_j\omega_k$, $i = 1,2,3$, where, taking into account (15.1.7), we have $K'_{O1} = I_1\omega_1$, $K'_{O2} = I_2\omega_2$, $K'_{O3} = I_3\omega_3$, we may write the system (15.1.21), (14.1.54) in the form

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = \ldots = \frac{\mathrm{d}x_6}{X_6} = \mathrm{d}t\,, \tag{15.1.41}$$

considered in the preceding subsection. Let us assume now that for the system

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = \ldots = \frac{\mathrm{d}x_6}{X_6}\,, \tag{15.1.41'}$$

which does not contain the time explicitly, we succeeded to determine the independent first integrals $f_k(x_1, x_2, \ldots, x_6) = C_k$, $k = 1,2,\ldots,5$, $C_k = \text{const}$, which form a fundamental system of first integrals (the rank of the matrix $\left[\partial f_k / \partial x_j\right]$, $k = 1,2,\ldots,5$, $j = 1,2,\ldots,6$, is 5); we can thus express five of the variables as a function of the sixth one (e.g., $x_k = x_k(x_6, C_1, C_2, \ldots C_5)$, $k = 1,2,\ldots,5$), so that the system (15.1.41) be reduced to the differential equation with separate variables $\mathrm{d}x_6 = X_6(x_6, C_1, C_2, \ldots, C_5)\,\mathrm{d}t$. By a quadrature, we get $f(x_6) = t + \tau$, $\tau = \text{const}$; noting that $\mathrm{d}f / \mathrm{d}x_6 = \mathrm{d}t / \mathrm{d}x_6 = 1 / X_6 \neq 0$, the theorem of implicit functions leads to $x_6 = x_6(t + \tau)$, obtaining then $x_k = x_k(t + \tau, C_1, C_2, \ldots, C_5)$, $k = 1,2,\ldots,5$, too. Hence, to integrate the system of differential equations (15.1.21), (14.1.54) it is sufficient to determine five independent first integrals, not depending on time. Noting that

$$\sum_{i=1}^{6} \frac{\partial X_i}{\partial x_i} = 0\,, \tag{15.1.41''}$$

and using the theory of the last multiplier, it results that it is sufficient to know four independent first integrals f_1, f_2, f_3, f_4 of the considered differential system to can determine a fifth first integral too, independent on the other first integrals; a corresponding integrant factor is $[D]_4^{-1}$, where we use the notation of the preceding subsection.

Euler's system (15.1.21) can be written in the vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\cdot(\mathbf{I}_O\boldsymbol{\omega}) = Mg\mathbf{i}'_3 \cdot \boldsymbol{\rho}\,. \tag{15.1.21''}$$

A scalar product by $\mathbf{i}'_3$ leads to $[\mathrm{d}(\mathbf{I}_O\boldsymbol{\omega})/\mathrm{d}t]\cdot\mathbf{i}'_3 = 0$ or $\mathrm{d}[(\mathbf{I}_O\boldsymbol{\omega})\cdot\mathbf{i}'_3]/\mathrm{d}t = 0$, so that

$$(\mathbf{I}_O\boldsymbol{\omega})\cdot\mathbf{i}'_3 = K'_{O3'}, \tag{15.1.42}$$

where, taking into account (15.1.7), we notice that $K'_{O3'}$ represents the constant projection of the moment of momentum on the fixed axis Ox'_3 (distinct from the component K'_{O3} of the moment of momentum $\mathbf{K}'_O$ along the Ox_3-axis). We obtain thus a scalar first integral of the moment of momentum (conservation of the component of the moment of momentum along the local vertical) in the form

$$I_1\omega_1\alpha_1 + I_2\omega_2\alpha_2 + I_3\omega_3\alpha_3 = K'_{O3'}. \tag{15.1.42'}$$

By a scalar product of the equation (15.1.21") by $\boldsymbol{\omega}$, we get

$$(\mathbf{I}_O\dot{\boldsymbol{\omega}})\cdot\boldsymbol{\omega} = Mg(\boldsymbol{\omega},\mathbf{i}'_3,\boldsymbol{\rho}) = Mg(\boldsymbol{\omega}\times\mathbf{i}'_3)\cdot\boldsymbol{\rho} = -Mg\dot{\mathbf{i}}'_3\cdot\boldsymbol{\rho} = -Mg\frac{\partial(\mathbf{i}'_3\cdot\boldsymbol{\rho})}{\partial t},$$

the differentiation taking place with respect to the movable frame of reference; by integration, we obtain

$$(\mathbf{I}_O\boldsymbol{\omega})\cdot\boldsymbol{\omega} = -2Mg\mathbf{i}'_3\cdot\boldsymbol{\rho} + 2h, \tag{15.1.43}$$

where h represents the energy constant. It results thus the first integral of the mechanical energy (it can be obtained also from the theorem of kinetic energy (15.1.12)) in the form

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = -2Mg(\rho_1\alpha_1 + \rho_2\alpha_2 + \rho_3\alpha_3) + 2h. \tag{15.1.43'}$$

The third first integral will be

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \tag{15.1.44}$$

which is justified because $\mathbf{i}'_3$ is a unit vector.

Taking into account the above results, we can state that the problem of integration of the system of equations (15.1.21), (14.1.54) is reduced to *the problem of finding a fourth first integral of this system*. This problem has been studied by H. Poincaré, Ed. Husson, P. Burgatti, F. Quatela and others. In the same period (the last two decades of the XIXth century), H. Burns dealt with algebraic first integrals for the problem of n particles. Starting from this problem, H. Poincaré passed to the existence of uniform solutions of the equations of motion which contain a small parameter ν; in the case of the rigid solid with a fixed point O one takes $\nu = Mg\rho$, the small parameter being thus the product of the weight Mg of the solid by the distance $\overline{OC} = \rho$ from the fixed point to the centre of mass (a quantity of the nature of an energy). H. Poincaré showed thus that, in case of a small parameter ν and of some arbitrary initial conditions, it is necessary that the ellipsoid of inertia at the fixed point be of rotation so that a fourth algebraic first integral does exist. In several papers at the beginning of XXth century (including his doctor thesis from 1906), Ed. Husson showed that this result remains valid for an arbitrary parameter ν; moreover, if the ellipsoid of inertia corresponding to the fixed point is not of rotation, then, for arbitrary initial conditions, it exist a new algebraic first integral of the problem only if $\nu = 0$ (hence, if $\rho = 0$ or $M = 0$, neglecting the own weight of the rigid solid). We may state

Theorem 15.1.7 (*Husson*) *In the problem of the heavy rigid solid with a fixed point O, governed by the geometric equations* (14.1.54) *and by the dynamic equations* (15.1.21), *in case of arbitrary initial conditions* (15.1.19), (15.1.19'), *besides the three first integrals* (15.1.42–15.1.44), *there exists a fourth first integral, algebraic function of* $\omega_1, \omega_2, \omega_3, \alpha_1, \alpha_2, \alpha_3$, *which does not depend explicitly on t, if and only if the fixed point O is just the centre of mass* C ($O \equiv C$, *hence* $\rho = 0$, *the Euler-Poinsot case*) *or if the ellipsoid of inertia is of rotation* ($I_1 = I_2$ *and* $\rho_1 = \rho_2 = 0$, *the Lagrange-Poisson case*; $I_1 = I_2 = 2I_3$ *and* $\rho_3 = 0$, *the Sonya-Kovalevsky case*).

If there exists an algebraic first integral in the hypothesis $I_1 = I_2$, then – as it has been shown by R. Liouville – there exists also a first integral in the form of a homogeneous polynomial of first degree in $\omega_1, \omega_2, \omega_3$ and of second degree in $\alpha_1, \alpha_2, \alpha_3$; consequently, in the considered case, any algebraic first integral is an algebraic combination of homogeneous polynomials. Using this result, P. Burgatti gave an elementary demonstration to Husson's theorem.

As it can be easily seen, the Euler-Poinsot case differs somewhat from the other two cases of integrability. Indeed, in this case Euler's equations are of the form (15.1.21'), so that the moment of momentum $\mathbf{K}'_C = \mathbf{I}_C \boldsymbol{\omega}$ is constant with respect to the inertial frame of reference $\mathscr{R}'$, its magnitude being conserved with respect to the inertial frame $\mathscr{R}$ too; we can write the first integral

$$|\mathbf{I}_C \boldsymbol{\omega}|^2 = \left(I_1^{(C)}\right)^2 \omega_1^2 + \left(I_2^{(C)}\right)^2 \omega_2^2 + \left(I_3^{(C)}\right)^2 \omega_3^2 = K_C'^2, \qquad (15.1.45)$$

to which we associate the first integral (15.1.43') in the form

$$I_1^{(C)} \omega_1^2 + I_2^{(C)} \omega_2^2 + I_3^{(C)} \omega_3^2 = 2h. \qquad (15.1.43'')$$

Thus, the integration of the system of differential equations (15.1.21') (and, analogously, the integration of the system (15.1.40)) is reduced to a quadrature, so that this system may be separately considered. We pass then to the integration of the differential system (14.1.54) in the form

$$\frac{\mathrm{d}x_4}{X_4} = \frac{\mathrm{d}x_5}{X_5} = \frac{\mathrm{d}x_6}{X_6}, \qquad (15.1.41''')$$

where we know the variable coefficients $\omega_i = \omega_i(t)$, $i = 1,2,3$, and for which we have at our disposal only the first integral (15.1.44). We cannot apply the theory of the last multiplier because the time appears explicitly in the functions X_4, X_5, X_6; but we can write other two first integrals, starting from the conservation of the moment of momentum with respect to a fixed frame of reference. One can show that the integration of the system of equations (14.1.15), which gives Euler's angles, is reduced to the integration of an equation of Riccati type; to integrate this equation (by reducing to quadratures) one can determine two particular integrals, which can be considered to be the searched first integrals.

If we give up the generality concerning the initial conditions and if we allow some particular initial conditions, then we can get also other cases of integrability (by quadratures), e.g.: the Hess's case, the Goryachev-Chaplygin case, the Bobylev-Steklov case etc.

15.1.2 The Euler-Poinsot case

L. Euler considered in 1758 the case in which the moment of the given external forces with respect to the fixed pole vanishes ($\mathbf{M}_O = \mathbf{0}$) (e.g., the case in which the given external forces are reduced to a vanishing resultant or to a resultant passing through the fixed point); the case mentioned above (the case in which the rigid solid is fixed at the centre of mass ($O \equiv C$), being acted upon only by its own weight) is thus a particular case of that with which we will deal in what follows.

One can imagine also other cases of loading of the rigid solid leading to the same system of differential equations. Let us suppose, for instance, that each point of position vector $\mathbf{r}$ and of mass $\mathrm{d}m$ of the rigid solid $\mathscr{S}$ is acted upon by a system of elastic forces (it is attracted or repulsed by a discrete mechanical system $\overline{\mathscr{S}}$ of p fixed particles of masses m_j and of position vectors $\mathbf{r}_j$, $j = 1,2,...,p$, the magnitudes of the attractive or repulsive forces being in direct proportion to the masses and to the distances, respectively); hence, these forces are of the form (the coefficient of proportionality K is positive for attraction and negative for repulsion)

$$K\mathrm{d}m\sum_{j=1}^{p} m_j\left(\mathbf{r}_j - \mathbf{r}\right) = K\overline{M}(\overline{\boldsymbol{\rho}} - \mathbf{r})\,\mathrm{d}m\,,$$

where $\overline{M}$ is the mass of the system $\overline{\mathscr{S}}$, while $\overline{\boldsymbol{\rho}}$ is the position vector of the mass centre $\overline{C}$ (the influence of the system $\overline{\mathscr{S}}$ is replaced by the influence of the mass centre $\overline{C}$, at which the mass $\overline{M}$ is considered to be concentrated). The resultant of these forces is ($\mu(\mathbf{r})$ is the density of the rigid solid of volume V)

$$\mathbf{R} = K\overline{M}\int_V \mu(\mathbf{r})(\overline{\boldsymbol{\rho}} - \mathbf{r})\,\mathrm{d}V = K\overline{M}M(\overline{\boldsymbol{\rho}} - \boldsymbol{\rho})\,.$$

In particular, if $O \equiv \overline{C}$, then we have $\mathbf{R} = -K\overline{M}M\boldsymbol{\rho}$. As well, the moment of the respective forces with respect to the fixed point is given by

$$\mathbf{M}_O = K\overline{M}\int_V \mu(\mathbf{r})\mathbf{r}\cdot(\overline{\boldsymbol{\rho}} - \mathbf{r})\,\mathrm{d}V = -K\overline{M}\overline{\boldsymbol{\rho}}\cdot\int_V \mu(\mathbf{r})\mathbf{r}\,\mathrm{d}V = -K\overline{M}M\overline{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}\,.$$

In the same case we have $\mathbf{M}_{\overline{C}} = \mathbf{0}$ and we are in the Euler case. Analogously, if $O \equiv C$ we are in the same case ($\mathbf{M}_C = \mathbf{0}$, even if the point C is mobile with respect to an inertial frame of reference); in case of forces of attraction ($K > 0$), the mass centre C describes an ellipse of centre $\overline{C}$ (e.g., the case of the elliptic oscillator), the motion of rotation about this centre being governed by Euler's equations.

In 1834, L. Poinsot has made a profound synthetic study of the case considered by Euler, obtaining a particularly elegant geometric representation of the motion; this case of integrability (considered to be *the* I*st case of integrability*) is thus called *the Euler-Poinsot case*. If $O \equiv C$, then the motion corresponding to this case is an inertial one, because – as we have seen in Sect. 14.1.1.9 – it takes place (excepting the translation) also in the case of the free rigid solid for which the given forces are in equilibrium (their torsor vanishes at any point of the space) if $\boldsymbol{\omega}^0 \neq \mathbf{0}$. If $\mathbf{R} = \mathbf{0}$, then the motion remains inertial about a point O distinct from C too. Introducing the elliptic functions, C.G.J. Jacobi gave a final form to the solution of this problem, expressing the direction cosines of the axes of the frame of reference $\mathscr{R}$ with respect to the frame $\mathscr{R}'$ as uniform functions of time, while – in 1883 – Ch. Hermite reduced the determination of these cosines to the integration of an equation of Lamé, determining analytically all the elements of Poinsot's solution.

After a general study of motion of the rigid solid with a fixed point in the considered case, one passes to the determination of its position with respect to the fixed frame of reference $\mathscr{R}'$; the geometric study of the motion made by Poinsot and MacCullagh is then presented, as well as some complementary results.

15.1.2.1 Kinematic Solution of the Motion

We begin the study of the motion by the dynamical equations (15.1.40), corresponding to an arbitrary fixed point O, which specify the motion of the rigid solid fixed at the above mentioned point (the angular velocity $\boldsymbol{\omega}$ about this fixed point); the principal axes of inertia of the rigid solid at O have been chosen as co-ordinate axes of the non-inertial frame of reference $\mathscr{R}$.

We notice that, multiplying the first equation (15.1.40) by $I_1\omega_1$, the second one by $I_2\omega_2$ and the third one by $I_3\omega_3$ and summing, we obtain a first integral of the form (15.1.45); analogously, multiplying the first equation (15.1.40) by ω_1, the second one by ω_2 and the third one by ω_3 and summing, it results a first integral of the form (15.1.43"). The theorem of kinetic energy (15.1.12) leads to $\mathrm{d}T'/\mathrm{d}t = 0$, hence to $T' = h = \mathrm{const}$ (the elementary work of the given forces vanishes); one obtains a new interpretation of the first integral of the mechanical energy, which becomes a first integral of the kinetic energy. The constants K'_O and T' which intervene in these first integrals, are, obviously, positive; we agree to denote them in the form $K'_O = I\Omega$, $2T' = I\Omega^2$, where I is a quantity of the nature of a moment of inertia, while Ω is a quantity of the nature of an angular velocity (we have $I = K'^2_O / 2T'$, $\Omega = 2T' / K'_O$).

Because the moment of momentum $\mathbf{K}'_O = \mathbf{I}_O \boldsymbol{\omega}$ is constant with respect to the fixed frame of reference $\mathscr{R}'$, it results that its direction is constant in time with respect to this frame. Taking into account the relation (15.1.8), we notice that $2T' = \mathbf{K}'_O \cdot \boldsymbol{\omega} = K'_O \omega \cos(\mathbf{K}'_O, \boldsymbol{\omega})$, hence

$$\Omega = \omega \cos(\mathbf{K}'_O, \boldsymbol{\omega}), \tag{15.1.46}$$

and we can state

Theorem 15.1.8 (*Lagrange, Poinsot*) *In the Euler-Poinsot motion, the projection of the instantaneous angular velocity on the invariant direction of the moment of momentum is constant in time.*

As a matter of fact, one obtains thus an interesting interpretation for the constant Ω. The relation (15.1.8"), where we have introduced the instantaneous axis of rotation Δ too, allows to write $I\Omega^2 = I_\Delta \omega^2$, so that

$$I_\Delta = I \cos^2(\mathbf{K}'_O, \boldsymbol{\omega}). \tag{15.1.46'}$$

One obtains thus a relation which gives a remarkable interpretation for the constant I and puts in evidence the variation of the moment of inertia I_Δ. Using the vector $\mathbf{J}_O$ associated to the moment of inertia tensor $\mathbf{I}_O$ and defined by (15.1.17), one gets the relation $I = J_O^2$ too. Obviously, the constants I and Ω are specified by the initial conditions.

In this case, the motion is governed by the dynamical system

$$\begin{aligned} I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 &= I^2 \Omega^2, \\ I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 &= I \Omega^2, \end{aligned} \tag{15.1.47}$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = 0, \tag{15.1.47'}$$

the equation (15.1.47') being one of the three equations (15.1.40). We associate the last three initial conditions (15.1.19) to these equations. The ratio of the two finite relations (15.1.47) is written in the form

$$\frac{I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2}{I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2} = I. \tag{15.1.48}$$

Assuming that the principal moments of inertia are ordered in the form $I_1 \geq I_2 \geq I_3$ and noting that

$$\frac{a_1}{b_1} \geq \frac{a_1 + a_2}{b_1 + b_2} \geq \frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} \geq \frac{a_2 + a_3}{b_2 + b_3} \geq \frac{a_3}{b_3}$$

if

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \frac{a_3}{b_3}, \quad a_i, b_i > 0, \quad i = 1,2,3,$$

we can write also

$$I_1 \geq \frac{I_1^2 \omega_1^2 + I_2^2 \omega_2^2}{I_1 \omega_1^2 + I_2 \omega_2^2} \geq I \geq \frac{I_2^2 \omega_2^2 + I_3^2 \omega_3^2}{I_2 \omega_2^2 + I_3 \omega_3^2} \geq I_3 . \tag{15.1.48'}$$

In this subsection, we assume – at the beginning – that I differs from the principal moments of inertia and that the ellipsoid of inertia is not of rotation ($I_1 > I$, $I_2 > I_3$, $I \neq I_2$). In this case, from the relations (15.1.47) it results

$$\begin{aligned} \omega_1^2 &= \frac{I_2 (I_2 - I_3)}{I_1 (I_1 - I_3)} (\beta_2^2 - \omega_2^2), \quad \beta_2^2 = \frac{I (I - I_3)}{I_2 (I_2 - I_3)} \Omega^2, \\ \omega_3^2 &= \frac{I_2 (I_1 - I_2)}{I_3 (I_1 - I_3)} (\overline{\beta}_2^2 - \omega_2^2), \quad \overline{\beta}_2^2 = \frac{I (I_1 - I)}{I_2 (I_1 - I_2)} \Omega^2 . \end{aligned} \tag{15.1.49}$$

The differential equation (15.1.47') becomes

$$\dot{\omega}_2^2 = \frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3} (\beta_2^2 - \omega_2^2)(\overline{\beta}_2^2 - \omega_2^2), \tag{15.1.50}$$

hence a differential equation of first order for the unknown function $\omega_2 = \omega_2(t)$. We notice that

$$\beta_2^2 - \overline{\beta}_2^2 = \frac{I (I_1 - I_3)(I - I_2)}{I_2 (I_2 - I_3)(I_1 - I_2)} \Omega^2 ,$$

so that $\operatorname{sign}(\beta_2 - \overline{\beta}_2) = \operatorname{sign}(\beta_2^2 - \overline{\beta}_2^2) = \operatorname{sign}(I - I_2)$, where we assume that $\beta_2, \overline{\beta}_2 > 0$ too.

To fix the ideas, we suppose that $I < I_2$; it results $\beta_2 < \overline{\beta}_2$, while the relation (15.1.49) allows to state that $|\omega_2| \leq \beta_2$, inequality which holds during the motion. We also point out that $\omega_1 = 0$ for $\omega_2 = \pm\beta_2$, while ω_3 preserves a constant sign (because $\omega_3^2 > 0$). We denote (we take into account that $I_1 < I_2 + I_3$; see the relation (3.1.24) too)

$$\begin{aligned} \overline{\beta}_1^2 &= \frac{I_2 (I_2 - I_3)}{I_1 (I_1 - I_3)} \beta_2^2 = \frac{I (I - I_3)}{I_1 (I_1 - I_3)} \Omega^2 < \beta_2^2 , \\ \beta_3^2 &= \frac{I_2 (I_1 - I_2)}{I_3 (I_1 - I_3)} \overline{\beta}_2^2 = \frac{I (I_1 - I)}{I_3 (I_1 - I_3)} \Omega^2 < \overline{\beta}_2^2 . \end{aligned} \tag{15.1.49'}$$

In this case, the relations (15.1.49) correspond to two ellipses in the plane $O\omega_1\omega_2$ and in the plane $O\omega_3\omega_2$, respectively, of equations ($0 < \overline{\beta}_1 < \beta_2 < \overline{\beta}_2$, $0 < \beta_3 < \overline{\beta}_2$)

$$\frac{\omega_1^2}{\beta_1^2} + \frac{\omega_2^2}{\beta_2^2} = 1\,, \quad \frac{\omega_3^2}{\beta_3^2} + \frac{\omega_2^2}{\beta_2^2} = 1 \tag{15.1.49''}$$

and to a hyperbola in the plane $O\omega_3\omega_1$, of equation ($0 < \overline{\beta}_3 < \beta_3$, $0 < \beta_1 < \overline{\beta}_2$)

$$\frac{\omega_3^2}{\overline{\beta}_3^2} - \frac{\omega_1^2}{\beta_1^2} = 1\,, \quad \beta_1^2 = \frac{I\,(I_2 - I)}{I_1\,(I_1 - I_2)}\Omega^2\,, \quad \overline{\beta}_3^2 = \frac{I\,(I_2 - I)}{I_3\,(I_2 - I_3)}\Omega^2\,. \tag{15.1.49'''}$$

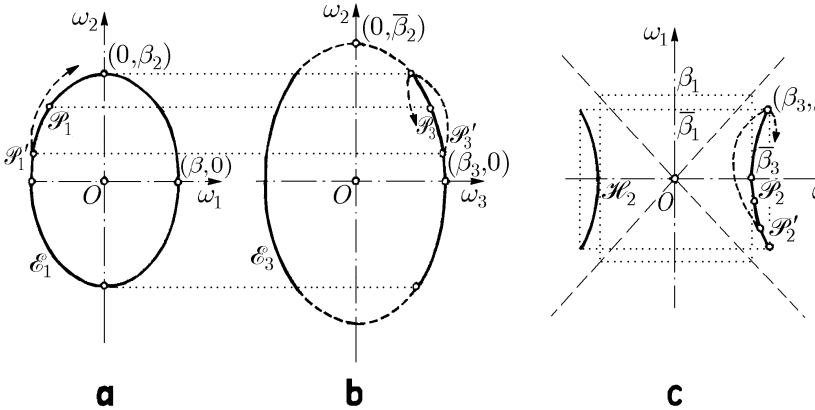


Fig. 15.5 The ellipse $\mathscr{E}_1$ (a), the ellipse $\mathscr{E}_3$ (b) and the hyperbola $\mathscr{H}_2$ (c) in the motion of a rigid solid with a fixed point

We assume that at the initial moment $t = t_0$ we have $\omega_2 = \omega_2^0 \geq 0$; to make a choice, we suppose, as well, that, at this moment, $\omega_1 < 0$ and $\omega_3 > 0$ (if ω_2 begins to grow, starting from the initial moment, we have $\dot{\omega}_2 > 0$ and the equation (15.1.47') shows that $\omega_1\omega_3 < 0$). The point $\mathscr{P}_1(\omega_1, \omega_2)$ describes the whole ellipse $\mathscr{E}_1$, the interior remaining at the right (Fig. 15.5a), because, from the first equation (15.1.40), it results that $\dot{\omega}_1 > 0$ if $\omega_2\omega_3 > 0$; correspondingly, the point $\mathscr{P}_3(\omega_3, \omega_2)$ describes only an arc of the ellipse $\mathscr{E}_3$ (Fig. 15.5b), because $|\omega_2| \leq \beta_2 < \overline{\beta}_2$, while the point $\mathscr{P}_2(\omega_3, \omega_1)$ describes also an arc of the hyperbola $\mathscr{H}_2$ (Fig. 15.5c), because $|\omega_1| \leq \overline{\beta}_1$, $\omega_3 \leq \beta_3$ (for $I = I_2$ the hyperbola degenerates into its asymptotes, the point (β_3, β_1) being situated on one of them). By separation of variables, from (15.1.50), one obtains

$$t - t_0 = \frac{1}{p}\int_{\omega_2^0/\beta_2}^{\omega_2/\beta_2} \frac{\mathrm{d}z}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}\,, \tag{15.1.51}$$

where we have denoted

$$p^2 = \frac{(I_2 - I_3)(I_1 - I)I}{I_1 I_2 I_3}\Omega^2, \quad k^2 = \left(\frac{\beta_2}{\overline{\beta}_2}\right)^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{(I_1 - I)(I_2 - I_3)}. \tag{15.1.52}$$

Denoting $\omega_2 = \beta_2 \sin\kappa$ and introducing the elliptic integral of the first kind $F(\kappa, k)$, given by (7.1.41), where κ is the amplitude, while k is the modulus of the integral, we may write the relation (15.1.51) in the form

$$t = t_0 + \frac{1}{p}\left[F(\kappa, k) - F\left(\kappa^0, k\right)\right], \tag{15.1.51'}$$

where $\sin\kappa^0 = \omega_2^0 / \beta_2$. Denoting $u = p(t - t_0)$, we can write

$$u = F(\kappa, k) - F\left(\kappa^0, k\right) \tag{15.1.51''}$$

too. Without any loss of generality, we assume that $\omega_2^0 = 0$ (the points $\mathscr{P}_1^0$ and $\mathscr{P}_3^0$ are at the extremities of the minor diameters of the ellipses $\mathscr{E}_1$ and $\mathscr{E}_3$, respectively). It results, $\kappa^0 = F\left(\kappa^0, k\right) = 0$, so that

$$u = F(\kappa, k), \tag{15.1.51'''}$$

where $u = \arg\kappa$, $\kappa = \operatorname{am} u$. Let us introduce *Jacobi's elliptic functions*: *the sinus amplitude* ($\operatorname{sn} u = \sin\kappa$), *the cosinus amplitude* ($\operatorname{cn} u = \cos\kappa$) and *the delta amplitude* ($\operatorname{dn} u = \sqrt{1 - k^2 \sin^2\kappa}$); the signs of $\operatorname{cn} u$ and $\operatorname{dn} u$ are chosen so that $\operatorname{cn} u = \operatorname{dn} u = 1$ for $u = 0$. One can introduce *the tangent amplitude* ($\operatorname{tanam} u = \operatorname{sn} u / \operatorname{cn} u$) too. We mention also the differential relations

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \quad \frac{\mathrm{d}}{\mathrm{d}u}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \quad \frac{\mathrm{d}}{\mathrm{d}u}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

We can express the components of the rotation angular velocity vector in the form (we notice that $\omega_1^0 = -\overline{\beta}_1$, $\omega_2^0 = 0$, $\omega_3^0 = \beta_3$)

$$\omega_1(t) = -\overline{\beta}_1 \operatorname{cn} p(t - t_0), \quad \omega_2(t) = \beta_2 \operatorname{sn} p(t - t_0), \quad \omega_3(t) = \beta_3 \operatorname{dn} p(t - t_0), \tag{15.1.53}$$

where we took into account the relations (15.1.49'') and the second notation (15.1.52). To $\kappa = \pi / 2$ corresponds $\omega_2 = \beta_2$ and we denote

$$T = K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}, \tag{15.1.54}$$

where $K(k)$ is the complete elliptic integral of the first kind. The component ω_1 varies between $-\overline{\beta}_1$ and $\overline{\beta}_1$ with the period $4T/p$ $(\operatorname{cn}0 = \cos\kappa^0 = 1$, $\operatorname{cn} pT = \cos(\pi/2) = 0)$, while the component ω_2 varies between β_2 and $-\beta_2$ with the same period $(\operatorname{sn}0 = \sin\kappa^0 = 0,\ \operatorname{sn} pT = \sin(\pi/2) = 1)$; the component ω_3 is varying between the limits β_3 and $\beta_3\sqrt{\overline{\beta}_2^2 - \beta_2^2}\,/\,\overline{\beta}_2 = \sqrt{I(I_2 - I)/I_3(I_2 - I_3)}\varOmega$ with the period $2T/p$ $(\mathrm{dn}0 = 1, \mathrm{dn}pT = \sqrt{1-k^2} = k'$, where k' is *the complementary modulus*).

If $I > I_2$, then we have $\beta_2 > \overline{\beta}_2$ too; an analogous study, in which we assume also that $t_0 = 0$ and $\omega_2^0 = 0$, leads to

$$\omega_1(t) = -\overline{\beta}_1 \operatorname{dn} \overline{p}(t - t_0), \quad \omega_2(t) = \overline{\beta}_2 \operatorname{sn} \overline{p}(t - t_0), \quad \omega_3(t) = \beta_3 \operatorname{cn} \overline{p}(t - t_0), \tag{15.1.55}$$

with

$$\overline{p}^2 = \frac{(I_1 - I_2)(I - I_3)I}{I_1 I_2 I_3}\varOmega^2, \quad \overline{k}^2 = \frac{1}{k^2} = \frac{(I_1 - I)(I_2 - I_3)}{(I - I_3)(I_1 - I_2)}. \tag{15.1.55'}$$

The component ω_1 varies between $-\overline{\beta}_1$ and $-\overline{\beta}_1\sqrt{\beta_2^2 - \overline{\beta}_2^2}\,/\,\beta_2$ $= -\sqrt{I(I - I_2)/I_1(I_1 - I_2)}\varOmega$ with the period $2\overline{T}/\overline{p}$; the component ω_2 varies between $\overline{\beta}_2$ and $-\overline{\beta}_2$ with the period $4\overline{T}/\overline{p}$, while the component ω_3 varies between β_3 and $-\beta_3$ with the same period. In this case $\overline{T} = K(\overline{k})$.

If we associate the relation

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \tag{15.1.47''}$$

to the first integrals (15.1.47), then we can calculate the components of the vector $\boldsymbol{\omega}$ as functions of ω in the form (the determinant of the coefficients is of Vandermonde type)

$$\begin{aligned}
\omega_1^2 &= \frac{I_2 I_3}{(I_1 - I_2)(I_1 - I_3)}(\omega^2 - \gamma_1^2), \\
\omega_2^2 &= \frac{I_3 I_1}{(I_2 - I_3)(I_2 - I_1)}(\omega^2 - \gamma_2^2), \\
\omega_3^2 &= \frac{I_1 I_2}{(I_3 - I_1)(I_3 - I_2)}(\omega^2 - \gamma_3^2),
\end{aligned} \tag{15.1.56}$$

where we have introduced the notations

$$\gamma_1^2 = \frac{I(I_2 + I_3 - I)}{I_2 I_3}\varOmega^2, \quad \gamma_2^2 = \frac{I(I_3 + I_1 - I)}{I_3 I_1}\varOmega^2, \quad \gamma_3^2 = \frac{I(I_1 + I_2 - I)}{I_1 I_2}\varOmega^2. \tag{15.1.56'}$$

Multiplying the first equation (15.1.40) by ω_1 / I_1, the second one by ω_2 / I_2 and the third one by ω_3 / I_3 and summing, it results

$$\omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 + \omega_3 \dot{\omega}_3 = \left(\frac{I_2 - I_3}{I_1} + \frac{I_3 - I_1}{I_2} + \frac{I_1 - I_2}{I_3} \right) \omega_1 \omega_2 \omega_3$$

or

$$\omega \dot{\omega} = \frac{(I_1 - I_2)(I_2 - I_3)(I_1 - I_3)}{I_1 I_2 I_3} \omega_1 \omega_2 \omega_3 . \tag{15.1.57}$$

Taking into account (15.1.56), we obtain, finally,

$$\frac{\mathrm{d}(\omega^2)}{\mathrm{d}t} = 2\omega\dot{\omega} = \pm 2\sqrt{(\omega^2 - \gamma_1^2)(\omega^2 - \gamma_2^2)(\omega^2 - \gamma_3^2)} , \tag{15.1.57'}$$

where we take the sign + or the sign – as ω^2 increases or decreases in time. The magnitude of the angular velocity vector $\boldsymbol{\omega}$ can be thus expressed by means of Weierstrass's elliptic function $\mathscr{P}(t)$, having thus a real period; we notice that ω^2 has values contained between $\min(\gamma_1^2, \gamma_3^2)$ and γ_2^2. Unlike F. Lindemann, who uses Jacobi's functions, J. Haug presents thus another approach (with a certain symmetry character) of Euler's problem for the rigid solid with a fixed point, which – obviously – leads to the same results.

15.1.2.2 Determination of the Position of the Rigid Solid

To specify the position of the movable frame of reference $\mathscr{R}$ (hence, of the rigid solid) with respect to the fixed frame $\mathscr{R}'$ it is sufficient to determine Euler's angles ψ, θ and φ. We notice that, during the motion, the moment of momentum with respect to the pole O, in the inertial frame of reference, is conserved in time ($\mathbf{K}'_O = \overrightarrow{\mathrm{const}}$); without any loss of generality, we can choose the Ox'_3-axis along the direction of this vector, so that $\mathbf{K}'_O = K'_O \mathbf{i}'_3$. The direction cosines of the unit vector $\mathbf{i}'_3$ with respect to the frame $\mathscr{R}$ are thus $\alpha_i = K'_{Oi} / K'_O$, $i = 1,2,3$; taking into account the relations (5.2.36) and the relations $K'_{O1} = I_1 \omega_1$, $K'_{O2} = I_2 \omega_2$, $K'_{O3} = I_3 \omega_3$, $K'_O = I\Omega$, we may write

$$\begin{aligned} I_1 \omega_1 &= I\Omega \sin\theta \sin\varphi , \\ I_2 \omega_2 &= I\Omega \sin\theta \cos\varphi , \\ I_3 \omega_3 &= I\Omega \cos\theta . \end{aligned} \tag{15.1.58}$$

The angles θ and φ are obtained easily, being given by

$$\tan\theta = \frac{\sqrt{I_1^2 \omega_1^2 + I_2^2 \omega_2^2}}{I_3 \omega_3} , \quad \tan\varphi = \frac{I_1 \omega_1}{I_2 \omega_2} . \tag{15.1.59}$$

To calculate the angle of precession ψ, we use the first relation (14.1.15); taking into account the previous relations, we can write

$$\dot{\psi} = \frac{\omega_1 \sin\varphi + \omega_2 \cos\varphi}{\sin\theta} = I\Omega\left(\frac{\sin^2\varphi}{I_1} + \frac{\cos^2\varphi}{I_2}\right) = \frac{I_1\omega_1^2 + I_2\omega_2^2}{I\Omega\sin^2\theta}$$

$$= \frac{I\Omega\left(I_1\omega_1^2 + I_2\omega_2^2\right)}{I_1^2\omega_1^2 + I_2^2\omega_2^2} = \frac{I\Omega\left(I\Omega^2 - I_3\omega_3^2\right)}{I^2\Omega^2 - I_3^2\omega_3^2} = \frac{I\Omega}{I_3}\left[1 - \frac{I\Omega^2(I - I_3)}{I^2\Omega^2 - I_3^2\omega_3^2}\right] > 0 \quad (15.1.59')$$

and the angle ψ is obtained by a quadrature.

Assuming that $I < I_2$, we use the solution (15.1.53); there result G.R. Kirchhoff's formulae

$$\cos\theta = \sqrt{\frac{I_3(I_1 - I)}{I(I_1 - I_3)}}\,\mathrm{dn}\,p(t - t_0), \quad \tan\varphi = -\sqrt{\frac{I_1(I_2 - I_3)}{I_2(I_1 - I_3)}}\frac{\mathrm{cn}\,p(t - t_0)}{\mathrm{sn}\,p(t - t_0)},$$

$$\dot{\psi} = \frac{I\left[I_2 - I_3 + (I_1 - I_2)\,\mathrm{sn}^2\,p(t - t_0)\right]}{I_1(I_2 - I_3) + I_3(I_1 - I_2)\,\mathrm{sn}^2\,p(t - t_0)}\Omega \quad (15.1.59'')$$

$$= \frac{I\Omega}{I_3}\left[1 - \frac{(I_1 - I_3)(I_2 - I_3)}{I_1(I_2 - I_3) + I_3(I_1 - I_2)\,\mathrm{sn}^2\,p(t - t_0)}\right] > 0\,.$$

To be precise, we notice that $\cos\theta$ is varying between the inferior limit $k'\sqrt{I_3(I_1 - I)/I(I_1 - I_3)} = \sqrt{I_3(I_2 - I)/I(I_2 - I_3)}$ and the superior one $\sqrt{I_3(I_1 - I)/I(I_1 - I_3)}$ (obviously, both subunitary), $\tan\varphi$ varies on the whole real axis, while $\dot{\psi}$ is varying between the inferior limit $I\Omega/I_1$ and the superior one $I\Omega/I_2$. Such considerations are due to N. Lindskog and W. von Tannenberg. From the second relation (14.1.15), we get

$$\dot{\theta} = -\frac{(I_1 - I_2)\,\omega_1\omega_2}{I\Omega\sin\theta} = -\frac{(I_1 - I_2)\,\omega_1\omega_2}{\sqrt{I_1^2\omega_1^2 + I_2^2\omega_2^2}}\,.$$

Hence, $\operatorname{sign}\dot{\theta} = -\operatorname{sign}(\omega_1\omega_2)$ and $\dot{\theta}(t + 4T) = \dot{\theta}(t)$. It results thus that $\theta(t + 4T) = \theta(t) + \mathrm{const}$; but $\cos\theta$ is a periodic function of time, of period $2T$ (the period of the function $\mathrm{dn}pt$), which never vanishes ($\mathrm{dn}\,pt \neq 0$). Hence, the angle of nutation is periodic too ($\theta(t + 2T) = \theta(t)$). In what concerns the function $\tan\varphi$, this one is periodic with respect to time, of period $4T$ (the period of the functions $\mathrm{cn}pt$ and $\mathrm{sn}pt$), and can take the value zero too; from (15.1.58) it results that $\varphi(t)$ can vary by $\pm 2\pi$. But, from the third relation (14.1.15) we obtain

$$\dot{\varphi} = \omega_3 - \frac{(I_1 \omega_1^2 + I_2 \omega_2^2)\cot\theta}{I\Omega \sin\theta} = \frac{[I_1(I_1 - I_2)\omega_1^2 + I_2(I_2 - I_3)\omega_2^2]\omega_3}{I_1^2 \omega_1^2 + I_2^2 \omega_2^2} > 0\,;$$

hence, $\operatorname{sign}\dot{\varphi} = \operatorname{sign}\omega_3$; we have $\dot{\varphi}(t + 4T) = \dot{\varphi}(t)$, wherefrom $\varphi(t + 4T) = \varphi(t) + 2\pi$ (because $\omega_3 > 0$, the function $\varphi(t)$ is an increasing one), the angle of proper rotation growing continuously with 2π. In what concerns the angle of precession, we notice that $\dot{\psi}(t + 2T) = \dot{\psi}(t)$, so that $\psi(t + 2T) - \psi(t) = \psi_0 > 0$, $\psi_0 = \text{const}$ ($\dot{\psi} > 0$, hence ψ is an increasing function); the angle of precession is no more a periodic function of time (as a matter of fact, as well as the angle of proper rotation).

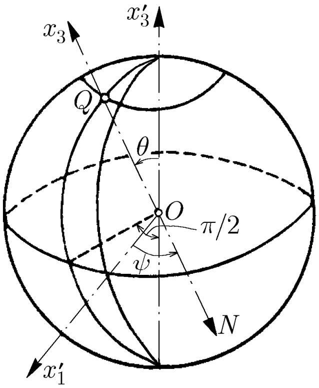


Fig. 15.6 Euler's angles on a unit sphere of centre O

Let Q be the trace of the Ox_3-axis on a sphere of centre O and unit radius (to fix the ideas); if $Ox_1'x_2'$ is the equatorial plane, then the angle θ is the colatitude, while the angle $\psi - \pi/2$ is the longitude with respect to the meridian plane $Ox_1'x_3'$ (Fig. 15.6). During the motion of the rigid solid, the point Q (hence, the axis Ox_3) has a motion of nutation, coming back on the same parallel circle, after a period $4T$, as well as a motion of precession, which brings no more Q on the same meridian (after a period $2T$ or $4T$). On the other hand, the motion of proper rotation about the Ox_3-axis contributes to the growing of the corresponding angle by 2π. Finally, we see that after the period $4T$ (which is period for all the three components ω_i, $i = 1,2,3$, with respect to the movable frame of reference) the rotation angular velocity vector $\boldsymbol{\omega}$ takes the same position with respect to the rigid solid (with respect to the movable frame $\mathscr{R}$), but not with respect to the fixed frame $\mathscr{R}'$, because neither the frame $\mathscr{R}$ (hence neither the rigid solid) does not come back at the same position with respect to the frame $\mathscr{R}'$.

We may put in evidence the poles of the function $\dot{\psi}$ in the form (15.1.59"), introducing a constant argument $\mathrm{i}c$, $\mathrm{i} = \sqrt{-1}$, $c \in \mathbb{R}$, specified by one of the equivalent relations

$$\mathrm{sn}^2 \mathrm{i}c = -\frac{I_1 (I_2 - I_3)}{I_3 (I_1 - I_2)}, \quad \mathrm{cn}^2 \mathrm{i}c = \frac{I_2 (I_1 - I_3)}{I_3 (I_1 - I_2)}, \quad \mathrm{dn}^2 \mathrm{i}c = \frac{I (I_1 - I_3)}{I_3 (I_1 - I)}.$$

Taking into account the notation (15.1.52), we obtain (we can choose the sign of the argument $\mathrm{i}c$ so that to take the sign + in the right member)

$$\mathrm{i}\,\mathrm{sn}\,\mathrm{i}c\,\mathrm{cn}\,\mathrm{i}c\,\mathrm{dn}\,\mathrm{i}c = \frac{I\Omega}{pI_3} \frac{(I_1 - I_3)(I_2 - I_3)}{I_3 (I_1 - I_2)}.$$

It results (we denote $\tau = pt$)

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{I\Omega}{pI_3} + \frac{\mathrm{i}\,\mathrm{sn}\,\mathrm{i}c\,\mathrm{cn}\,\mathrm{i}c\,\mathrm{dn}\,\mathrm{i}c}{\mathrm{sn}^2 \mathrm{i}c - \mathrm{sn}^2 \tau}.$$

To can integrate this equation with separate variables, we introduce the theta functions $\vartheta_j = \vartheta_j (v, \chi)$, $j = 1,2,3,4$, as solutions of the partial differential equations

$$\frac{\partial^2 \vartheta_j}{\partial v^2} = 4\pi \frac{\partial \vartheta_j}{\partial \chi}, \quad j = 1,2,3,4, \tag{15.1.60}$$

in the form

$$\begin{aligned} \vartheta_1 (v, \chi) &= 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^{(2n-1)^2/4} \sin (2n-1) \pi v, \\ \vartheta_2 (v, \chi) &= 2 \sum_{n=1}^{\infty} q^{(2n-1)^2/4} \sin (2n-1) \pi v, \\ \vartheta_3 (v, \chi) &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2n\pi v, \\ \vartheta_4 (v, \chi) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n\pi v, \end{aligned} \tag{15.1.60'}$$

where $q = \mathrm{e}^{-\pi\chi}$, while $\chi = K'/K$, $K(k)$ being the complete elliptic integral of first kind (15.1.54) and $K'(k) = K(k')$, $k' = \sqrt{1 - k^2}$. We notice that $\chi = \chi(k)$, being thus constant for a fixed *k*; in this case, $\vartheta_i = \vartheta_i (v)$, $i = 1,2,3,4$. We introduce also the functions

$$\Theta(u) = \vartheta_4 (v), \quad H(u) = \vartheta_1 (v), \quad \Theta_1 (u) = \vartheta_3 (v), \quad H_1 (u) = \vartheta_2 (v), \quad v = u/2K; \tag{15.1.60''}$$

in this case, we have

$$\operatorname{sn} u = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}, \quad \operatorname{cn} u = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}, \quad \operatorname{dn} u = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}. \tag{15.1.60'''}$$

One can show that

$$\operatorname{sn}^2 u_1 - \operatorname{sn}^2 u_2 = \frac{\Theta^2(0)}{k} \frac{H(u_1 - u_2) H(u_1 + u_2)}{\Theta^2(u_1) \Theta^2(u_2)}.$$

Calculating the logarithmic derivative with respect to u_1, we can write

$$2 \frac{\operatorname{sn} u_1 \operatorname{cn} u_1 \operatorname{dn} u_1}{\operatorname{sn}^2 u_1 - \operatorname{sn}^2 u_2} = -\frac{2}{\Theta(u_1)} \frac{\mathrm{d}\Theta(u_1)}{\mathrm{d}u_1} + \frac{1}{H(u_1 - u_2)} \frac{\mathrm{d}H(u_1 - u_2)}{\mathrm{d}u_1} + \frac{1}{H(u_1 + u_2)} \frac{\mathrm{d}H(u_1 + u_2)}{\mathrm{d}u_1}.$$

Putting $u_1 = \mathrm{i}c$, $u_2 = \tau$, introducing the real constant

$$\lambda = \frac{I\Omega}{pI_3} - \frac{\mathrm{i}}{\Theta(\mathrm{i}c)} \frac{\mathrm{d}\Theta(\mathrm{i}c)}{\mathrm{d}u} \tag{15.1.61}$$

and integrating with respect to τ (we assume that the axes are chosen so that $\psi(0) = 0$), we obtain the angle of precession in the form ($\tau = p(t - t_0)$)

$$\psi(t) = \lambda p(t - t_0) + \frac{\mathrm{i}}{2} \ln \frac{H(p(t - t_0) + \mathrm{i}c)}{H(p(t - t_0) - \mathrm{i}c)}. \tag{15.1.61'}$$

The position of the rigid solid with respect to the inertial (fixed) frame of reference $\mathcal{R}'$ is thus entirely specified by Euler's angles ψ, θ and φ. The sines and the cosines of these angles are uniform functions of time or square roots of such functions; e.g.,

$$\sin\theta = \frac{v}{\Theta(pt)} \sqrt{H(p(t - t_0) + \mathrm{i}c) H(p(t - t_0) - \mathrm{i}c)}. \tag{15.1.61''}$$

These results have been obtained by O.I. Somov in a form appropriate to that above. Jacobi calculated the direction cosines of the axes of the non-inertial frame of reference $\mathcal{R}$ with respect to the inertial frame $\mathcal{R}'$ as uniform functions of time. Other results have been given by A. Cayley and A.G. Greenhill.

We calculate, after R. Grammel, the mean values of Euler's angles, as well as their variations around these values. Thus, the second relation (15.1.59'') can be written in the form (we denote $\beta = I_2^2 \beta_2^2 / I_1^2 \beta_1^2$)

$$\cot\varphi = -\sqrt{\frac{I_2(I_1 - I_3)}{I_1(I_1 - I_3)}} \frac{\operatorname{sn} p(t - t_0)}{\operatorname{cn} p(t - t_0)} = -\sqrt{\beta} \frac{\operatorname{sn} p(t - t_0)}{\operatorname{cn} p(t - t_0)} = -\sqrt{\frac{\beta}{k'}} \frac{H(p(t - t_0))}{H_1(p(t - t_0))},$$

so that

$$\cot\varphi = -\sqrt{\frac{\beta}{k'}}\,\frac{\sin\omega_\varphi(t-t_0) - q^2\sin 3\omega_\varphi(t-t_0) + q^6\sin 5\omega_\varphi(t-t_0) - \ldots}{\cos\omega_\varphi(t-t_0) + q^2\cos 3\omega_\varphi(t-t_0) + q^6\cos 5\omega_\varphi(t-t_0) + \ldots},$$

with $\omega_\varphi = \pi p/2K$; neglecting the powers of q, we obtain the approximate formula

$$\cot\varphi = -\sqrt{\frac{\beta}{k'}}\tan\omega_\varphi(t-t_0).$$

By differentiation, we get $(1+\cot^2\varphi)\dot{\varphi} = \omega_\varphi\sqrt{\beta/k'}\,[1+\tan^2\omega_\varphi(t-t_0)]$, wherefrom

$$\dot{\varphi} = 2\omega_\varphi\frac{\sqrt{k'\beta}}{k'+\beta}R(t),$$

$$R(t) = [1+\varepsilon\cos 2\omega_\varphi(t-t_0)]^{-1} = 1 + \sum_{n=1}^{\infty}(-1)^n\varepsilon^n\cos^n 2\omega_\varphi(t-t_0),$$

because $|\varepsilon| < 1$, $\varepsilon = (k'-\beta)/(k'+\beta)$; using the formulae

$$\cos^{2n-1}\alpha = 2^{2-n}\sum_{j=0}^{n-1}C_{2n-1}^{j}\cos[2(n-j)-1]\alpha,$$

$$\cos^{2n}\alpha = 2^{1-2n}\left[\frac{1}{2}C_{2n}^{n} + \sum_{j=0}^{n-1}C_{2n}^{j}\cos 2(n-j)\alpha\right],$$

where the symbol C_m^p represents *the number of combinations of* m *things* p *at a time*, we obtain

$$\int_{t_0}^{t}R(\tau)\mathrm{d}\tau = A_0(t-t_0) + \frac{1}{2\omega_\varphi}\sum_{n=1}^{\infty}(-1)^n A_n\sin 2n\omega_\varphi(t-t_0),$$

the coefficients of the expansion into a series being given by

$$A_0 = \sum_{\nu=0}^{\infty}\frac{1}{2^{2\nu}}C_{2\nu}^{\nu}\varepsilon^{2\nu}, \quad A_n = \frac{1}{2^{n-1}n}\sum_{\nu=0}^{\infty}\frac{1}{2^{2\nu}}C_{n+2\nu}^{\nu}\varepsilon^{n+2\nu}, \quad n > 0.$$

For the first coefficients it results

$$A_0 = \frac{1}{\sqrt{1-\varepsilon^2}} = \frac{k'+\beta}{2\sqrt{k'\beta}}, \quad A_1 = \frac{2}{\varepsilon}(A_0 - 1), \quad A_2 = \frac{1}{\varepsilon}A_1 - A_0,$$

so that one can use a recurrence formula for the other coefficients. Finally, the proper rotation is given by

$$\varphi(t) = \varphi_0 + \omega_\varphi(t-t_0) + \tilde{\varphi}(t), \quad \tilde{\varphi}(t) = \frac{\sqrt{k'\beta}}{k'+\beta}\sum_{n=1}^{\infty}(-1)^n A_n\sin 2n\omega_\varphi(t-t_0), \tag{15.1.62}$$

where ω_φ is *the mean value of the proper rotation angular velocity*, while $\tilde{\varphi}(t)$ represents the oscillation of period $2K/p$ around *the mean value* $\varphi_0 + \omega_\varphi (t - t_0)$.

Starting from (15.1.59'), we can write

$$\dot{\psi} = \frac{I\Omega}{2I_1 I_2}\left[I_1 + I_2 + (I_1 - I_2)\frac{1 - \tan^2\varphi}{1 + \tan^2\varphi} \right]$$
$$= \frac{I\Omega}{2I_1 I_2}\left[I_1 + I_2 - (I_1 - I_2)\frac{k'\cos^2\omega_\varphi (t - t_0) - \beta\sin^2\omega_\varphi (t - t_0)}{k'\cos^2\omega_\varphi (t - t_0) + \beta\sin^2\omega_\varphi (t - t_0)} \right]$$
$$= \frac{I\Omega}{2I_1 I_2}\left\{ I_1 + I_2 - (I_1 - I_2)\left[\frac{k' + \beta}{k' - \beta} - \frac{4k'\beta}{k'^2 - \beta^2} R(t) \right] \right\},$$

in the frame of the order of approximation considered above; by integration, we obtain the precession in the form

$$\begin{aligned} \psi(t) &= \psi_0 + \omega_\psi (t - t_0) + \tilde{\psi}(t), \\ \omega_\psi &= \frac{1}{\sqrt{k'} + \sqrt{\beta}}\left[\frac{I}{I_1}\sqrt{k'} + \frac{I}{I_2}\sqrt{\beta} \right]\Omega, \\ \tilde{\psi}(t) &= \frac{I(I_1 - I_2)}{I_1 I_2}\frac{k'\beta}{k'^2 - \beta^2}\frac{\Omega}{\omega_\varphi}\sum_{k=1}^{\infty}(-1)^n A_n \sin 2n\omega_\varphi (t - t_0), \end{aligned} \tag{15.1.62'}$$

where ω_ψ is *the mean value of the angular velocity of precession*, while $\tilde{\psi}(t)$ is the oscillation of period $2K/p$ around *the mean value* $\psi_0 + \omega_\psi (t - t_0)$.

Noting that $\sqrt{I_3 (I_1 - I)/I(I_1 - I_3)} = I_3\omega_3^0 / I\Omega = \cos\theta_0$, corresponding to the initial moment t_0, the first formula (15.1.59") and the last formula (15.1.60"') lead to

$$\gamma = \cos\theta = \sqrt{k'}\cos\theta_0 \frac{1 + 2q\cos 2\omega_\varphi (t - t_0)}{1 - 2q\cos 2\omega_\varphi (t - t_0)},$$

where we have retained only the first power of q in the expansion into series (15.1.60'), in conformity to the notations (15.1.60"). Expanding the above ratio into a power series and proceeding as in the preceding cases, we can represent *the nutation* in the form

$$\begin{aligned} \cos\theta(t) &= \gamma_0 + \tilde{\gamma}(t), \\ \gamma_0 &= (1 + 4q^2)\sqrt{k'}\cos\theta_0, \\ \tilde{\gamma}(t) &= 4q\sqrt{k'}\cos\theta_0 \left[3q^2\cos 2\omega_\varphi (t - t_0) + \sum_{n=1}^{\infty} q^{n-1}\cos 2n\omega_\varphi (t - t_0) \right], \end{aligned} \tag{15.1.62"}$$

where γ_0 is a *mean value*, while $\tilde{\gamma}(t)$ represents *the oscillation* of period $2K/p$ around this mean value.

We have seen in Sect. 14.1.1.3 that the position of the rigid solid with respect to the fixed frame of reference $\mathscr{R}'$ can be specified also by means of the Cayley-Klein parameters α, β, $\gamma = -\overline{\beta}$, $\delta = \overline{\alpha}$ (the upperlining indicates the complex conjugate); F. Klein showed that in the Euler-Poinsot case these parameters are elliptic functions of second kind which, both at the numerator and at the denominator, have only one theta function.

15.1.2.3 Geometric representation of the motion after Poinsot

Setting up the ellipsoid of inertia $\mathscr{E}$ (Poinsot's ellipsoid) at the point O, one obtains a geometric representation of the motion of the rigid solid with a fixed point; the position vector of a point P of the ellipsoid is situated along the angular velocity vector $\boldsymbol{\omega}$ at this point, being specified by the formula (15.1.15). The plane Π tangent to the ellipsoid at

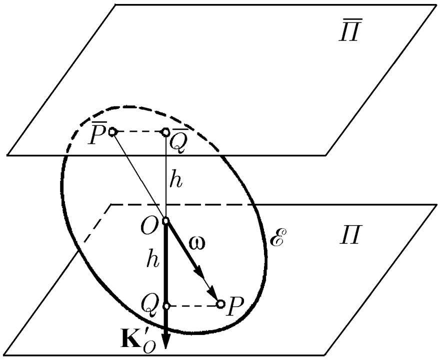


Fig. 15.7 The motion of rolling and pivoting without sliding of Poinsot's ellipsoid on one of Laplace's planes

the point P, called *pole*, is normal to the moment of momentum $\mathbf{K}'_O$ at the pole O. The distance $h = |\overrightarrow{OQ}|$ from the point O to this plane is given by (15.1.16), being constant in time (Fig. 15.7) (these results hold for any motion of the rigid solid with a fixed point; see Sect. 15.1.1.2 too). But, in the considered Euler-Poinsot case, the vector $\mathbf{K}'_O$ is constant in time with respect to the fixed frame of reference $\mathscr{R}'$, so that the plane Π is of constant normal, being situated at a constant distance from the point O; hence, the plane Π is fixed with respect to the frame $\mathscr{R}'$. The instantaneous rotation axis pierces Poinsot's ellipsoid at a second point $\overline{P}$ and the support of $\mathbf{K}'_O$ is normal also to the plane $\overline{\Pi}$ tangent at $\overline{P}$ to the ellipsoid. Projecting the constant vector $\mathbf{K}'_O$ on the normal to a given plane, one obtains a constant; the planes parallel to this plane are called *Laplace planes*. The planes Π and $\overline{\Pi}$ belong to a family of invariable planes of Laplace. The point P of contact is on the instantaneous axis of rotation, having thus a null velocity; we can state

Theorem 15.1.9 (*Poinsot*) *In the Euler-Poinsot case, the rigid solid with a fixed point is moving so that Poinsot's ellipsoid* $\mathscr{E}$ *corresponding to the fixed point has a slidingless rolling and pivoting motion on one of Laplace's planes. The magnitude* ω *of the rotation instantaneous angular velocity is in direct proportion to the magnitude of the position vector of the point* P *at which the instantaneous axis of rotation pierces the ellipsoid.*

We can express the distance h also in the remarkable form

$$h = \frac{K}{\sqrt{I}} = \frac{K}{J_0}, \tag{15.1.16'}$$

which allows an interesting interpretation of the constant I; as well, we have

$$\overrightarrow{OP} = h\sqrt{\frac{I}{2T'}}\boldsymbol{\omega} = \frac{h}{\Omega}\boldsymbol{\omega} = \frac{h}{\cos(\mathbf{K}'_O, \boldsymbol{\omega})}\operatorname{vers}\boldsymbol{\omega}. \tag{15.1.15'}$$

Because $|\overrightarrow{OP}|$ is contained between the semi-minor axis $K/\sqrt{I_1}$ and the semi-major axis $K/\sqrt{I_3}$, we get $I_3 < I_\Delta = I\cos^2(\mathbf{K}'_O, \boldsymbol{\omega}) < I_1$, justifying once more the inequality $I \geq I_3$. On the other hand, $h \leq |\overrightarrow{OP}|$ and $h \geq K/\sqrt{I_1}$ (because the point Q is exterior to the ellipsoid or at the most on it), so that $K/\sqrt{I_1} \leq K/\sqrt{I} \leq K/\sqrt{I_3}$; finally, it results $I_3 \leq I \leq I_1$.

We have used, in the above exposition, the ellipsoid of inertia $\mathscr{E}$, represented with respect to the principal axes of inertia taken as axes of the non-inertial frame of reference $\mathscr{R}$ in the form

$$I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2 = K^2 = Ih^2, \tag{15.1.63}$$

where $K > 0$ is a constant which specifies the units (K has the dimensional equation $[K] = M^{1/2}L^2$) and is conveniently determined (see Chap. 3, Sect. 1.2.6 too). Giving various values to the constant K, one can use different ellipsoids of Poinsot; for instance, one can take $K = 1$ (in this case, the unit has dimension), the respective ellipsoid being obtained from (15.1.63) by a similitude with the ratio $\sqrt{K}$. One can use also the second equation (15.1.47), with $K^2 = I\Omega^2$; in this case, the position vector of the pole P is just $\boldsymbol{\omega}$. The equation of the tangent plane to the ellipsoid $\mathscr{E}$ at the pole P $(K\omega_i/\sqrt{2T'}, i = 1,2,3)$ is given by

$$I_1\omega_1 x_1 + I_2\omega_2 x_2 + I_3\omega_3 x_3 = K\sqrt{2T'} = K\Omega\sqrt{I} = I\Omega h, \tag{15.1.63'}$$

where x_1, x_2, x_3 are the co-ordinates of a point of the Π-plane.

15.1.2.4 Geometric representation of the motion after Mac Cullagh

Starting from the ellipsoid of gyration $\mathscr{E}'$ (the ellipsoid reciprocal to the ellipsoid of inertia), introduced in Chap. 3, Sect. 1.2.6 and used in Sect. 14.1.1.6, J. Mac Cullagh succeeded to give, in 1840, another intuitive image of the motion of the rigid solid with a fixed point. We will use the equation (3.1.105) of the ellipsoid of gyration, with respect to the principal axes of inertia, in the movable frame of reference $\mathscr{R}$, in the form

$$\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3} = \frac{1}{M}, \tag{15.1.64}$$

where we take $K = R^2\sqrt{M}$ (a point $P' \in \mathscr{E}'$ is the inverse of the point $Q \in \Pi$ from the preceding subsection, with respect to the sphere (O, R)); we put thus in evidence the principal moments of inertia, which is more convenient in the following calculations.

Noting that $K'_{O1} = I_1\omega_1$, $K'_{O2} = I_2\omega_2$, $K'_{O3} = I_3\omega_3$, the equations of the support $\mathscr{D}$ of the vector $\mathbf{K}'_O$, fixed with respect to the frame of reference $\mathscr{R}'$, are

$$\frac{x_1}{I_1\omega_1} = \frac{x_2}{I_2\omega_2} = \frac{x_3}{I_3\omega_3} = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{\sqrt{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2}}. \tag{15.1.65}$$

Assuming that x_i^K, $i = 1,2,3$, are the co-ordinates of one of the points P'_K, $\bar{P}'_K \equiv \mathscr{D} \cap \mathscr{E}'$ and eliminating these co-ordinates between the equations (15.1.64), (15.1.65), we obtain

$$\frac{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2}{I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2}\overline{OP'_K}^2 = \frac{1}{M},$$

wherefrom

$$|\overrightarrow{OP'_K}| = \sqrt{\frac{I}{M}} = \frac{J_O}{\sqrt{M}} = i, \tag{15.1.65'}$$

i being a constant quantity of the nature of a radius of gyration, corresponding to the constant quantity I. We notice that $|\overrightarrow{OP'_K}|$ is contained between the semi-minor axis and the semi-major axis of the ellipsoid $\mathscr{E}'$, so that $i_3 \le i \le i_1$ (so as $I_3 \le I \le I_1$), justifying thus once more both inequalities established before. Hence, the points P'_K and $\bar{P}'_K$, situated on the invariable axis $\mathscr{D}$ and diametrically opposed in the ellipsoid of gyration $\mathscr{E}'$, are fixed with respect to the frame of reference $\mathscr{R}'$ ($|\overrightarrow{OP'_K}| = |\overrightarrow{O\bar{P}'_K}| = \text{const}$) (Fig. 15.8); to fix the ideas, the position vector of the point P'_K will be $J_O/\sqrt{M}$, of components $x_1^K = J_O I_1\omega_1 / K'_O\sqrt{M}$,

$x_2^K = J_O I_2 \omega_2 / K_O' \sqrt{M}$, $x_3^K = J_O I_3 \omega_3 / K_O' \sqrt{M}$. The equation of the plane Π', tangent to the ellipsoid $\mathscr{E}'$ at the point P_K', is given by

$$\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \frac{K_O'}{J_O \sqrt{M}}, \tag{15.1.64'}$$

the normal OQ' to this plane having the direction parameters ω_i, $i = 1,2,3$; hence, the angular velocity vector $\boldsymbol{\omega}$ is normal to this plane. Moreover, the distance h' from the point O to this plane is given by

$$h' = \frac{K_O'}{J_O \sqrt{M} \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}} = \frac{K_O'}{J_O \sqrt{M} \omega}. \tag{15.1.64''}$$

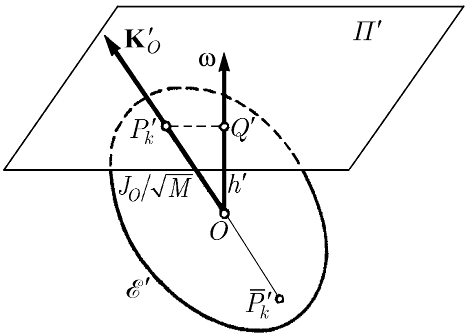


Fig. 15.8 The inertial motion of the rigid solid with a fixed point in Mac Cullagh's geometric representation

Hence, we can state

Theorem 15.1.10 (*J. Mac Cullagh*) *The inertial motion of a rigid solid about a fixed point O of it, in the Euler-Poinsot case, takes place so that the ellipsoid of gyration $\mathscr{E}'$ corresponding to this point passes through the fixed points P_K' and $\overline{P}_K'$, situated on the invariable axis $\mathscr{D}$. The rotation angular velocity $\boldsymbol{\omega}$ is normal to the plane tangent to the ellipsoid of gyration $\mathscr{E}'$ at one of the fixed points, while its magnitude ω is in inverse proportion to the distance h' from the fixed point O to the plane Π'.*

15.1.2.5 The Polhode

The locus of the point P at which the instantaneous axis of rotation pierces the ellipsoid of inertia $\mathscr{E}$ (rigidly linked to the rigid solid $\mathscr{S}$) is a curve $\mathscr{P}$ called *polhode*, intersection of the ellipsoid with the polhodic cone $\mathscr{C}_p$ (the locus of the instantaneous axes of rotation with respect to the frame of reference $\mathscr{R}$), hence a directrix of this cone. Analogously, the locus of the point P on the fixed plane Π, tangent to the ellipsoid $\mathscr{E}$ at this point, is a curve $\mathscr{H}$, called *herpolhode*, intersection of the plane Π with the herpolhodic cone $\mathscr{C}_h$ (the locus of the instantaneous axes of rotation with respect to the frame $\mathscr{R}'$) (in Fig. 15.9 we represent the polhode and the herpolhode corresponding to the point P and the plane Π; for the point P' and the plane Π' one obtains analogous results). Obviously, the two curves are tangent one to the other, measuring equal lengths between corresponding points, because of the slidingless rolling of the polhodic cone over the herpolhodic one (and of the polhode over the herpolhode).

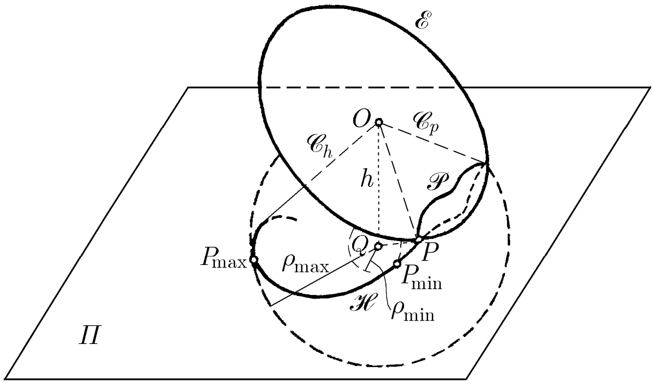


Fig. 15.9 The polhode $\mathscr{P}$ and the herpolhode $\mathscr{H}$ in the motion of a rigid solid with a fixed point

If we eliminate Ω^2 between the first integrals (15.1.47), then we find

$$I_1(I_1 - I)\omega_1^2 + I_2(I_2 - I)\omega_2^2 + I_3(I_3 - I)\omega_3^2 = 0\,. \tag{15.1.66}$$

Assuming that $I_1 > I_2 > I_3$, one cannot have $I > I_1$ or $I < I_3$, because the left member would be strictly negative or strictly positive, the cone being imaginary; hence, $I_3 \leq I \leq I_1$, result previously obtained. Taking into account (15.1.15'), the equation of the instantaneous axis is written in the form $x_1/\omega_1 = x_2/\omega_2 = x_3/\omega_3 = h/\Omega$ in the frame of reference $\mathscr{R}$, the equation of the polhodic cone (in the same frame) being

$$I_1(I_1 - I)x_1^2 + I_2(I_2 - I)x_2^2 + I_3(I_3 - I)x_3^2 = 0\,. \tag{15.1.66'}$$

Because $P \equiv \mathscr{E} \cap \mathscr{C}_p$, the polhode will have the equations (15.1.63), (15.1.66); hence, this is an algebraic curve of the fourth degree with two distinct closed branches and with properties of central symmetry (with respect to the co-ordinate planes too), the polhodic cone having the same axes of symmetry as the ellipsoid $\mathscr{E}$. Multiplying the equation (15.1.63) by I and summing with the equation (15.1.66'), we obtain the equation of an ellipsoid $\mathscr{E}_k$, called *kinetic ellipsoid* (due to the signification of the first integral (15.1.47)),

$$I_1^2 x_1^2 + I_2^2 x_2^2 + I_3^2 x_3^2 = I^2 h^2 = IK^2 . \tag{15.1.67}$$

The ellipsoid $\mathscr{E}_k$ is coaxial with the ellipsoid $\mathscr{E}$, the two ellipsoids having as intersection the polhodic curve.

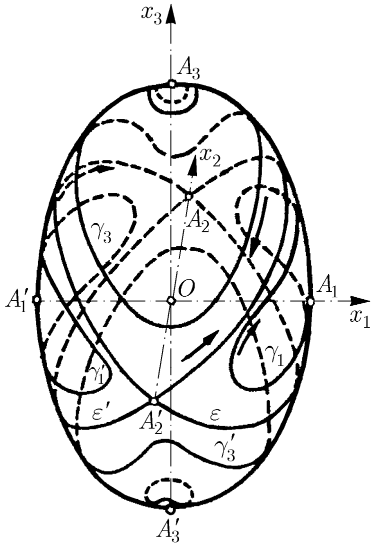


Fig. 15.10 Polhodes drawn on the ellipsoid of inertia $\mathscr{E}$

Concerning the polhodic cone, we notice that the first coefficient of the equation (15.1.66) is always positive, while the last one is always negative; in what concerns the second coefficient, this one is positive or negative as $I < I_2$ or $I > I_2$, respectively, depending thus on the initial conditions. In the first case, $h > K\sqrt{I_2}$ and the polhodic cone contains the Ox_3-axis; in the second case, $h < K\sqrt{I_2}$ and the polhodic cone contains the Ox_1-axis. If $I = I_2$, hence if $h = K/\sqrt{I_2}$, then the polhodic cone degenerates in two planes (we use the notations (14.1.49') and notice that $\beta_2 = \overline{\beta}_2 = \Omega$)

$$x_1 = \pm x_3 \frac{\sqrt{I_3 (I_2 - I_3)}}{\sqrt{I_1 (I_1 - I_2)}} = \pm \frac{\overline{\beta}_1}{\beta_3} x_3 , \tag{15.1.68}$$

which pass through the mean axis (the axis $A_2 A_2'$) of the ellipsoid $\mathscr{E}$, the corresponding polhode being constituted of two ellipses (ε and ε') for which this axis is a common one (Fig. 15.10) (the only case in which the two branches of the polhode have common points). If $I = I_1$ or $I = I_3$ we have $x_2 = x_3 = 0$ or $x_1 = x_2 = 0$, respectively, the polhode being reduced to the points A_1 and A_1' (extremities of the minor axis of the ellipsoid) or to the points A_3 and A_3' (extremities of the major axis of the ellipsoid), respectively. If $I_2 < I < I_1$, then the polhode is formed of two closed curves γ_1 and

γ_1', which surround the points A_1 and A_1', respectively; as well, if $I_3 < I < I_2$, then the polhode is formed of two closed curves γ_3 and γ_3', which surround the points A_3 and A_3', respectively (Fig. 15.10). In the hypothesis considered in Sect. 15.1.2.1 ($\omega_2(t_0) = \omega_2^0 > 0$, $\omega_1(t_0) < 0$, $\omega_3(t_0) > 0$), the pole P travels through the polhodes in the sense indicated in Fig. 15.10 (on the curves on which the sense has not been indicated, that one is obtained by symmetry). To have a clearer image of the polhode, we consider also its projection on the three planes of co-ordinates, in the frame of reference $\mathscr{R}$. Thus, projecting on the plane Ox_1x_2 (we eliminate x_3 between the equations (15.1.63) and (15.1.66')) for various values of I (hence, for various initial conditions), we obtain a family of coaxial ellipses of equations (cylinders of elliptic section which pierce the ellipsoid of inertia after polhodes)

$$I_1(I_1 - I_3)x_1^2 + I_2(I_2 - I_3)x_3^2 = I(I - I_3)h^2.$$

Taking into account the notations (15.1.49), (15.1.49'), we can write these equations also in the form

$$\frac{x_1^2}{\beta_1^2} + \frac{x_2^2}{\beta_2^2} = \frac{h^2}{\Omega^2} = \frac{K^2}{I\Omega^2}. \tag{15.1.68'}$$

Moreover, starting from the first equation (15.1.49") and using the relation (15.1.15'), we find again these equations, the respective ellipses being equivalent. We notice that for $I > I_2$ one obtains arcs of ellipse, bounded by the ellipse of inertia $I_1x_1^2 + I_2x_2^2 = K^2$, while for $I \leq I_2$ there result complete ellipses (Fig. 15.11a). Projecting the polhode on the plane Ox_2x_3 (we eliminate x_1 between the equations (15.1.63) and (15.1.66')) and using the same notations, we get – analogously – a family of coaxial ellipses (cylinders of elliptic section, which pierce the ellipsoid of inertia after polhodes)

$$\frac{x_2^2}{\beta_2^2} + \frac{x_3^2}{\beta_3^2} = \frac{h^2}{\Omega^2} = \frac{K^2}{I\Omega^2}. \tag{15.1.68''}$$

Taking into account the relation (15.1.15'), one observes that these ellipses are equivalent with the second ellipse (15.1.49"). For $I < I_2$ there result arcs of ellipse, bounded by the ellipse of inertia $I_2x_2^2 + I_3x_3^2 = K^2$, while for $I \geq I_2$ there result complete ellipses (Fig. 15.11b). Finally, the projection of the polhode on the plane Ox_3x_1 (one eliminates x_2 between the equations (15.1.63) and (15.1.66')), with the same notations, leads to two conjugate coaxial hyperbolae (cylinders of hyperbolic section, the traces of which on the ellipsoid of inertia are the polhodes)

$$\frac{x_3^2}{\beta_3^2} - \frac{x_1^2}{\beta_1^2} = \frac{h^2}{\Omega^2} = \frac{K^2}{I\Omega^2}, \tag{15.1.68'''}$$

which degenerate in two straight lines, specified by the equations (15.1.68), for $I = I_2$. If $I < I_2$, then the polhode is formed of two arcs of hyperbola, the real axis of which is the axis $A_1 A_1'$ (Fig. 15.11c). Taking into account the relation (15.1.15'), one observes that these hyperbolae are equivalent to the hyperbola (15.1.49''').

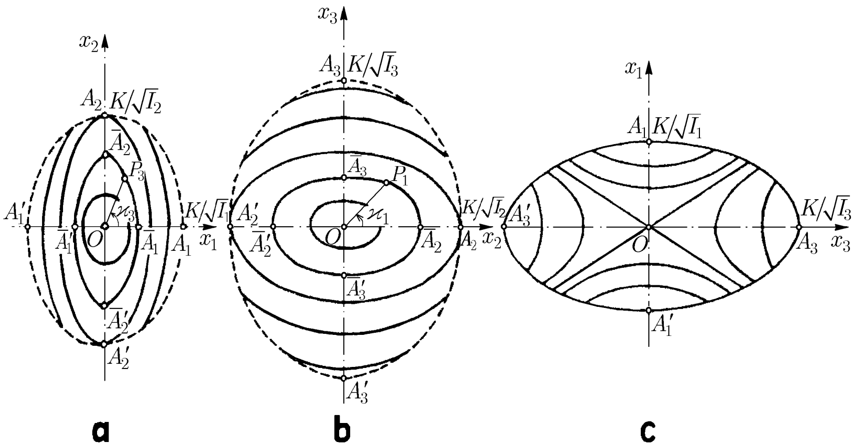


Fig. 15.11 Projections of a polhode on the planes Ox_1x_2 (**a**), Ox_2x_3 (**b**) and Ox_3x_1 (**c**)

Concluding, there are two families of polhodes, separated by the singular polhode corresponding to the case $I = I_2$ (ε and ε'); a family of polhodes (γ_1 and γ_1') surround the extremities of the minor axis of the ellipsoid of inertia, while the other family of polhodes (γ_3 and γ_3') surround the extremities of the major axis of this ellipsoid. Through each point of the ellipsoid passes a polhode and only one. The polhodes being plotted on the ellipsoid, to can determine that one which corresponds to given initial conditions it is sufficient to know the piercing point P_0 of the support of the angular velocity vector $\boldsymbol{\omega}$ on the ellipsoid of inertia at the initial moment; the searched polhode is that which passes through P_0, while the plane tangent to the ellipsoid at P_0 is the plane Π.

15.1.2.6 The Herpolhode

Studies to determine the herpolhode have been made by W. Hess, Sparre, G. Darboux, A.G. Greenhill, G. Halphen, J.N. Franke, A. Mannheim, A. de St. Germain, A.H. Résal, P. Barbarin, E. Lacour, A. Petrus etc. In what follows, we use polar co-ordinates in the plane Π, taking as pole of reference the point Q, projection of the fixed point O on the fixed plane Π (Fig. 15.9). The vector radius is given by

$$\overline{QP} = \rho = \sqrt{\overline{OP}^2 - h^2} = \frac{h}{\Omega}\sqrt{\omega^2 - \Omega^2} = \frac{K}{\sqrt{I}\Omega}\sqrt{\omega^2 - \Omega^2}\,. \tag{15.1.69}$$

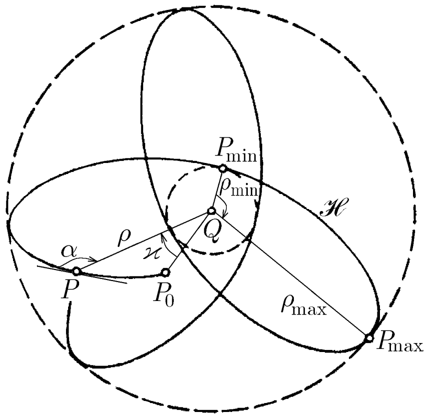


Fig. 15.12 The herpolhode drawn in a circular annulus

Taking into account that $K/\sqrt{I_1} \le |\overrightarrow{OP}| \le K/\sqrt{I_3}$, it results that ρ is inferior and superior bounded, so that the herpolhode is a curve contained in a circular annulus $\mathscr{C}$ ($\rho_{\mathrm{min}} \le \rho \le \rho_{\mathrm{max}}$) (Fig. 15.12). We get

$$\omega^2 = \Omega^2 + I\Omega^2 \frac{\rho^2}{K^2} \tag{15.1.69'}$$

from (15.1.69), while by replacing in (15.1.57') we may write

$$\frac{\mathrm{d}(\rho^2)}{\mathrm{d}t} = \pm 2\frac{\Omega\sqrt{I}}{K}\sqrt{[\rho^2 - \delta_1^2 \operatorname{sign}(I_2 - I)](\delta_2^2 - \rho^2)[\rho^2 - \delta_3^2 \operatorname{sign}(I - I_2)]}\,, \tag{15.1.70}$$

where we have introduced the notations

$$\delta_1^2 = \frac{|I_2 - I|(I - I_3)}{I_2 I_3 I}K^2\,,\ \delta_2^2 = \frac{(I - I_3)(I_1 - I)}{I_3 I_1 I}K^2\,,\ \delta_3^2 = \frac{(I_1 - I)|I_2 - I|}{I_1 I_2 I}\,. \tag{15.1.70'}$$

If $I < I_2$, then we have $\rho_{\mathrm{min}} = \delta_1$, while if $I > I_2$, then we have $\rho_{\mathrm{min}} = \delta_3$; in both cases, $\rho_{\mathrm{max}} = \delta_2$. Integrating the differential equation (15.1.70), one can express the vector radius ρ by means of elliptic functions.

We get the same result noting that $\rho^2 = \overline{OP}^2 - h^2 = x_1^2 + x_2^2 + x_3^2 - h^2$; using the equations of the cylinders (15.1.68')–(15.1.68''') and eliminating the variables x_2 and x_3 or the variables x_1 and x_2, we obtain

$$\rho^2 = \frac{1}{I_2 I_3}\left[(I_1 - I_2)(I_1 - I_3)x_1^2 + (I_2 - I)(I - I_3)h^2\right]$$
$$= \frac{1}{I_1 I_2}\left[(I_1 - I_3)(I_2 - I_3)x_3^2 + (I_1 - I)(I - I_2)h^2\right].$$

Considering the ellipse $\bar{A}_1\bar{A}_2\bar{A}_1'\bar{A}_2'$ (Fig. 15.11a), which is entirely travelled through for $I < I_2$, we see that $\rho_{\min}$ corresponds to the points $\bar{A}_2$ and $\bar{A}_2'$ $(x_1 = 0)$, while $\rho_{\max}$ to the points $\bar{A}_1$ and $\bar{A}_1'$ $(x_1 = h^2 I(I - I_3)/I_1(I_1 - I_3))$; as well, noting that the ellipse $\bar{A}_2\bar{A}_3\bar{A}_2'\bar{A}_3'$ (Fig. 15.11b) is entirely travelled through for $I > I_2$, we find that $\rho_{\min}$ corresponds to the points $\bar{A}_2$ and $\bar{A}_2'$ $(x_3 = 0)$, while $\rho_{\max}$ to the points $\bar{A}_3$ and $\bar{A}_3'$ $(x_3 = h^2 I(I_1 - I)/I_3(I_1 - I_3))$. Hence, the arc of helpolhode $\overset{\frown}{P_{\min}QP_{\max}}$ corresponds to a quarter of the ellipse $\bar{A}_1\bar{A}_2\bar{A}_1'\bar{A}_2'$ or of the ellipse $\bar{A}_2\bar{A}_3\bar{A}_2'\bar{A}_3'$, as $I < I_2$ or $I > I_2$, respectively. To a complete travelling through of the polhode $\mathscr{P}$ by the point P corresponds on the herpolhode $\mathscr{H}$ an arc of curve of length $4\overset{\frown}{P_{\min}QP_{\max}}$ (between corresponding points one has equal lengths on the curves $\mathscr{P}$ and $\mathscr{H}$), while the angle described by the radius vector QP is $4\overset{\frown}{P_{\min}QP_{\max}}$. If the measure in radians of the angle $\overset{\frown}{P_{\min}QP_{\max}}$ is not commensurable with π, then *the herpolhode is an open curve*; the pole P does not take again the same position, at the same moment, on the ellipsoid $\mathscr{E}$ and on the plane Π. If, in particular, the measure in radians of the angle is commensurable with π, then *the herpolhode is a closed curve*.

Taking into account the formula (5.1.16') (see Chap. 5, Sect. 1.1.4 too), we can state that the double of the areal velocity of the point P in the plane Π is equal to the projection on the fixed direction of the moment of momentum $\mathbf{K}_O'$ of the moment with respect to the fixed point O of the velocity of the point P (calculated with respect to the same point O), that is the moment of this velocity with respect to the OQ-axis, given by (we take into account the formula (15.1.15') and the notations previously introduced)

$$\frac{\mathbf{K}_O'}{K_O'} \cdot \left[\left(\frac{h}{\Omega}\boldsymbol{\omega}\right) \times \left(\frac{h}{\Omega}\dot{\boldsymbol{\omega}}\right)\right] = \frac{h^2}{\Omega^2 K_O'}(\mathbf{K}_O', \boldsymbol{\omega}, \dot{\boldsymbol{\omega}}) = \frac{K^2}{I^2\Omega^3}\begin{vmatrix} I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \dot{\omega}_1 & \dot{\omega}_2 & \dot{\omega}_3 \end{vmatrix}.$$

Consequently, we can write

$$\rho^2 \dot{\kappa} = \frac{K^2}{I^2 \Omega^3} \begin{vmatrix} I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ [(I_2 - I_3)/I_1]\omega_2\omega_3 & [(I_3 - I_1)/I_2]\omega_3\omega_1 & [(I_1 - I_2)/I_3]\omega_1\omega_2 \end{vmatrix}$$

in polar co-ordinates ρ, κ, where we have used Euler's equations (15.1.40). Developing the determinant after the last line, it results

$$\rho^2 \dot{\kappa} = \frac{K^2}{I^2 \Omega^3} \left[\frac{(I_2 - I_3)^2}{I_1} \omega_2^2 \omega_3^2 + \frac{(I_3 - I_1)^2}{I_2} \omega_3^2 \omega_1^2 + \frac{(I_1 - I_2)^2}{I_3} \omega_1^2 \omega_2^2 \right].$$

Taking into account (15.1.56), we can write

$$\begin{aligned} \rho^2 \dot{\kappa} = \frac{I_1 I_2 I_3}{I^2 \Omega^3} K^2 \bigg[& \frac{1}{(I_1 - I_2)(I_1 - I_3)} (\gamma_2^2 - \omega^2)(\omega^2 - \gamma_3^2) \\ & + \frac{1}{(I_2 - I_3)(I_1 - I_2)} (\omega^2 - \gamma_3^2)(\omega^2 - \gamma_1^2) \\ & + \frac{1}{(I_1 - I_3)(I_2 - I_3)} (\omega^2 - \gamma_1^2)(\gamma_2^2 - \omega^2) \bigg], \end{aligned}$$

while, by means of the relation (15.1.69'), we have

$$\begin{aligned} \rho^2 \dot{\kappa} = & \frac{I_1 I_2 I_3 \Omega}{K^2 (I_2 - I_3)(I_1 - I_3)(I_1 - I_2)} \{ (I_2 - I_3)(\delta_2^2 - \rho^2)[\rho^2 - \delta_3^2 \operatorname{sign}(I - I_2)] \\ & + (I_1 - I_3)[\rho^2 - \delta_3^2 \operatorname{sign}(I - I_2)][\rho^2 - \delta_1^2 \operatorname{sign}(I_2 - I)] \\ & + (I_1 - I_2)[\rho^2 - \delta_1^2 \operatorname{sign}(I_2 - I)](\delta_2^2 - \rho^2) \}. \end{aligned}$$

Effecting the calculations, we obtain the differential equation

$$\rho^2 \dot{\kappa} = \Omega [\rho^2 + \delta^2 \operatorname{sign}(I - I_2)], \tag{15.1.71}$$

where

$$\delta^2 = \frac{(I_1 - I)(I - I_3)|I - I_2|}{I_1 I_2 I_3 I} K^2, \tag{15.1.71'}$$

which determines the angle κ. We notice that $\delta^2 \operatorname{sign}(I - I_2) > 0$ for $I > I_2$; if $I < I_2$ one finds $\rho_{\min}^2 = \delta_1^2 > \delta^2$. Hence, $\dot{\kappa} > 0$, the angle κ being thus increasing in time; *the herpolhode surrounds* thus *the point Q always in the same sense, attaining successively the internal and the external circles of the annulus* $\mathscr{C}$.

Eliminating the time between the equations (15.1.70) and (15.1.71), we can write the differential equation of the herpolhode in the form

$$\frac{\sqrt{I}}{K}\mathrm{d}\kappa = \frac{1}{h}\mathrm{d}\kappa = \pm\frac{\left[\rho^2 + \delta^2 \operatorname{sign}(I - I_2)\right]\mathrm{d}\rho}{\rho\sqrt{\left[\rho^2 - \delta_1^2 \operatorname{sign}(I_2 - I)\right]\left(\delta_2^2 - \rho^2\right)\left[\rho^2 - \delta_3^2 \operatorname{sign}(I - I_2)\right]}}\,. \tag{15.1.72}$$

One observes thus that between the constants which intervene in this equation take place the relations

$$\delta^2 = \left(1 - \frac{I}{I_1}\right)\delta_1^2 = \left|1 - \frac{I}{I_2}\right|\delta_2^2 = \left(\frac{I}{I_3} - 1\right)\delta_3^2\,. \tag{15.1.72'}$$

The equation (15.1.72) is with separate variables and can be integrated by a quadrature with the aid of elliptic functions. The angle α made by the radius vector QP with the tangent to the herpolhode (Fig. 15.12) is given by $\tan\alpha = \rho\mathrm{d}\kappa/\mathrm{d}\rho$. We notice that for $\rho = \rho_{\min}$ or for $\rho = \rho_{\max}$, hence at the points $P_{\min}$ and $P_{\max}$, respectively, this tangent tends to infinity; hence, *the herpolhode* is tangent to the circles which form the annulus $\mathscr{C}$. The curvature at a point P is given by

$$\frac{2\mathrm{d}\kappa/\mathrm{d}\rho + \rho\mathrm{d}^2\kappa/\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\kappa/\mathrm{d}\rho)^3}{\left[1 + \rho^2(\mathrm{d}\kappa/\mathrm{d}\rho)^2\right]^{3/2}}\,,$$

in polar co-ordinates. We study the sign of the numerator of this expression or, which is equivalent, the sign of the function (we take into account that $\mathrm{d}\kappa/\mathrm{d}\rho \neq 0$)

$$f(\rho^2) = \frac{2}{\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\ln\frac{\mathrm{d}\kappa}{\mathrm{d}\rho} + \left(\frac{\mathrm{d}\kappa}{\mathrm{d}\rho}\right)^2,$$

because the denominator is always positive. Taking into account the equation (15.1.72), the notations (15.1.70'), (15.1.71') and the relations (15.1.72'), we can write

$$f(\rho^2) = \frac{2}{\rho^2 + \delta^2\operatorname{sign}(I - I_2)} + \frac{I_1(I_2 + I_3 - I_1)}{(I_1 - I_2)(I_1 - I_3)}\frac{1}{\rho^2 - \delta_1^2\operatorname{sign}(I_2 - I)}$$
$$+\frac{I_2(I_3 + I_1 - I_2)}{(I_2 - I_3)(I_1 - I_2)}\frac{1}{\delta_2^2 - \rho^2} + \frac{I_3(I_1 + I_2 - I_3)}{(I_1 - I_3)(I_2 - I_3)}\frac{1}{\rho^2 - \delta_3^2\operatorname{sign}(I - I_2)}\,.$$

By reduction to a common denominator, one finds that the numerator is a polynomial of second degree in ρ^2 (the coefficient of ρ^6 vanishes), which has two real zeros (we notice that $f(0) = 0$). Because $\rho_{\min}^2 < \rho^2 < \rho_{\max}^2$ (excepting the points of tangency with the circular annulus, for which we have equality), where $\rho_{\min}^2 = \delta_1^2$ for $I < I_2$

(we have $\delta^2 < \delta_1^2$ too, in conformity to the formula (15.1.72'), while $\operatorname{sign}(I - I_2) = -1$) and $\rho_{\min}^2 = \delta_3^2$ for $I > I_2$ (we have $\operatorname{sign}(I - I_2) = 1$ too), while $\rho_{\max}^2 = \delta_2^2$, it results that $f(\rho^2) > 0$ for any point of the herpolhode; in conclusion, *this curve has always the concavity directed towards the fixed point* Q (hence without points of inflection, as it has been stated by Poinsot in his memories).

15.1.2.7 Permanent Rotations

We notice that the system (15.1.40) has three obvious systems of particular solutions

$$\omega_i = \omega_i^0, \quad \omega_i^0 = \mathrm{const}, \quad \omega_j = \omega_k = 0, \quad i \neq j \neq k \neq i, \quad i,j,k = 1,2,3, \tag{15.1.73}$$

which must correspond to the initial conditions. If, e.g., we consider the solution $\omega_1 = \omega_1^0$, $\omega_2 = \omega_3 = 0$, this one must verify the system (15.1.40) at any moment t, hence also at the initial moment; thus, the initial conditions must be of the same form. In this case, the theorem of existence and uniqueness ensures that the motion of the rigid solid is a *uniform* (finite) *rotation* about the Ox_1-axis ($\boldsymbol{\omega} = \omega \mathbf{i}_1 = \overrightarrow{\mathrm{const}}$); because the derivative with respect to time of the vector $\boldsymbol{\omega}$ is the same in the two frames of reference ($\mathscr{R}'$ and $\mathscr{R}$), it results that the vector $\boldsymbol{\omega}$ is constant also with respect to the inertial frame (the direction of the Ox_1-axis remains constant with respect to this frame too). Analogously, $\boldsymbol{\omega} = \omega \mathbf{i}_2$ and $\boldsymbol{\omega} = \omega \mathbf{i}_3$, respectively, can represent uniform rotations about the corresponding axes if the initial conditions are compatible with these solutions. The respective axes of rotation are called *permanent axes of rotation*, the corresponding rotations being *permanent rotations*. This result corresponds to the Theorem 14.2.1, obtained as a particular case of motion of the rigid solid about a fixed axis.

We consider now the permanent rotations as limit cases of the general results obtained above, corresponding to the situation in which the inequalities concerning the position of the constant I with respect to the principal moments of inertia (not equal one to the others) become equalities. Thus, if $I = I_3$, then the relation (15.1.66) leads to $\omega_1(t) = \omega_2(t) = 0$, because $I \neq I_1$ and $I \neq I_2$, while the second relation (15.1.49) shows that $\omega_3(t) = \Omega = \mathrm{const}$; moreover, the relations (15.1.48') lead to the same result (the last relations can take place only if $\omega_2 = 0$, while the first relations lead to $\omega_1 = 0$). We are thus in the case (15.1.73) for $i = 3$, $j = 1$ and $k = 2$. As we have seen, the Ox_3-axis is fixed with respect to the rigid solid (hence with respect to the frame of reference $\mathscr{R}$ too) and with respect to the fixed frame $\mathscr{R}'$. From (15.1.58) it results $\sin\theta \sin\varphi = \sin\theta \cos\varphi = 0$, $\cos\theta = 1$, hence $\theta = 0$; the angle φ, as well as the angle ψ given by (15.1.59'), will be arbitrary (the planes $Ox_1'x_2'$ and Ox_1x_2 are superposed). Thus, the Ox_3-axis coincides with the Ox_3'-axis, the support of the moment of momentum $\mathbf{K}_O'$; the formula (15.1.46) leads to the same result if we make $\omega = \omega_3 = \Omega$, noting that $\boldsymbol{\omega}$ and $\mathbf{K}_O'$ have the same direction (including the same sense). We can choose the angle ψ arbitrarily ($\psi(t) = \text{const}$, eventually $\psi = 0$), the angle $\varphi(t) = \Omega(t - t_0) + \varphi_0$ corresponding to a uniform rotation about the fixed axis (Fig. 15.13a). We have seen that, in this case, the polhode is reduced to the points A_3 and A_3' at which the polhodic cone, degenerated into a straight line along the major axis of the ellipsoid of inertia pierces this ellipsoid. Because we are in the case $I < I_2$, it results that $\rho_{\min} = \delta_1 = 0$, $\rho_{\max} = \delta_2 = 0$, so that the herpolhode is reduced to the point A_3 (or to the point A_3'), the plane Π being tangent to the ellipsoid at this point.

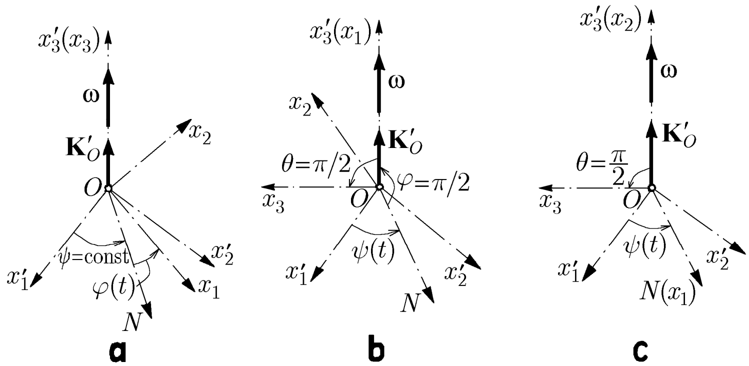


Fig. 15.13 Permanent rotations about the fixed axis Ox_3' if it coincides with the Ox_3 -axis (**a**), the Ox_1 -axis (**b**) or the Ox_2 -axis (**c**)

If $I = I_1$, then the relation (15.1.66) leads to $\omega_2(t) = \omega_3(t) = 0$, because $I \neq I_2$ and $I \neq I_3$; we obtain, analogously, $\omega_1(t) = \Omega = \text{const}$. We find thus again the case (15.1.73) for $i = 1$, $j = 2$, $k = 3$. The Ox_1 -axis coincides with the fixed axis Ox_3', because $\sin\theta\sin\varphi = 1$, $\sin\theta\cos\varphi = \cos\theta = 0$ (so that $\theta = \varphi = \pi/2$), the angular velocity vector $\boldsymbol{\omega}$ being along the moment of momentum $\mathbf{K}_O'$ too (in direction and, obviously, sense). Using the relation (15.1.59'), we notice that the uniform rotation about the fixed axis specified by the angle of precession $\psi(t) = \Omega(t - t_0) + \psi_0$, corresponding to the initial conditions too (Fig. 15.13b). The polhodic cone is reduced to a straight line along the minor axis of the ellipsoid of inertia and pierces this ellipsoid at the extremities A_1 and A_1' of the respective axis; the polhode is thus formed by the points A_1 and A_1'. Being in the case $I > I_2$, we have $\rho_{\min} = \rho_3 = 0$, $\rho_{\max} = \rho_2 = 0$, the herpolhode being reduced to the point A_1 (or to the point A_1') and the plane Π being tangent to the ellipsoid at this point.

Finally, if $I = I_2$, then the notations (15.1.56') lead to $\gamma_1^2 = \gamma_3^2 = \Omega^2$, $\gamma_2^2 = I_2\Omega^2(I_3 + I_1 - I_2)/I_3 I_1$, while the equation (15.1.57') becomes

$$\frac{\mathrm{d}(\omega^2)}{\mathrm{d}t} = \pm 2(\omega^2 - \Omega^2)\sqrt{\gamma_2^2 - \Omega^2}\,,$$

so that $\varOmega^2 \leq \omega^2 \leq \gamma_2^2$. We denote $\omega^2 = \varOmega^2 \cos^2 \chi + \gamma_2^2 \sin^2 \chi$, $0 \leq \chi \leq \pi$ (as a matter of fact, $\chi_0 \leq \chi \leq \pi$, $\chi_0 \geq 0$); it results (we have taken the sign + before the radical, corresponding to an increasing of ω^2)

$$\frac{\mathrm{d}\chi}{\sin \chi} = \lambda \mathrm{d}t, \quad \lambda = \sqrt{\gamma_2^2 - \varOmega^2} = \sqrt{\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}}\varOmega .$$

By integration, we have

$$\ln \tan \frac{\chi}{2} = \lambda(t + \tau), \quad \tau = \frac{1}{\lambda} \ln \tan \frac{\chi_0}{2} - t_0 ,$$

wherefrom

$$\tan \frac{\chi}{2} = \mathrm{e}^{\lambda(t+\tau)}, \quad \omega^2 = \gamma_2^2 - \left(\gamma_2^2 - \varOmega^2\right) \tanh^2 \lambda(t + \tau). \tag{15.1.74}$$

Then, the relations (15.1.56) lead to

$$\omega_1^2 = \frac{\beta_1^2}{\cosh^2 \lambda(t + \tau)}, \quad \omega_2^2 = \varOmega^2 \tanh^2 \lambda(t + \tau), \quad \omega_3^2 = \frac{\beta_3^2}{\cosh^2 \lambda(t + \tau)}, \tag{15.1.74'}$$

where we took into account (15.1.49') and we noticed that $\beta_2 = \overline{\beta}_2 = \varOmega$. In the hypothesis in Sect. 15.1.2.1 ($\omega_2(t_0) > 0$, $\omega_1(t_0) < 0$, $\omega_3(t_0) > 0$) we have

$$\omega_1(t) = -\frac{\beta_1}{\cosh \lambda(t + \tau)}, \quad \omega_2(t) = \varOmega \tanh \lambda(t + \tau), \quad \omega_3(t) = \frac{\beta_3}{\cosh \lambda(t + \tau)}, \tag{15.1.74''}$$

observing that the pole P travels through an arc of ellipse in an infinite time ($\lim\limits_{t \to \infty} \omega_1 = 0 - 0$, $\lim\limits_{t \to \infty} \omega_3 = 0 + 0$, $\lim\limits_{t \to \infty} \omega_2 = \varOmega - 0$), stopping at the point A_2. Starting from (15.1.51) and making $p = \lambda$, $k = 1$, $\beta_2 = \varOmega$, it results

$$t - t_0 = \frac{1}{\lambda} \int_{\omega_2^0 / \varOmega}^{\omega_2 / \varOmega} \frac{\mathrm{d}z^2}{1 - z^2},$$

wherefrom, by a change of variable $z = \tanh \kappa$, we find again the above results (we use the relations (15.1.49'') too). We notice that, for $t \to \infty$, the relations (15.1.58), (15.1.59') lead to $\sin\theta \sin\varphi \to 0$, $\sin\theta \cos\varphi \to 1$, $\dot{\psi} \to \varOmega$, wherefrom $\theta \to \pi/2$, $\varphi \to 0$, $\psi \to \varOmega(t - t_0) + \psi_0$ (we have put in evidence the initial conditions too); thus, the Ox_2-axis tends to the fixed axis Ox_3' (being thus also a fixed axis), the rotation taking place about the moment of momentum $\mathbf{K}_O'$ (Fig. 15.13c). If the initial conditions correspond to the motion determined for $t \to \infty$, then we obtain a permanent rotation of the form (15.1.73) about the mean axis of the ellipsoid $\mathscr{E}$ (the mean principal axis of rotation) for $i = 2$, $j = 3$, $k = 1$.

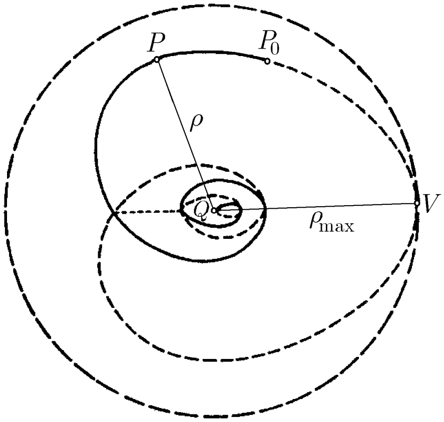


Fig. 15.14 The double spiral of a herpolhode in the limit case in which the circular annulus is reduced to a circle

We have seen, in this case, that the polhode is formed by two ellipses (ε and ε'), contained in the planes (15.1.68) in which degenerates the polhodic cone; as a matter of fact, the case which corresponds to the formulae (15.1.74") leads to an arc of ellipse ε, the sense of travelling through towards the point A_2 being that indicated in Fig. 15.10. In what concerns the herpolhode, we notice that $\rho_{\min} = \delta_1 = \delta_3 = 0$, while $\rho_{\max} = \delta_2 = K\sqrt{(I_1 - I_2)(I_2 - I_3)/I_1 I_2 I_3}$, hence the circular annulus is reduced to a circle. The equation (15.1.72) of the herpolhode becomes (we have $\delta = 0$)

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\rho} = \pm \frac{h}{\rho\sqrt{\rho_{\max}^2 - \rho^2}}.$$

By integration, we get

$$\frac{\rho_{\max}}{\rho} = \pm \cosh\left(\frac{\rho_{\max}}{h}(\kappa - \kappa_0)\right), \tag{15.1.75}$$

hence a double spiral, with an axis of symmetry, which tends asymptotically towards the point Q (the vertex V corresponds to $\rho_{\max}$, being the only point of tangency with the circle) (Fig. 15.14).

As a matter of fact, the point P travels through only an arc of spiral, from the point P_0 (corresponding to the initial position) till the asymptotic point Q (to which the pole arrives in an infinite time, although the length of the spiral is finite), through which passes the rotation axis.

In conclusion, *in the particular case in which the rigid solid begins to rotate about a principal axis of inertia, corresponding to the fixed point, this motion continues indefinitely; the axis of rotation is fixed in the solid and in the space* (with respect to the frames of reference $\mathscr{R}$ and $\mathscr{R}'$, respectively). Assuming now that ω_i / ω, $i = 1,2,3$, are the constant direction cosines of an arbitrary permanent axis of rotation Δ with respect to the frame of reference $\mathscr{R}$, the formula (15.1.47") shows that $\omega = \mathrm{const}$, so that the components ω_i, $i = 1,2,3$, must be constant too; Euler's equations (15.1.40) become

$$(I_2 - I_3)\omega_2\omega_3 = 0, \quad (I_3 - I_1)\omega_3\omega_1 = 0, \quad (I_1 - I_2)\omega_1\omega_2 = 0,$$

and hold only if two of the components of the vector $\boldsymbol{\omega}$ vanish (if the principal moments of inertia are distinct). Hence, *the principal axes of inertia are the only instantaneous axes of rotation which remain fixed in the rigid solid* (as well as in space – the frame of reference $\mathscr{R}'$). Analogously, imposing the condition $\omega = \mathrm{const}$ (hence $\dot{\omega} = 0$), the relation (15.1.57') shows that we can have only $\omega = \gamma_1$, or $\omega = \gamma_2$ or $\omega = \gamma_3$, while from (15.1.56) it results that we have $\omega_1 = 0$ or $\omega_2 = 0$ or $\omega_3 = 0$ (as a matter of fact, to this conclusion leads also the relation (15.1.57)). The equations (15.1.40) show then that *the only axes of uniform rotation are the principal axes of inertia*.

Poinsot's geometric representation allows also an intuitive study of the stability of the three possible permanent axes of rotation. The quantity I of the nature of a moment of inertia depends on the initial conditions (15.1.19) for the angular velocity $\boldsymbol{\omega}$. Taking into account (15.1.48), we have

$$I = \frac{I_1^2\left(\omega_1^0\right)^2 + I_2^2\left(\omega_2^0\right)^2 + I_3^2\left(\omega_3^0\right)^2}{I_1\left(\omega_1^0\right)^2 + I_2\left(\omega_2^0\right)^2 + I_3\left(\omega_3^0\right)^2}, \tag{15.1.76}$$

so that I depends continuously on the initial conditions; if $\boldsymbol{\omega}^0 = \mathbf{0}$, then we are in the case of equilibrium, while I is non-determinate (it is a quantity of dynamic character). Let us assume that $\omega_1^0 > 0$, $\omega_2^0 = \omega_3^0 = 0$; in this case $I = I_1$, the polhode reducing to the point A_1 (if $\omega_1^0 < 0$, then there corresponds the point A_1') on the ellipsoid $\mathscr{E}$.

By an arbitrary variation of the initial conditions, the point P moves away from the point A_1 (because I will vary with respect to I_1), describing a polhode (closed curve) around this point. The plane tangent at P to the ellipsoid of inertia will have a variation with respect to the plane Π tangent to the same ellipsoid at the vertex A_1, while the herpolhode will be contained in a circular annulus of non-zero radii. All these variations are of the same order of magnitude, as it can easily verified (as a matter of fact, as the polhodic and herpolhodic cones). If we assume that $\omega_3^0 > 0$, $\omega_1^0 = \omega_2^0 = 0$, then we have $I = I_3$, the polhode being reduced to the point A_3 on the ellipsoid $\mathscr{E}$; in case of an arbitrary variation of the initial conditions we can make analogous considerations. But if $\omega_2^0 > 0$, $\omega_3^0 = \omega_1^0 = 0$, then we have $I = I_2$, while the polhode is reduced to the point A_2 (intersection of the ellipses ε and ε') on the ellipsoid of inertia. To an arbitrary variation of the initial conditions, the quantity I will vary with respect to I_2, while the pole P describes a polhode around one of the points A_3 and A_3' (or A_1 and A_1'), which leads to a corresponding herpolhode; obviously, the variations corresponding to the polhode and to the herpolhode are no more of the order of magnitude of the variations of the initial conditions. In conclusion, we can state

Theorem 15.1.11 *A permanent rotation about the major or of the minor axis of the ellipsoid of inertia corresponding to the fixed point represents a stable motion, while a permanent rotation about the mean axis of this ellipsoid constitutes a labile motion.*

We notice that the ellipses ε and ε' (which pierce at the points A_2 and A_2') divide the ellipsoid in four zones, each one of them containing one of the points A_1, A_1', A_3 or A_3'. After Bour, one can take as *measure of the stability of the rotation* about the axis $A_1'A_1$, for instance, the ratio between the area of the zone which contains one of the points A_1 or A_1' and half of the area of the ellipsoid $\mathscr{E}$. One observes that for I_1 close to I_2 (hence, for an ellipsoid of inertia close to an ellipsoid of rotation) the area of the zone which contains the point A_1 is small with respect to half of the area of the ellipsoid, hence the considered measure is small, the stability of the rotation about the Ox_2-axis being small too.

Projecting the pole P of the polhode on the plane Ox_2x_3 at P_1 and taking into account (15.1.15'), we notice that the angle χ_1 formed by OP_1 with the Ox_2-axis is given by $\tan\chi_1 = \omega_3 / \omega_2$ (Fig. 15.11b); differentiating with respect to time, we obtain $(\mathrm{d}\tan\chi_1 / \mathrm{d}t = (1 + \tan^2\chi_1)\dot{\chi}_1)$

$$\dot{\chi}_1 = \frac{\omega_2\dot{\omega}_3 - \omega_3\dot{\omega}_2}{\omega_2^2 + \omega_3^2} .$$

Using the equations (15.1.40) and then the second equation (15.1.49"), we get, finally,

$$\dot{\chi}_1 = \frac{I(I_1 - I)\Omega^2}{I_2 I_3(\omega_2^2 + \omega_3^2)}\omega_1 . \tag{15.1.77}$$

Analogously, if P_3 is the projection of the pole P on the plane Ox_1x_2, we can write

$$\dot{\chi}_3 = -\frac{I(I - I_3)\Omega^2}{I_1 I_2 \left(\omega_1^2 + \omega_2^2\right)}\omega_3 \tag{15.1.77'}$$

for the angle χ_3 formed by OP_3 with the Ox_1-axis (Fig. 15.11a). It results that $\operatorname{sign}\dot{\chi}_1 = \operatorname{sign}\omega_1$ and $\operatorname{sign}\dot{\chi}_3 = -\operatorname{sign}\omega_3$. Let us suppose firstly that the stationary rotation takes place counterclockwise (positive sense) about the Ox_1-axis, so that $\omega_1 > 0$, hence that $\dot{\chi}_1 > 0$ too; in the perturbed motion, the instantaneous axis of rotation will be rotated about the Ox_1-axis in the same sense, so that the pole P will describe the polhode which surrounds the point A_1 counterclockwise too. If the permanent rotation takes place about the Ox_3-axis, counterclockwise (positive sense) too, so that $\omega_3 > 0$, hence $\dot{\chi}_3 > 0$, then, in the perturbed motion, the instantaneous axis of rotation will rotate about the Ox_3-axis clockwise (negative sense). Obviously, we obtain similar results if we take $\omega_1 < 0$ or $\omega_3 < 0$ (see Fig. 15.10 too). Hence, we can state

Theorem 15.1.12 *The permanent rotations about the minor axis and about the major axis, respectively, of the ellipsoid of inertia corresponding to the fixed point are different because, in the perturbed motion, the pole of the instantaneous axis of rotation describes a polhode in the same sense as the rotation of the rigid solid, in the first case, and in the opposite sense with respect to the rotation of the rigid solid, in the second case, respectively.*

This result can be easily put in evidence with the aid of Maxwell's top.

15.1.2.8 Case of an Ellipsoid of Inertia of Rotation

In case of an ellipsoid of inertia of rotation (we suppose that the axis of rotation is the major axis of the ellipsoid, $I_1 = I_2 = J > I > I_3$), of equation

$$J\left(x_1^2 + x_2^2\right) + I_3 x_3^2 = K^2, \tag{15.1.78}$$

the system of equations (15.1.40) is reduced to (we suppose that $\omega_3^0 > 0$)

$$\dot{\omega}_1 - p\omega_2 = 0, \quad \dot{\omega}_2 - p\omega_1 = 0, \quad \omega_3 = \omega_3^0, \quad \omega_3^0 = \text{const}, \quad p = \left(1 - \frac{I_3}{J}\right)\omega_3^0. \tag{15.1.79}$$

We get thus

$$\ddot{\omega}_1 + p^2\omega_1 = 0, \quad \ddot{\omega}_2 + p^2\omega_2 = 0. \tag{15.1.79'}$$

It results (we are in the case $I < I_2$, while $k = 0$, the elliptic functions becoming circular ones)

$$\begin{aligned} \omega_1(t) &= \omega_1^0 \cos p(t - t_0) + \omega_2^0 \sin p(t - t_0), \\ \omega_2(t) &= \omega_2^0 \cos p(t - t_0) - \omega_1^0 \sin p(t - t_0), \end{aligned} \tag{15.1.79''}$$

as well as $\omega^2(t) = \omega_0^2 = \text{const}$, $\omega^2 = \omega_i \omega_i$, $\omega_0^2 = \omega_i^0 \omega_i^0$; ω_i^0, $i = 1,2,3$, are the components of the rotation angular velocity vector at the initial moment $t = t_0$.

As a matter of fact, the first integrals (15.1.47) take the form

$$J^2\left(\omega_1^2 + \omega_2^2\right) + I_3^2 \omega_3^2 = I^2 \Omega^2,$$
$$J\left(\omega_1^2 + \omega_2^2\right) + I_3 \omega_3 = I \Omega^2,$$

wherefrom

$$\omega_1^2 + \omega_2^2 = \frac{I(I - I_3)}{J(J - I_3)} \Omega^2 = \overline{\beta}_1^2 = \left(\omega_1^0\right)^2 + \left(\omega_2^0\right)^2,$$
$$\omega_3^2 = \frac{I(J - I)}{I_3(J - I_3)} \Omega^2 = \beta_3^2 = \left(\omega_3^0\right)^2. \qquad (15.1.79''')$$

We are thus led to the same result. We can write

$$p = \frac{I}{J}\sqrt{\frac{(J - I)(J - I_3)}{I_3 I}}\Omega \qquad (15.1.79^{\text{iv}})$$

in this case; we have $k = 0$ too, so that the elliptic functions in (15.1.53) become circular functions ($\text{dn}\, pt = 1, \text{cn}^2\, pt + \text{sn}^2\, pt = 1$). We notice also that

$$\omega^2(t) = I\Omega^2 \frac{(J + I_3 - I)}{J I_3} = \gamma_1^2 = \gamma_2^2 = \omega_0^2,$$
$$\frac{\omega_3}{\omega} = \cos(\mathbf{i}_3, \boldsymbol{\omega}) = \frac{J(J - I)}{(J - I_3)(J + I_3 - I)} = \frac{\omega_3^0}{\omega_0} = \text{const};$$

hence, the instantaneous angular velocity $\boldsymbol{\omega}$ is constant in magnitude and makes a constant angle with the axis of symmetry Ox_3 of the ellipsoid of inertia relative to the fixed point O. As well, assuming that the Ox_3'-axis is along the fixed direction of the moment of momentum $\mathbf{K}_O'$, the formula (15.1.46) shows that the angle formed by the vector $\boldsymbol{\omega}$ with the fixed axis Ox_3' is constant too. Hence, the polhodic cone $\mathscr{C}_p$ is a cone of rotation of equation

$$J(J - I)\left(x_1^2 + x_2^2\right) = I_3(I - I_3)x_3^2, \qquad (15.1.78')$$

the axis of symmetry being Ox_3, while the herpolhodic cone $\mathscr{C}_h$ is a cone of rotation too, for which the axis of symmetry is Ox_3' (Fig. 15.15a). As a consequence, the polhodes are circles, intersections of the cone (15.1.78') by the planes $x_3 = \text{const}$. As well, the herpolhodes are circles too, specified by the radius (we notice that $\rho_{\max} = \rho_{\min} = \delta_1 = \delta_2$, the circular annulus being reduced to a circle, which is just the herpolhode) $\rho = K\sqrt{(J - I)(I - I_3)/IJI_3}$. From (15.1.72') one observes that $\delta < \delta_1$, $\delta = \text{const}$, while the formula (15.1.71) leads to

$\dot{\kappa} = \Omega \left(\rho^2 - \delta^2 \right) / \rho^2 = \Omega I / J = \text{const}$, so that the motion of the pole P on the herpolhode is uniform; obviously, the motion of the pole P on the polhode has the same property, because the polhode rolls without sliding over the herpolhode.

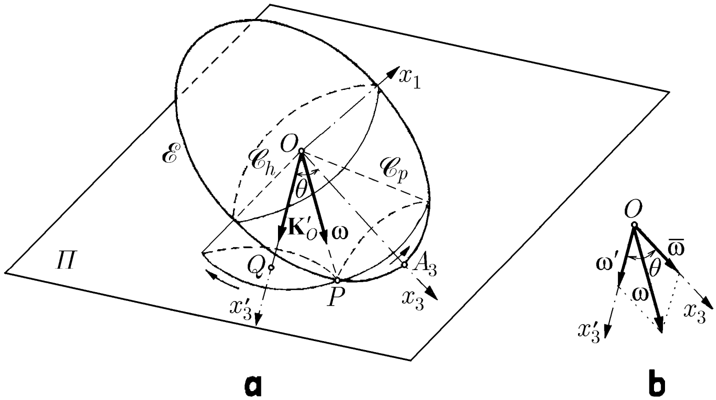


Fig. 15.15 The polhodic and herpolhodic exterior tangent cones (**a**). The canonical decomposition of the vector $\boldsymbol{\omega}$

The relations (15.1.58) become

$$J\omega_1 = J\left[\omega_1^0 \cos p(t - t_0) + \omega_2^0 \sin p(t - t_0) \right] = I\Omega \sin\theta \sin\varphi ,$$
$$J\omega_2 = J\left[\omega_2^0 \cos p(t - t_0) - \omega_1^0 \sin p(t - t_0) \right] = I\Omega \sin\theta \cos\varphi ,$$
$$I_3\omega_3 = I_3\omega_3^0 = I\Omega \cos\theta .$$

Hence, it results

$$\cos\theta(t) = \frac{I_3}{I} \frac{\omega_3^0}{\Omega} = \sqrt{\frac{I_3 (J - I)}{I(J - I_3)}} = \cos\theta_0 , \tag{15.1.80}$$

the angle of nutation θ between the axes of the two cones being thus constant ($\theta(t) = \theta_0$). Because the plane which passes through OP and is tangent to the two cones is normal to the meridian planes formed by OP with the Ox'_3-axis and with the Ox_3-axis, respectively, it results that these meridian planes coincide; consequently, in its motion, the Ox_3-axis describes a cone having as axis the support of the vector $\mathbf{K}'_O$ (Fig. 15.15a).

Using the notations

$$\omega_1^0 = \sqrt{\left(\omega_1^0\right)^2 + \left(\omega_2^0\right)^2} \sin\gamma = \overline{\beta}_1 \sin\gamma ,$$
$$\omega_2^0 = \sqrt{\left(\omega_1^0\right)^2 + \left(\omega_2^0\right)^2} \cos\gamma = \overline{\beta}_1 \cos\gamma , \quad \tan\gamma = \frac{\omega_1^0}{\omega_2^0} , \tag{15.1.81}$$

we get $\tan\varphi = \tan(p(t - t_0) + \gamma)$, wherefrom

$$\varphi(t) = p(t - t_0) + \varphi_0\,, \quad \varphi_0 = \arctan\frac{\omega_1^0}{\omega_2^0}\,, \tag{15.1.81'}$$

the motion of proper rotation (about the Ox_3-axis) being uniform. If we make $I_1 = I_2 = J$ in (15.1.59'), then we obtain $\dot{\psi} = \Omega I / J$; we notice that $\dot{\psi} = \dot{\kappa}$, which was to be expected. We are thus led to

$$\psi(t) = \frac{I}{J}\Omega(t - t_0) + \psi_0\,, \tag{15.1.81''}$$

corresponding to the uniform motion of precession (about the Ox_3'-axis), called also *regular precession*. These results allow the canonical decomposition of the angular velocity vector $\boldsymbol{\omega}$ in the form (we use another notation, different from that corresponding to the decomposition in the frames of reference $\mathscr{R}'$ and $\mathscr{R}$ and we notice that $\dot{\theta} = 0$; as a matter of fact, the relations $\omega_3 = \varpi + \omega' \cos\theta$, $\omega_3' = \omega' + \varpi\cos\theta$)

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \boldsymbol{\varpi} = \omega'\mathbf{i}_3' + \varpi\mathbf{i}_3 = \dot{\psi}\mathbf{i}_3' + \dot{\varphi}\mathbf{i}_3\,, \tag{15.1.81'''}$$

where the vector $\boldsymbol{\omega}'$ is constant, while the vector $\boldsymbol{\varpi}$ is constant only in magnitude (Fig. 15.15b). If we make $I_1 = I_2 = J$ in the formulae (15.1.62)–(15.1.62''), then we find $k = 0$, $k' = \beta = 1$, hence $\varepsilon = 0$, $K = \pi/2$, so that $\omega_\varphi = p$, $q = 0$; in this case, the proper rotation, the precession and the nutation are reduced to their mean values, which correspond to the values given by (15.1.80), (15.1.81'), (15.1.81''). As closer we are from the case of symmetry considered above (or as the motion of the rigid solid is closer to a stable rotation about the Ox_3'-axis) as quicker tends to zero the parameter q (hence, as better is the approximation made by deducing the formulae (15.1.62)–(15.1.62'')). In exchange, the error growth as we are closer to the particular case $k = 1$, hence as the motion is closer to a (labile) rotation about the mean principal axis of inertia. In the case considered by us, the ellipsoid of rotation is a prolate spheroid (the major axis of the ellipsoid is axis of symmetry); we notice that, in this case, $\dot{\varphi} = p > 0$ and $\dot{\psi} > 0$, the polhodic and the herpolhodic cones being exterior tangent. The sense of motion of the pole P on the polhode is indicated by the relation (15.1.77'), which leads to $\dot{\chi}_3 = -p = -\dot{\varphi}$ (result which had to be expected), while the sense of the motion of the same pole on the herpolhode is specified by the relation $\dot{\psi} = \dot{\kappa}$ (Fig. 15.15a).

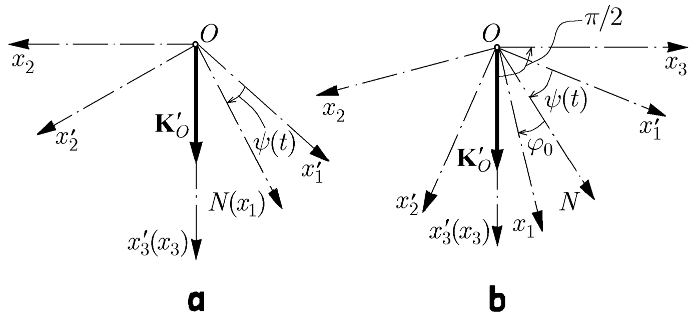


Fig. 15.16 Permanent rotation about the stable major axis Ox_3 and about an arbitrary principal axis in the equatorial plane (**b**) of the prolate spheroid

If $I = I_3 < J$, then we obtain $\omega_3^0 = \Omega$, $p = (J - I_3)\Omega / J$, $\omega_1^0 = \omega_2^0 = 0$, so that $\omega_1(t) = \omega_2(t) = 0$, $\omega_3(t) = \omega(t) = \Omega$; then, $\cos\theta = 1$, hence $\theta = 0$ and the axes Ox_3 and Ox'_3 coincide. The angle φ is non-determinate (it can be taken equal to zero), while the angle of precession is given by $\psi(t) - \psi_0 = \Omega I_3 (t - t_0)/J$, the precession being uniform. The polhode and the herpolhode are reduced to the point A_3 (we have $\rho = 0$). The motion is a permanent rotation about the major axis Ox_3 of the prolate spheroid, which is a stable axis of rotation (Fig. 15.16a); this is the case of the gyroscope, which will be studied in the next chapter. If $I = J > I_3$, then we get $\omega_3^0 = 0$, $(\omega_1^0)^2 + (\omega_2^0)^2 = \omega_0^2 = \Omega^2$ and $p = 0$; as well, $\omega_1^0 = \Omega \sin\varphi_0$, $\omega_2^0 = \Omega\cos\varphi_0$ and $\varphi = \varphi_0$. Then, $\psi(t) = \Omega(t - t_0) + \psi_0$, the precession being uniform; we notice that $\cos\theta = 0$, hence $\theta = \pi/2$, as well as $p = 0$ too. The motion is a permanent rotation about an arbitrary principal axis in the equatorial plane of *the prolate spheroid*, which coincides with the Ox'_3-axis, being thus fixed in space (Fig. 15.16b); the polhode and the herpolhode are reduced to a point A on the equator of the prolate spheroid and in the plane Π tangent to the ellipsoid at this point. At a small perturbation of the position of the instantaneous axis of rotation, the herpolhode will be a curve in the vicinity of the point A with $\rho_{\max}$ of the same order of magnitude; in exchange, the polhode will be a circle parallel to the equatorial circle and close to this one. Because, in the rigid solid, the axis moves much away from the initial position, it results that this one is a labile axis of rotation.

In the case in which $I_1 > I > I_2 = I_3 = J$, the axis of symmetry of the ellipsoid being its minor axis, the latter one is an *oblate spheroid* of equation

$$I_1 x_1^2 + J(x_2^2 + x_3^2) = K^2. \tag{15.1.82}$$

We can make a study analogue to that above, obtaining

$$\omega_1(t) = \omega_1^0 = \sqrt{\frac{I(I-J)}{I_1(I_1-J)}}\,\Omega\,, \quad \omega_2^2(t) + \omega_3^2(t) = \left(\omega_2^0\right)^2 + \left(\omega_3^0\right)^2 = \frac{I(I_1-I)}{J(I_1-J)}\,\Omega^2$$
(15.1.82')

and also

$$\begin{aligned}\omega_2(t) &= \omega_2^0 \cos \overline{p}(t-t_0) - \omega_3^0 \sin \overline{p}(t-t_0)\,,\\ \omega_3(t) &= \omega_3^0 \cos \overline{p}(t-t_0) + \omega_2^0 \sin \overline{p}(t-t_0)\,,\end{aligned} \qquad (15.1.82'')$$

where

$$\overline{p} = \frac{I_1 - J}{J}\,\omega_1^0 = \frac{I}{J}\sqrt{\frac{(I-J)(I_1-J)}{I_1 I}}\,\Omega\,. \qquad (15.1.82''')$$

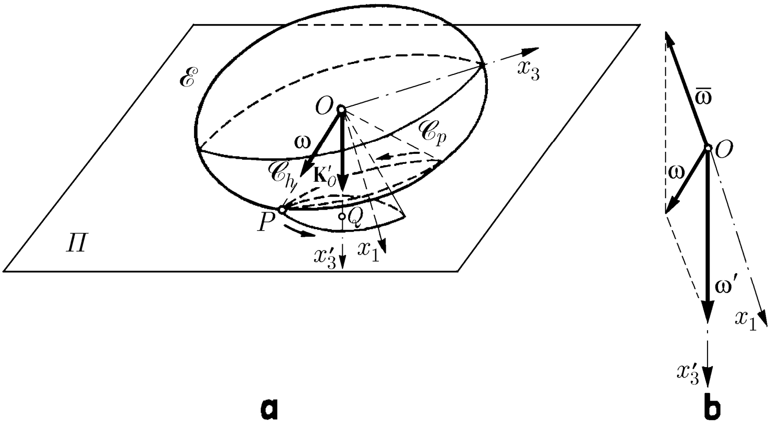


Fig. 15.17 The polhodic and herpolhodic interior tangent cones (**a**). The canonical decomposition of the vector $\boldsymbol{\omega}$

The Ox_1-axis is an axis of symmetry for the polhodic cone, while the Ox'_3-axis is an axis of symmetry for the herpolhodic cone (Fig. 15.17,a); the fixed angle made by the two axes is given by $\cos(\mathbf{i}'_3, \mathbf{i}_1) = \sin\theta\cos\varphi = I_1\omega_1^0 / I\Omega = \sqrt{I_1(I-J)/I(I_1-J)}$. The angular velocity vector $\boldsymbol{\omega}$ is decomposed in the form (Fig. 15.17b)

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \overline{\boldsymbol{\omega}} = \omega'\mathbf{i}'_3 + \overline{\omega}\mathbf{i}_1 = \dot{\kappa}\mathbf{i}'_3 - \overline{p}\mathbf{i}_1\,, \qquad (15.1.83)$$

where $\dot{\kappa} = \Omega I / J > 0$, and, if we take into account (15.1.77), then $\overline{p} = \dot{\chi}_1 > 0$; thus, the motion of the point P on the herpolhode and on the polhode is specified (Fig. 15.17a). From (15.1.82), (15.1.82'), it results $\omega^2(t) = I\Omega^2(I_1 + J - I)/I_1 J = \omega_0^2$. The relation (15.1.46) allows to write $\cos(\boldsymbol{\omega}, \mathbf{i}'_3) = \Omega/\omega = \Omega/\omega_0 = \text{const}$; as well, we have $\cos(\boldsymbol{\omega}, \mathbf{i}'_1) = \omega_1/\omega = \omega_1^0/\omega_0 = \text{const}$. Noting that $\omega_1^0 < \Omega$, it results $\cos(\boldsymbol{\omega}, \mathbf{i}_1) < \cos(\boldsymbol{\omega}, \mathbf{i}'_3)$, wherefrom $\sphericalangle(\boldsymbol{\omega}, \mathbf{i}_1) > \sphericalangle(\boldsymbol{\omega}, \mathbf{i}'_3)$. In conclusion, the herpolhodic cone and the polhodic cone are *interior tangent*, the first one being interior to the second one. The minor axis of the spheroid (axis of symmetry) is stable in case of a permanent rotation about it; in exchange, the axes in the equatorial plane are labile axes of rotation.

If $I_1 = I_2 = I_3 = I = J$, then the ellipsoid of inertia is a sphere, any axis of it being a principal axis of inertia, as well as an indifferent axis of rotation.

15.1.2.9 Another Geometric Representation of the Motion after Poinsot

The geometric interpretation given by Poinsot to the problem of Euler is remarkable by its simplicity and clarity, as well because it is an intuitive method; in exchange, this interpretation does not put time in evidence, and there is no one element to vary in direct proportion to the variable t. To eliminate partially this lack, Poinsot imagines a new geometric representation, starting from the decomposition of the angular velocity vector $\boldsymbol{\omega}$ in two components: $\boldsymbol{\omega}_{||}$ along the direction of the moment of momentum $\mathbf{K}'_O$ and $\boldsymbol{\omega}_{\perp}$ along a direction normal to this one; the vector $\boldsymbol{\omega}$ and its components are contained in a plane specified by the Ox-axis, normal to this plane (Fig. 15.18). We notice that $\boldsymbol{\omega}_{||} = \overrightarrow{\mathrm{const}}$ (we have $|\boldsymbol{\omega}_{||}| = \Omega$ too), while $\boldsymbol{\omega}_{\perp}$ varies both in direction and in modulus, being contained, for any t, in the fixed plane Π_h, normal to the vector $\mathbf{K}'_O$ (its locus with respect to the frame of reference $\mathscr{R}'$), which will be called *herpolhodic plane* (degenerate herpolhodic cone). With respect to the frame $\mathscr{R}$, the locus of the component $\boldsymbol{\omega}_{\perp}$ will be a *second polhodic cone* $\mathscr{C}_p^{\perp}$ *of Poinsot*.

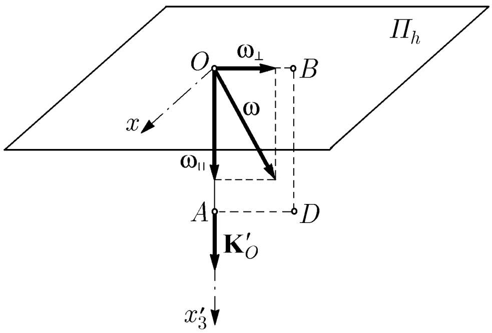


Fig. 15.18 The herpolhodic plane

Let D be a point on the support of the vector $\boldsymbol{\omega}$, while A and B are its projections on the supports of the vectors $\boldsymbol{\omega}_{||}$ and $\boldsymbol{\omega}_{\perp}$, respectively. We denote the co-ordinates of the point B by x_i and the co-ordinates of the point D by $\lambda\omega_i$, $\lambda = \mathrm{const}$, $i = 1,2,3$, with respect to the frame of reference $\mathscr{R}$; we notice that the quantities $I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$ may be used as direction parameters of the straight line OA, so that we can write

$x_1 = \omega_1(\lambda + \lambda' I_1)$, $x_2 = \omega_2(\lambda + \lambda' I_2)$, $x_3 = \omega_3(\lambda + \lambda' I_3)$, $\lambda' = \mathrm{const}$. The equation of the plane Π_h is written in the form (orthogonality condition)

$$I_1\omega_1 x_1 + I_2\omega_2 x_2 + I_3\omega_3 x_3 = 0. \tag{15.1.84}$$

Eliminating the co-ordinates x_1, x_2, x_3 and taking into account (15.1.47), we get $\lambda + \lambda' I = 0$. Whence, it results $x_1 = \lambda'(I_1 - I)\omega_1$, $x_2 = \lambda'(I_2 - I)\omega_2$, $x_3 = \lambda'(I_3 - I)\omega_3$; the relation (15.1.66), which is verified by the components of the vector $\boldsymbol{\omega}$, allows now to write the equation of the second polhodic cone $\mathscr{C}_p^\perp$ of Poinsot in the form

$$\frac{I_1}{I_1 - I}x_1^2 + \frac{I_2}{I_2 - I}x_2^2 + \frac{I_3}{I_3 - I}x_3^2 = 0, \tag{15.1.84'}$$

this one being thus a cone of second degree. In the motion of the rigid solid, the cone $\mathscr{C}_p^\perp$ *is rolling with sliding* on the plane Π_h, because *it is rotating together with the rigid solid about the* Ox_3'*-axis with the constant velocity* Ω; assuming that the plane Π_h has a uniform motion of rotation, of angular velocity Ω, about the Ox_3'-axis, the rolling of the cone $\mathscr{C}_p^\perp$ is *slidingless*.

In this last case, in the motion of the rigid solid, the angle of rotation in the plane Π_h is in direct proportion with the interval of time $t - t_0$; associating to the plane Π_h a hand along the Ox-axis, we can measure the time by means of the motion of this hand on a fixed dial. If, by a clock mechanism, the plane Π_h would have a uniform rotation about the Ox_3'-axis (with the angular velocity Ω), transferring a motion to the cone $\mathscr{C}_p$ (by friction or by some gearing), the rigid solid (rigidly linked to this cone) would have an inertial motion about the fixed point O. The building up of *the Darboux-Koenig herpolhodograph*, which allows to obtain a complete kinematic image of the motion of the rigid solid, is based on these considerations. But the practical sensibility of the apparatus is relatively small, the ellipsoid of inertia $\mathscr{E}$ being not an arbitrary one, because its semi-axes must verify the relations (3.1.96); indeed, one can show that the ratio of the smallest to the greatest magnitude of the angular velocity varies thus between $\sqrt{2}/2$ and 1. As a matter of fact, the magnitude of the angular velocity has thus a certain stability and the herpolhode has not points of inflection.

15.1.2.10 Poinsot Type Motions. The Sylvester Theorems. Volterra's Problem

If $b_1 = K^2/I_1$, $b_2 = K^2/I_2$, $b_3 = K^2/I_3$ are the squares of the semi-axes of the ellipsoid of inertia, then the equation (15.1.63) reads

$$\frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \frac{x_3^2}{b_3} = 1, \tag{15.1.85}$$

where $b_1 < b_2 < b_3$; in this case, Euler's equations (15.1.40) take the form

$$\begin{aligned}
&\dot{\omega}_1 + b_1\left(\frac{1}{b_3} - \frac{1}{b_2}\right)\omega_2\omega_3 = 0\,,\\
&\dot{\omega}_2 + b_2\left(\frac{1}{b_1} - \frac{1}{b_3}\right)\omega_3\omega_1 = 0\,,\\
&\dot{\omega}_3 + b_3\left(\frac{1}{b_2} - \frac{1}{b_1}\right)\omega_1\omega_2 = 0\,,
\end{aligned} \tag{15.1.86}$$

while the two first integrals (15.1.47) read

$$\begin{aligned}
&\frac{\omega_1^2}{b_1^2} + \frac{\omega_2^2}{b_2^2} + \frac{\omega_3^2}{b_3^2} = \frac{K_O'^2}{K^2} = \frac{I^2\Omega^2}{K^4} = \frac{\Omega^2}{b^2}\,,\\
&\frac{\omega_1^2}{b_1} + \frac{\omega_2^2}{b_2} + \frac{\omega_3^2}{b_3} = \frac{2T'}{K^2} = \frac{I\Omega^2}{K^2} = \frac{\Omega^2}{b}\,,
\end{aligned} \tag{15.1.86'}$$

where b is the square of the distance from the centre O to the fixed plane Π over which is rolling the ellipsoid of inertia $\mathscr{E}$, given by (15.1.16'). The conditions

$$\frac{1}{b_1} < \frac{1}{b_2} + \frac{1}{b_3}\,,\quad \frac{1}{b_2} < \frac{1}{b_3} + \frac{1}{b_1}\,,\quad \frac{1}{b_3} < \frac{1}{b_1} + \frac{1}{b_2}\,, \tag{15.1.86''}$$

corresponding to the conditions (3.1.103), must be verified, as well as the condition $b_1 < b < b_3$; only two of the three differences $b - b_1$, $b - b_2$, $b - b_3$ can have the same sign. We notice that the relation (15.1.15') allows to pass from the equation (15.1.85) to the second first integral (15.1.86'). The loci of the point P of tangency of the ellipsoid $\mathscr{E}$ with the plane Π on the ellipsoid and on the plane are, obviously, the polhode and the herpolhode, respectively.

Let us suppose now that the magnitudes b_1, b_2 and b_3 do not fulfil the conditions (15.1.86''), possibly being negative too (the ellipsoid (15.1.85) may be replaced by a unparted or by a parted hyperboloid, being, in general, a quadric with centre). In this case, the equations (15.1.86) lose their dynamical signification, no more representing the inertial motion of the rigid solid; but they keep their kinematical signification, corresponding to a motion of the rigid solid, the geometric image of which is represented by the rolling without sliding of a quadric Γ with centre over one of its tangent planes Π, fixed in space. A relation of the form $\overrightarrow{OP} = \mu\boldsymbol{\omega}$, $\mu = \text{const}$ takes further place. The polhode P will be the locus of the points which are on the quadric Γ and for which the planes Π, tangent to the quadric at various points of this curve, are at a constant distance from the centre O of the quadric. Such a motion is called a *Poinsot type motion*; it takes place only if two of the products $(b - b_2)(b - b_3)$, $(b - b_3)(b - b_1)$, $(b - b_1)(b - b_2)$ are negative, the third one being positive, as it results from the above inequalities. One observes that we have introduced the quantities

b_1, b_2, b_3 (instead of the semi-axes a_1, a_2, a_3) to can pass from Poinsot's motion to a motion of Poinsot type. Considerations in this direction are due to A. Clebsch, E.J. Routh and P. Appell.

Let be two motions of Poinsot type with the same centre and with the same principal axes of inertia, characterized by the principal moments of inertia I_1, I_2, I_3 and $\overline{I}_1, \overline{I}_2, \overline{I}_3$ and by the angular velocities $\boldsymbol{\omega}$ and $\overline{\boldsymbol{\omega}}$, respectively; two such motions for which takes place the relation $\boldsymbol{\omega} + \overline{\boldsymbol{\omega}} = \mathbf{0}$ are called *conjugate motions in the sense of Darboux*. Obviously, two such motions have the same polhodes. One can show that to any motion of Poinsot type kinetically possible corresponds always a conjugate motion in the sense of Darboux, also kinetically possible.

Assuming that the motion of Poinsot type is a *direct motion*, we can give a geometric interpretation to *the inverse motion of Poinsot type*, considered as a slidingless motion of a movable plane Π over a quadric with centre Γ, fixed in space, to which it is tangent, remaining at a constant distance from this centre. Obviously, we can represent this motion as a rolling without sliding of the herpolhodic cone $\mathscr{C}_h$ (movable now) over the polhodic cone $\mathscr{C}_p$ (fixed now). In case of conjugate motions in the Darboux sense, the two quadrics Γ and $\overline{\Gamma}$ are pierced along the same polhode. The corresponding inverse motions of Poinsot type can be represented by the rolling of the herpolhodic cones $\mathscr{C}_h$ and $\overline{\mathscr{C}}_h$, rigidly connected to these quadrics, over the same polhodic cone $\mathscr{C}_p$; the cones $\mathscr{C}_h$ and $\overline{\mathscr{C}}_h$ are tangent to the cone $\mathscr{C}_p$ along the same generatrix (support of the vector $\boldsymbol{\omega}$), hence they are tangent one to the other. The motion of a tangent plane Π with respect to a tangent plane $\overline{\Pi}$ is thus reduced to the rolling of the herpolhodic cone $\mathscr{C}_h$ over the herpolhodic cone $\overline{\mathscr{C}}_h$; the angular velocities of the planes Π and $\overline{\Pi}$ with respect to the inertial frame of reference are $\boldsymbol{\omega}$ and $-\boldsymbol{\omega}$, respectively, so that – in the relative motion of the two planes – the angular velocity will be $\pm 2\boldsymbol{\omega}$. One can state

Theorem 15.1.13 (*J.J. Sylvester*) *If at all the points of the polhode $\mathscr{P}$ we take equal segments of a line along the normals to the quadric Γ, their extremities will be on a new polhode $\mathscr{P}'$, on another quadric Γ', homofocal and homothetic to Γ and orthogonal to the set up normals.*

As a kinetic consequence of this theorem, one can state

Theorem 15.1.13' (*J.J. Sylvester*) *Let be a rigid solid which has a Poinsot type motion and to which has been imparted an angular velocity about the normal to the fixed plane Π over which the quadric Γ is rolling; the compound motion will be a motion of Poinsot type too, the new plane of rolling Π' being parallel to the plane Π.*

The corresponding quadric Γ' differs, obviously, from the quadric Γ.

Another generalization of the Euler-Poinsot problem is due to V. Volterra, which considered, in 1895, the system of equations

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + c_3\omega_2 - c_2\omega_3 &= 0\,,\\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 + c_1\omega_3 - c_3\omega_1 &= 0\,,\\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + c_2\omega_1 - c_1\omega_2 &= 0\,,
\end{aligned}
\tag{15.1.87}
$$

where c_1, c_2 and c_3 are constant quantities. One obtains the first integrals

$$
\begin{gathered}
I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T'\,,\\
(I_1\omega_1 + c_1)^2 + (I_2\omega_2 + c_2)^2 + (I_3\omega_3 + c_3)^2 = c^2\,,
\end{gathered}
\tag{15.1.87'}
$$

c being an integration constant. Starting from these results and from one of the equations (15.1.87), the integration of this differential system is reduced to quadratures, as in Sect. 15.1.2.1.

As well, in case of a moment $\mathbf{M}_O$ along the direction of the moment of momentum $\mathbf{K}'_O$, so that $\mathbf{M}_O = \lambda\mathbf{K}'_O$, where λ is a constant scalar ($M_{O1} = \lambda I_1\omega_1$, $M_{O2} = \lambda I_2\omega_2$, $M_{O3} = \lambda I_3\omega_3$), A.G. Greenhill and E. Padova performed a change of independent variable and of unknown functions of the form $\lambda t = \ln(1 + \lambda\bar{t})$, $\omega_i = (1 + \lambda\bar{t})\bar{\omega}_i$, $i = 1,2,3$, finding again the homogeneous equations of the Euler-Poinsot case for the unknown functions $\bar{\omega}_i = \bar{\omega}_i(\bar{t})$, $i = 1,2,3$.

15.2 The Case in which the Ellipsoid of Inertia is of Rotation. Other Cases of Integrability

In what follows, we study the Lagrange-Poisson case of integrability and the Sonya Kovalevsky one, for which the ellipsoid of inertia is of rotation (it is a prolate spheroid, having $I_1 = I_2 = J$). We consider then also other cases of integrability, corresponding to particular initial conditions, as well as other cases of loading in the dynamics of the rigid solid with a fixed point.

15.2.1 The Lagrange-Poisson Case

Returning to the problem of the rigid solid with a fixed point, subjected to its own weight **G**, we will consider the research made by J.-L. Lagrange in 1788; in this case, the ellipsoid of inertia is of rotation, the centre of mass C being situated on its axis of symmetry ($I_1 = I_2 = J > I_3$, hence the case of a prolate spheroid, and $\rho_1 = \rho_2 = 0$, $\rho_3 > 0$). The problem has been considered again, in 1815, by S.-D. Poisson, without taking into account the results due to Lagrange and without quoting him; this case of integrability (considered to be the II*nd case of integrability*) is called *the Lagrange-Poisson case*. Such a situation is encountered, for instance, in case of a homogeneous rigid solid of rotation around the Ox_3-axis; as well, we are situated in such a case if the solid is non-homogeneous, having a geometric as well as a mechanical axial symmetry (the density is of the form $\mu = \mu\left(\sqrt{x_1^2 + x_2^2}, x_3\right)$ with respect to the Ox_3-axis.

We present firstly a general study of the motion, determining the rotation angular velocity vector and putting stress on the motion of precession; we pass then to a geometric representation of the motion, using the results obtained by Poinsot in this direction.

15.2.1.1 Integration of the Equations of Motion

The motion of the heavy rigid solid with a fixed point is governed by Euler's equations (15.1.21); these equations take the form

$$\begin{aligned} J\dot{\omega}_1 + (I_3 - J)\,\omega_2\omega_3 &= Mg\rho_3\alpha_2\,, \\ J\dot{\omega}_2 + (J - I_3)\,\omega_3\omega_1 &= -Mg\rho_3\alpha_1\,, \\ \dot{\omega}_3 &= 0 \end{aligned} \tag{15.2.1}$$

in the Lagrange-Poisson case. We obtain

$$\omega_3(t) = \omega_3^0\,, \tag{15.2.1'}$$

the constant ω_3^0 being called *spin*, while the first integrals (15.1.42'), (15.1.43') take the form

$$\begin{aligned} J(\omega_1\alpha_1 + \omega_2\alpha_2) + I_3\omega_3^0\alpha_3 &= K'_{O3'}\,, \\ J\left(\omega_1^2 + \omega_2^2\right) + I_3\left(\omega_3^0\right)^2 &= -2Mg\rho_3\alpha_3 + 2h\,. \end{aligned} \tag{15.2.1''}$$

The first integrals (15.2.1'), (15.2.1"), together with the first integral (15.1.44), form the system of four first integrals necessary in the general theory to solve the problem.

Unlike the Euler-Poinsot case, the kinetic and the kinematic aspects cannot be separated in this case. It is convenient to replace the direction cosines $\alpha_1, \alpha_2, \alpha_3$ of the Ox'_3-axis with respect to the frame of reference $\mathscr{R}$ by the relations (5.2.36); it results

$$\begin{aligned} J(\omega_1\sin\varphi + \omega_2\cos\varphi)\sin\theta + I_3\omega_3^0\cos\theta &= K'_{O3'}\,, \\ J\left(\omega_1^2 + \omega_2^2\right) + I_3\left(\omega_3^0\right)^2 &= -2Mg\rho_3\cos\theta + 2h\,. \end{aligned} \tag{15.2.1'''}$$

Associating the relations (5.2.35) and eliminating the components ω_1 and ω_2, we get the system of equations

$$\begin{aligned} \dot{\psi}\sin^2\theta &= \alpha - a\omega_3^0\cos\theta\,, \\ \dot{\psi}\sin^2\theta + \dot{\theta}^2 &= \beta - b\cos\theta\,, \\ \dot{\psi}\cos\theta + \dot{\varphi} &= \omega_3^0\,, \end{aligned} \tag{15.2.2}$$

where we have introduced the notations $\alpha = K'_{O3'}/J$, $\beta = \left[2h - I_3\left(\omega_3^0\right)^2\right]/J$, $a = I_3/J > 0$, $b = 2Mg\rho_3/J > 0$; we notice that α and β are constants which depend on the initial conditions, while the constants a and b are functions only of the geometry and the mechanical properties of the rigid solid.

The system of differential equations (15.2.2) will determine Euler's angles $\psi = \psi(t)$, $\theta = \theta(t)$ and $\varphi = \varphi(t)$. Eliminating $\dot{\psi}$ between the first two relations, we obtain

$$\left(\alpha - a\omega_3^0 \cos\theta\right)^2 = (\beta - b\cos\theta)\sin^2\theta - \dot{\theta}^2 \sin^2\theta .$$

Denoting $u = \cos\theta$, it results the differential equation

$$\dot{u}^2 = P(u), \quad P(u) = (\beta - bu)(1 - u^2) - \left(\alpha - a\omega_3^0 u\right)^2, \tag{15.2.3}$$

wherefrom

$$t = t_0 + \int_{u_0}^{u} \frac{\mathrm{d}\xi}{\sqrt{P(\xi)}}, \tag{15.2.3'}$$

with $u_0 = \cos\theta_0$, $\theta_0 = \theta(t_0)$; the radical is taken with the same sign as $\dot{u}(t_0)$, assuming that $\dot{u}(t_0) \neq 0$, ($\dot{u}(t)$ has a continuous variation, beginning from $\dot{u}(t_0)$).

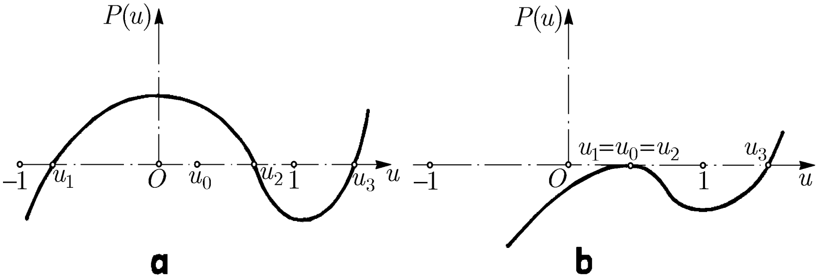


Fig. 15.19 The graphic $P(u)$ vs u. The cases $u_1 \neq u_2$ (**a**) and $u_1 = u_2 = u_0$ (**b**)

If $\theta_0 \neq 0$ and $\theta_0 \neq \pi$, then we have $u_0 \in (-1,1)$; we notice that $P(-\infty) < 0$, $P(\infty) > 0$ and $P(\pm 1) < 0$, in the hypothesis in which $\alpha \neq \pm a\omega_3^0$, hence $K'_{O3'} \neq \pm I_3\omega_3^0$. Because the equation (15.2.3) allows a solution only if $P(u_0) \geq 0$, it results that the polynomial $P(u)$ is of the form

$$P(u) = b(u - u_1)(u_2 - u)(u_3 - u), \tag{15.2.3''}$$

where u_1, u_2, u_3 are the real zeros of the polynomial of the third degree $P(u)$, so that $-1 < u_1 \leq u_0 \leq u_2 < 1 < u_3 < \infty$; the graphic of this polynomial is given in Fig. 15.19a (to draw this graphic, we assume that $\theta_0 \leq \pi/2$, hence $u_0 > 0$) for $u_1 \neq u_2$ and in Fig. 15.19b for $u_1 = u_2 = u_0$. Because $u_1 \leq u_2$, it results that $\theta_1 \geq \theta_2$ (we have $0 \leq \theta \leq \pi$), so that the nutation $\theta \in [\theta_2, \theta_1]$. We notice that the integral (15.2.3), (15.2.3') is of the form (7.1.4) to (7.1.5); following the same reasoning as in

Chap. 7, Sect. 1.1.1, we can state that $u(t)$ varies periodically between u_1 and u_2, the duration of a complete period being

$$T = 2\int_{u_1}^{u_2} \frac{\mathrm{d}u}{\sqrt{P(u)}}. \tag{15.2.3'''}$$

Hence, $u(t+T) = u(t)$ and $\dot{u}(t+T) = \dot{u}(t)$; it results, as well $\theta(t+T) = \theta(t)$.

By the change of variable $u = v/c + c'$, where $c = \sqrt[3]{b/4}$ and $c' = \left[\beta + a^2 \left(\omega_3^0\right)^2\right]/3b$, we can replace the polynomial $P(u)$ by a polynomial of the form $Q(v) = 4v^3 - g_2 v - g_3$, $g_2, g_3 = \mathrm{const}$; the relation (15.2.3') becomes

$$c(t - t_0) = \int_{v_0}^{v} \frac{\mathrm{d}\eta}{\sqrt{Q(\eta)}}, \tag{15.2.4}$$

where $v_0 = c(u_0 - c')$. Denoting $v = \mathscr{P}(\alpha; g_2, g_3)$, we introduce the variable α through the agency of *the elliptic function* $\mathscr{P}$ *of Weierstrass*, corresponding to the constants g_2 and g_3; this function verifies the differential equation

$$\left[\frac{\mathrm{d}\mathscr{P}(\alpha)}{\mathrm{d}\alpha}\right]^2 = 4\mathscr{P}^3(\alpha) - g_2\mathscr{P}(\alpha) - g_3,$$

so that $\alpha - \alpha_0 = c(t - t_0)$, where α_0 is given by $v_0 = \mathscr{P}(\alpha_0; g_2, g_3)$. The function $u(t)$ (and, implicitly, the angle of nutation $\theta(t)$) will be given by

$$u(t) = \cos\theta(t) = c' + \frac{1}{c}\mathscr{P}(c(t - t_0) + \alpha_0). \tag{15.2.4'}$$

We can introduce a new variable κ by the relation

$$\begin{gathered} u = u_1 \cos^2\kappa + u_2 \sin^2\kappa = u_1 + (u_2 - u_1)\sin^2\kappa \\ = u_2 - (u_2 - u_1)\cos^2\kappa = u_3 - (u_3 - u_1)\left(1 - k^2 \sin^2\kappa\right), \\ k = \frac{u_2 - u_1}{u_3 - u_1} < 1. \end{gathered} \tag{15.2.5}$$

Replacing in (15.2.3'), (15.2.3''), we get

$$p(t - t_0) = \int_{\kappa_0}^{\kappa} \frac{\mathrm{d}\chi}{\sqrt{1 - k^2 \sin^2\chi}}, \quad p = \frac{1}{2\sqrt{b(u_3 - u_1)}}, \tag{15.2.6}$$

where $\kappa = \kappa_0$ corresponds to $u = u_0$. Denoting $w = \sin\kappa$, we can write

$$p(t - t_0) = \int_{w_0}^{w} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta)}} \qquad (15.2.6')$$

too, with $w_0 = \sin\kappa_0$. By means of the elliptic integral of the first kind $F(\kappa,k)$, given by (7.1.41), one can use also the formula (15.1.51'). Introducing Jacobi's elliptic functions, it results, as well,

$$u(t) = u_1 \mathrm{cn}^2 p(t-t_0) + u_2 \mathrm{sn}^2 p(t-t_0) = u_1 + (u_2 - u_1)\mathrm{sn}^2 p(t-t_0)$$
$$= u_2 - (u_2 - u_1)\mathrm{cn}^2 p(t-t_0) = u_3 - (u_3 - u_1)\mathrm{dn}^2 p(t-t_0). \qquad (15.2.5')$$

Taking into account the relation (e_1, e_2, e_3 are the zeros of the polynomial $Q(v)$)

$$\mathscr{P}\left(\frac{\alpha}{\sqrt{e_1 - e_3}}\right) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(u;k)}, \qquad (15.2.7)$$

which links the elliptic function of Weierstrass to the Jacobi elliptic functions, we can pass from the formula (15.2.4') to the formulae (15.2.5'). The period (15.2.3''') can be expressed by a formula of the form (15.1.54). The motion of nutation is thus put in evidence. We can set up two circular cones of common axis Ox_3' and angles at the vertex $2\theta_1$ and $2\theta_2$, respectively; the Ox_3-axis describes a cone contained between these two cones (Fig. 15.20a). If the two mentioned cones form only one cone ($P(u) = 0$ has a double root, hence $\theta_1 = \theta_2$), then this cone will be a circular one. The other angles of Euler will be given by the equations (15.2.2), in the form

$$\dot{\psi} = \frac{\alpha - a\omega_3^0 u}{1-u^2}, \quad \dot{\varphi} = \omega_3^0 - \frac{(\alpha - a\omega_3^0 u)u}{1-u^2}. \qquad (15.2.8)$$

It results that $\dot{\psi}(t+T) = \dot{\psi}(t)$ and $\dot{\varphi}(t+T) = \dot{\varphi}(t)$, so that

$$\psi(t+T) = \psi(t) + \Psi_0, \quad \varphi(t+T) = \varphi(t) + \Phi_0, \qquad (15.2.8')$$

where Ψ_0 and Φ_0 are constant in time.

Research in this direction has been made by C.G.J. Jacobi, E. Lottner, O.I. Somov, C. Frenzel, A. Söderblom, J. Chrapan etc.

15.2.1.2 Motion of Precession. Regular Precession

As in Sect. 15.1.2.2, to can appreciate easier the motion of the rigid solid and to can make easier the determination of its position, we will consider the motion on the unit sphere of centre O of the point Q at which the movable Ox_3-axis pierces it (Fig.15.20a); the position of the point Q will be specified by the colatitude θ and by the longitude $\psi - \pi / 2$. The point Q will describe on the sphere a curve Γ of equation $\psi = \psi(\theta)$, contained between the parallels $\theta = \theta_1$ and $\theta = \theta_2$; taking into account (A.1.41') and adapting the notations, the element of arc on the curve will be given by the relation $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\psi^2 = \mathrm{d}s_\theta^2 + \mathrm{d}s_\psi^2$ (Fig. 15.20b). The angle V made by the curve Γ with a meridian circle at the point Q is given by $\tan V = \mathrm{d}s_\psi / \mathrm{d}s_\theta = \sin\theta\mathrm{d}\psi / \mathrm{d}\theta$. Due to $u = \cos\theta$, we have $\mathrm{d}\psi / \mathrm{d}\theta = -\sin\theta\mathrm{d}\psi / \mathrm{d}u = -\dot{\psi}\sin\theta\mathrm{d}t / \mathrm{d}u$; taking into account (15.2.3), (15.2.8), we may write

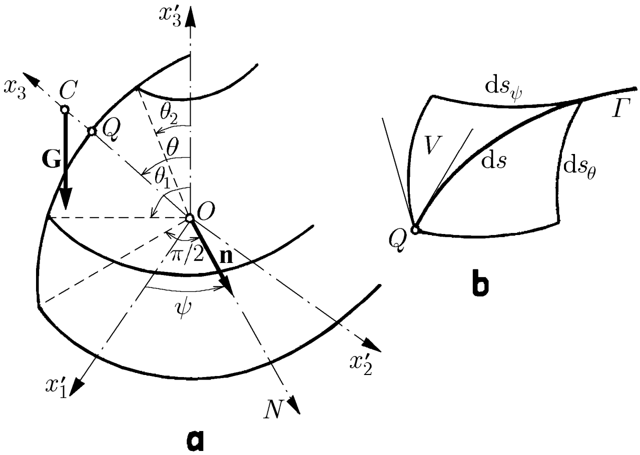


Fig. 15.20 Motion of precession on the unit sphere (**a**). The curve Γ on a spheric zone (**b**)

$$\tan V = \frac{\alpha - a\omega_3^0 u}{\sqrt{(1 - u^2)P(u)}}, \tag{15.2.9}$$

where we have no more mentioned the sign before the radical. Let us denote by $u' = \alpha / a\omega_3^0 = K'_{O3'} / I_3\omega_3^0 \neq \pm 1$ (corresponding to the previous hypothesis) the value of u which equates to zero $\tan V$; from (15.2.8) it results that $\dot{\psi}$ vanishes for $u = u'$ too. If $u' \in \complement[u_1, u_2]$, hence if u' is not a possible value for u, then $\dot{\psi}$ has always the same sign (it never vanishes), while ψ varies in the same sense (increasing or decreasing); the meridian plane is rotating in the same sense and the point Q describes the curve Γ (Fig.15.21a). We notice that for $u = u_1$ and $u = u_2$ we have $V = \pi / 2$, the curve being tangent to the parallels $\theta = \theta_1$ and $\theta = \theta_2$. If $u' \in (u_1, u_2)$, then $\dot{\psi}$ changes of sign when u passes through the value u'; the angle of precession varies in both senses, while the curve forms loops, being tangent to the parallels $\theta = \theta_1$ and $\theta = \theta_2$ (because $V = \pi / 2$) and normal to the parallel $\theta = \theta'$ (which corresponds to $u = u'$), because $\tan V = 0$. Obviously, the double points are on the same parallel and on the meridians which pass through the points of tangency of the curve Γ with the parallel $u = u'$ (Fig. 15.21b). It remains to consider the case in which $u' = u_1$ or $u' = u_2$; in this case $P(u') = 0$, while from (15.2.3) it results $u' = \beta / b$. We calculate the derivative $P'(u)$ starting from the relation (15.2.3) or from the relation (15.2.3"); in the first case, $P'(u_1) = -b(1 - u_1^2) < 0$ and $P'(u_2) = -b(1 - u_2^2) < 0$, while in the second case we have $P'(u_1) = b(u_2 - u_1)(u_3 - u_1) > 0$ and $P'(u_2) = -b(u_2 - u_1)(u_3 - u_2) < 0$. Hence, we cannot have $u' = u_1$, but only $u' = u_2$. The corresponding curve Γ is tangent to the parallel $\theta = \theta_1$. We notice that the difference $\alpha - a\omega_3^0 u$ is of the order of magnitude of $u_2 - u$, while $\sqrt{P(u)}$ is of the order of magnitude of $\sqrt{u_2 - u}$, so that $\tan V \to 0$ for $u \to u_2$, the curve Γ being tangent to the meridian circle on the parallel $\theta = \theta_2$. On the other hand, from (15.1.95) it results that $\operatorname{sign}\dot{\psi} = \operatorname{sign}(\alpha - a\omega_3^0 u) = \operatorname{sign}(u_2 - u) = 1$ for $u < u_2$, while $\dot{\psi}(u_2) = 0$; hence, the angle ψ is constantly increasing, the points at which the curve Γ reaches the parallel $\theta = \theta_2$ being cuspidal points (Fig. 15.21c).

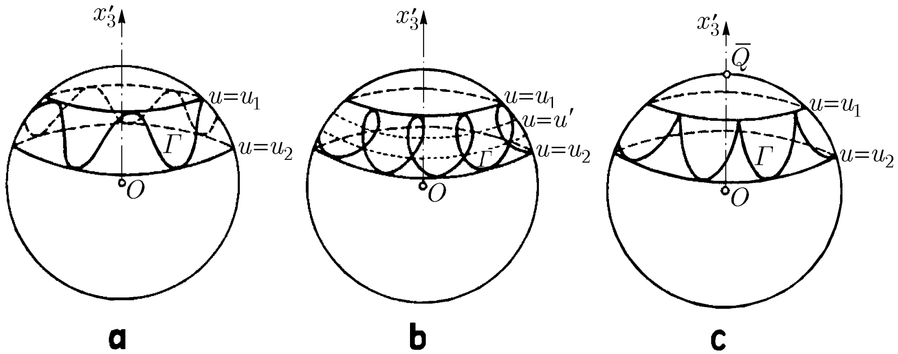


Fig. 15.21 The drawing of the curve Γ on a zone of the unit sphere. The cases $u' \in \complement[u_1, u_2]$ (**a**), $u' \in (u_1, u_2)$ (**b**) and $u' = u_1$ or $u' = u_2$ (**c**)

In conclusion, the point Q (hence the axis Ox_3 too) has a motion of nutation and a motion of precession between the two parallels $\theta = \theta_1$ and $\theta = \theta_2$, with the velocity $\mathbf{v} = \dot{\theta}\mathbf{n} + \dot{\psi}\sin\theta\mathbf{i}_3'$, $v^2 = \dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta$. Taking into account (15.2.8'), we can show that the curve Γ is, in general, an open curve, excepting the case in which Ψ_0 is commensurable with π.

It remains to consider also the case in which $u' = \alpha / a\omega_3^0 = K_{O3'}' / I_3\omega_3^0 = 1$. From (15.2.3) we notice that we can write

$$P(u) = (\beta - bu)(1 - u^2) - a^2(\omega_3^0)^2(u' - u)^2 .$$

Hence, if $u' = 1$, then we have $P(1) = 0$, so that $u_2 = \cos\theta_2 = 1$ and $\theta_2 = 0$, while the superior parallel $u = u_2$ is reduced to the piercing point $\overline{Q}$ of the Ox_3'-axis on the unit sphere; the curve Γ has the form drawn in Fig. 15.21c, the point $\overline{Q}$ being a multiple cuspidal point. Such a situation takes place, for instance, if the Ox_3-axis coincides with the Ox_3'-axis at the initial moment (hence $\theta_0 = 0$). We cannot have $u' = -1$ because, in this case, we would have $u_1 = -1$ too, hence $u' = u_1$, which we have seen that it is not possible.

The first formula (15.1.95) can be written also in the form

$$\mathrm{d}\psi = \frac{I_3\omega_3^0}{J}\frac{u' - u}{1 - u^2}\mathrm{d}t = \frac{I_3\omega_3^0}{J}\frac{u' - u}{1 - u^2}\frac{\mathrm{d}u}{\sqrt{P(u)}},$$

wherefrom

$$\frac{J\sqrt{b}}{I_3\omega_3^0}\Psi = \int_{u_1}^{u_2}\frac{u' - u}{1 - u^2}\frac{\mathrm{d}u}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}}, \tag{15.2.10}$$

$\Psi = \psi_2 - \psi_1$ representing the variation of the precession in the semi-period $T/2$. Projecting the curve Γ on the equatorial plane (normal to the Ox_3'-axis), one obtains a curve contained in a circular annulus and successively tangent to the limit circles; the angle at the centre in this plane, formed by two successive points of tangency, is the apsidal angle Ψ (the angle formed by two meridian planes which pass through two successive points of tangency to the parallels $u = u_1$ and $u = u_2$). If the above integral has a positive value (e.g., if $u' \geq u_2$), then $\operatorname{sign}(\psi_2 - \psi_1) = \operatorname{sign}\omega_3^0$, while if the integral has a negative value (e.g., if $u' < u_1$), then the precession verifies the relation $\operatorname{sign}(\psi_2 - \psi_1) = -\operatorname{sign}\omega_3^0$. Following a demonstration given by J.B. Diaz and F.T. Metcalf in 1962, if $u' \in (u_1, u_2)$, then we can write

$$\begin{aligned}
2J &= \int_{u_1}^{u_2}\frac{1 + u'}{1 + u}\frac{\mathrm{d}u}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}} - \int_{u_1}^{u_2}\frac{1 - u'}{1 + u}\frac{\mathrm{d}u}{\sqrt{(u - u_1)(u_2 - u)(u_3 - u)}} \\
&> \frac{1 + u'}{\sqrt{(1 + u_1)(1 + u_2)(1 + u_3)}}\int_{u_1}^{u_2}\frac{\sqrt{(1 + u_1)(1 + u_2)}\mathrm{d}u}{(1 + u)\sqrt{(u - u_1)(u_2 - u)}} \\
&\quad - \frac{1 - u'}{\sqrt{(1 - u_1)(1 - u_2)(u_3 - 1)}}\int_{u_1}^{u_2}\frac{\sqrt{(1 - u_1)(1 - u_2)}\mathrm{d}u}{(1 - u)\sqrt{(u - u_1)(u_2 - u)}},
\end{aligned}$$

where we have used the inequalities $\sqrt{u_3 - 1} < \sqrt{u_3 - u} < \sqrt{u_3 + 1}$. But from (15.1.90") one observes that

$$(1+u_1)(1+u_2)(1+u_3) = -\frac{P(-1)}{b} = \frac{\left(\alpha + a\omega_3^0\right)^2}{b} = \frac{a^2\left(\omega_3^0\right)^2(1+u')^2}{b};$$

analogously,

$$(1-u_1)(1-u_2)(u_3-1) = -\frac{P(1)}{b} = \frac{\left(\alpha - a\omega_3^0\right)^2}{b} = \frac{a^2\left(\omega_3^0\right)^2(1-u')^2}{b}.$$

It results thus

$$2J > \frac{\sqrt{b}}{a\omega_3^0}\left[\int_{u_1}^{u_2}\frac{\sqrt{(1+u_1)(1+u_2)}\mathrm{d}u}{(1+u)\sqrt{(u-u_1)(u_2-u)}} - \int_{u_1}^{u_2}\frac{\sqrt{(1-u_1)(1-u_2)}\mathrm{d}u}{(1-u_1)\sqrt{(u-u_1)(u_2-u)}}\right].$$

By the substitution $u = -v$ we notice that the second integral in the parenthesis is equal to the first one. Moreover, by the substitution $v = 1/(1+u)$ we have

$$\int\frac{\mathrm{d}u}{(1+u)\sqrt{(u-u_1)(u_2-u)}} = -\int\frac{\mathrm{d}v}{\sqrt{-(1+u_1)(1+u_2)v^2 + (2+u_1+u_2)v - 1}}$$
$$= \frac{1}{\sqrt{(1+u_1)(1+u_2)}}\arcsin\frac{2+u_1+u_2-2(1+u_1)(1+u_2)v}{u_2-u_1},$$

so that the two above equal integrals equate π; as a consequence, $J > 0$. We find thus again a proposition obtained in 1895 by J. Hadamard (corresponding to an affirmation of Halphen, based on fastidious demonstrations), by using the method of residues of the functions of complex variables; we can state

Theorem 15.2.1 (*Halphen-Hadamard*) *The apsidal angle* Ψ *has the same* sign *as the spin* ω_3^0 ($\operatorname{sign}\Psi = \operatorname{sign}\omega_3^0$) if $u' > u_1$ *and an opposite* sign ($\operatorname{sign}\Psi = -\operatorname{sign}\omega_3^0$) *if* $u' < u_1$.

We can show also that the sign of Ψ is the same as the sign of $\mathrm{d}\psi$ for $u = u_1$ (the lowest position of the point Q on the curve Γ).

Other researches in this direction have been made by A. Métral. Superior and inferior limits for the apsidal angle Ψ have been put in evidence by W. Kohn in 1946, being found again – using simpler methods – by Diaz and Metcalf in 1964.

Analogously, the second formula (15.2.8) allows to calculate the variation of the angle of proper rotation in the form ($\Phi = \varphi_2 - \varphi_1$)

$$\frac{\sqrt{b}}{\omega_3^0}\Phi = \int_{u_1}^{u_2}\left[1 - \frac{I_3}{J}\frac{(u'-u)u}{1-u^2}\right]\frac{\mathrm{d}u}{\sqrt{(u-u_1)(u_2-u)(u_3-u)}}. \qquad (15.2.10')$$

We assumed above that $u_1 < u_2$; in the case of a double root ($u_1 = u_2 = u_0$, Fig. 15.19b) we have $u(t) = u_0$, hence $\theta(t) = \theta_0$. From the relations (15.2.8) it results that $\dot{\psi}$ and $\dot{\varphi}$ are constant. Hence, the Ox_3-axis (rigidly linked to the rigid solid) describes a circular cone around the Ox_3'-axis, called cone of precession, with a constant angular velocity $\dot{\psi}$ (of precession), while the rigid solid is uniformly rotating (with the velocity of proper rotation $\dot{\varphi}$) about the Ox_3-axis. This is the case of a *regular precession* (a uniform motion of precession). The double root u_0 must verify the relations $P(u_0) = 0$ and $P'(u_0) = 0$; eliminating the differences $\beta - bu_0$ and $\alpha - a\omega_3^0 u_0$ and taking into account the relations (15.2.8) written for the initial moment ($\dot{\psi}_0 = \dot{\psi}(t_0)$ and $\dot{\varphi}_0 = \dot{\varphi}(t_0)$), we get the equivalent conditions

$$I_3\omega_3^0\dot{\psi}_0 - J\dot{\psi}_0^2\cos\theta = I_3\dot{\varphi}_0\dot{\psi}_0 - (J - I_3)\dot{\psi}_0^2\cos\theta_0 = Mg\rho_3\,, \tag{15.2.11}$$

which must be verified by the initial conditions in the case of the regular precession. If, in the Euler-Poinsot case, the ellipsoid of inertia would be of rotation, then the motion of precession would be always regular; but, in the Lagrange-Poisson case (the ellipsoid of inertia being always of rotation), the regular precession takes place only for particular initial conditions. If the conditions (15.2.11) are only approximately verified (e.g., in the motion of the Earth, when the angle θ is no more constant), then the motion is called *pseudoregular precession*.

As a matter of fact, we must mention that also other motions for which the imposed integrability conditions are only approximately verified have been considered. Thus, A. Pignedoli studied a pseudocase Lagrange-Poisson, in which the mass centre C is no more on the Ox_3-axis but is very close to this one, the properties of symmetry with respect to this axis being, as well, verified only approximately; A. Pignedoli dealt with an Euler-Poinsot pseudocase too for a heavy rigid solid, the centre C of which is very close to the fixed point O.

15.2.1.3 Particular Case. Analogy with the Spherical Pendulum

In the particular case in which the spin vanishes ($\omega_3^0 = 0$), the rotation angular velocity vector is contained, at any moment, in the plane Ox_1x_2 and has no component along the Ox_3-axis. The equation (15.2.3) reads

$$\dot{u}^2 = P(u)\,,\quad P(u) = (\beta - bu)\left(1 - u^2\right) - \alpha^2\,. \tag{15.2.12}$$

If we denote $u = -z/l$, where $l = J/M\rho_3$, then we find again the equation (7.1.62) of the spherical pendulum, in its motion on the sphere of centre O and radius l, studied in Chap. 7, Sect. 1.3.7; obviously, we must find a convenient interpretation for the corresponding constants. In the relation which specifies the change of variable appears the sign –, because the Oz-axis is along the descendent vertical, unlike the Ox_3'-axis taken along the ascendent vertical.

We also notice that $u' \to \pm\infty$, corresponding to the sign of $K'_{O3'}$, so that we are in the case of the curve Γ in Fig. 15.21a, as we have seen in the preceding subsection. The formula (15.2.10) is replaced by

$$\frac{J\sqrt{b}}{K'_{O3'}}\Psi = \int_{u_1}^{u_2} \frac{\mathrm{d}u}{(1-u^2)\sqrt{(u-u_1)(u_2-u)(u_3-u)}} . \tag{15.2.13}$$

One observes easily that the apsidal angle Ψ has the same sign as the constant component $K'_{O3'}$ of the moment of momentum. As it has been mentioned in Chap. 7, Sect. 1.3.7, in conformity with the results given by V. Puiseux in 1842 and by G. Halphen in 1885, we have $\pi/2 < \psi < \pi$. Results in this direction have been given by A. de Saint-Germain, L. Gérard, A. Weinstein and W. Kohn too. Passing from the sphere to a surface of rotation, J.L Synge showed that one cannot put in evidence limits of the apsidal angle for an arbitrary such surface.

As well, the variation of the angle of proper rotation is given by

$$\frac{J\sqrt{b}}{K'_{O3'}}\Phi = -\int_{u_1}^{u_2} \frac{u\mathrm{d}u}{(1-u^2)\sqrt{(u-u_1)(u_2-u)(u_3-u)}} . \tag{15.2.13'}$$

In conclusion, *the motion of the point Q on the unit sphere is analogous to the motion of a heavy particle constrained to stay all the time on a sphere* (case of the spherical pendulum).

15.2.1.4 Problem of the Regular Precession in the General Case of Motion of the Rigid Solid with a Fixed Point

Considering Euler's equations, written – in the general case – in the form (15.1.11), we put the problem to determine the forces which must act on the rigid solid with a fixed point, so that its motion be a regular precession, associated by a uniform proper rotation. Hence, let be a rigid solid $\mathscr{S}$ with a fixed point O, which has a motion of uniform proper rotation about the movable Ox_3-axis, with the angular velocity $\bar{\boldsymbol{\omega}}$, the Ox_3-axis having – at its turn – a uniform motion of precession about the fixed axis Ox'_3, with the angular velocity $\boldsymbol{\omega}'$. The angular velocity vector will be thus specified by (Fig. 15.22)

$$\boldsymbol{\omega} = \bar{\boldsymbol{\omega}} + \boldsymbol{\omega}' = \bar{\omega}^0 \mathbf{i}_3 + \omega'^0 \mathbf{i}'_3 , \tag{15.2.14}$$

where $\bar{\omega}^0 = \dot{\varphi}_0$, $\omega'^0 = \dot{\psi}_0$ correspond to the initial moment $t = t_0$; in this case, the angle of precession and the angle of rotation will be given by

$$\varphi(t) = \bar{\omega}^0 (t - t_0) + \varphi_0 , \quad \psi(t) = \omega'^0 (t - t_0) + \psi_0 , \tag{15.2.14'}$$

φ_0 and ψ_0 corresponding to the initial moment $t = t_0$. The angle of nutation $\theta = \sphericalangle(\boldsymbol{\omega}', \bar{\boldsymbol{\omega}}) = \sphericalangle(\mathbf{i}'_3, \mathbf{i}_3)$ is the constant angle formed by the unit vectors of the axes Ox'_3 and Ox_3, being given by

$$\theta(t) = \theta_0 , \tag{15.2.14''}$$

where θ_0 is the nutation at the initial moment $t = t_0$. In this case, the relations (5.2.35) give the components of the vector $\boldsymbol{\omega}$ along the axes of the movable frame of reference $\mathscr{R}$ in the form

$$\begin{aligned} \omega_1(t) &= \omega'^0 \sin\theta_0 \sin\left(\varpi^0 (t - t_0) + \varphi_0\right), \\ \omega_2(t) &= \omega'^0 \sin\theta_0 \cos\left(\varpi^0 (t - t_0) + \varphi_0\right), \\ \omega_3(t) &= \omega'^0 \cos\theta_0 + \varpi^0 . \end{aligned} \tag{15.2.15}$$

Knowing the components of the tensor of inertia $\mathbf{I}_O$ with respect to the frame $\mathscr{R}$, the formulae (15.1.11') allow to calculate the moment of the given forces with respect to the pole O, characterizing thus the loading which leads to the considered motion.

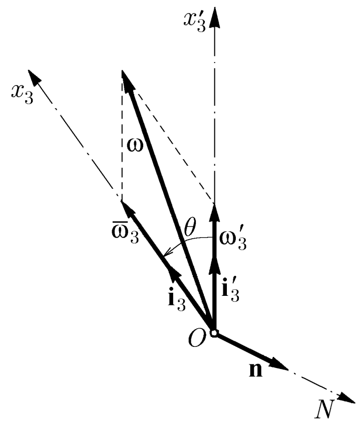


Fig. 15.22 Canonical decomposition of the vector $\boldsymbol{\omega}$ in the problem of the regular precession in the general motion of a rigid solid with a fixed point

In the particular case in which the Ox_3-axis is a principal axis of inertia of the rigid solid $\mathscr{S}$ with respect to the pole O (if $I_{31} = I_{32} = 0$), we can assume that the axes Ox_1 and Ox_2 are the other principal axes of inertia. Replacing the components of the vector $\boldsymbol{\omega}$ in the relations (15.1.11'') of Euler, we find the components of the moment $\mathbf{M}_O$ in the form

$$\begin{aligned} M_{O1}(t) &= \left[(I_1 - I_2 + I_3)\varpi^0 - (I_2 - I_3)\omega'^0 \cos\theta_0\right]\omega'^0 \\ &\quad \cdot \sin\theta_0 \cos\left(\varpi^0 (t - t_0) + \varphi_0\right), \\ M_{O2}(t) &= -\left[(I_2 + I_3 + I_1)\varpi^0 - (I_1 - I_3)\omega'^0 \cos\theta_0\right]\omega'^0 \\ &\quad \cdot \sin\theta_0 \sin\left(\varpi^0 (t - t_0) + \varphi_0\right), \\ M_{O3}(t) &= -\frac{1}{2}(I_1 - I_2)\left(\omega'^0\right)^2 \sin^2\theta_0 \sin 2\left(\varpi^0 (t - t_0) + \varphi_0\right). \end{aligned} \tag{15.2.16}$$

If the ellipsoid of inertia is an ellipsoid of rotation with respect to the Ox_3-axis (a prolate spheroid for which $I_1 = I_2 = J$), then we are led to

$$M_{O1}(t) = \left[I_3\varpi^0 - (J - I_3)\omega'^0 \cos\theta_0\right]\omega'^0 \sin\theta_0 \cos\left(\varpi^0 (t - t_0) + \varphi_0\right),$$
$$M_{O2}(t) = -\left[I_3\varpi^0 - (J - I_3)\omega'^0 \cos\theta_0\right]\omega'^0 \sin\theta_0 \sin\left(\varpi^0 (t - t_0) + \varphi_0\right),$$
$$M_{O3}(t) = 0. \tag{15.2.17}$$

In this case, the moment $\mathbf{M}_O$ is in the plane Ox_1x_2 (hence, $\mathbf{M}_O \perp \boldsymbol{\varpi}$) and one can easily see that $\mathbf{M}_O \cdot \boldsymbol{\omega} = M_{Oi}\omega_i = 0$ (hence $\mathbf{M}_O \perp \boldsymbol{\omega}$, so that we have $\mathbf{M}_O \perp \boldsymbol{\omega}'$ too); hence, the moment $\mathbf{M}_O$ is directed along the line of nodes ON (Fig. 15.22) and we can write

$$\mathbf{M}_O(t) = \left[I_3 - (J - I_3)\frac{\omega'^0}{\varpi^0}\cos\theta_0\right]\varpi^0\omega'^0 \sin\theta_0\,\mathbf{n}$$
$$= \left[I_3 - (J - I_3)\frac{\omega'^0}{\varpi^0}\cos\theta_0\right]\boldsymbol{\omega}' \times \boldsymbol{\varpi}. \tag{15.2.17'}$$

If the support of the angular velocity vector $\boldsymbol{\omega}$ is close to the Ox_3-axis, it results that – in a first approximation – one can neglect the ratio ω'^0 / ϖ^0 with respect to unity. We get thus

$$\mathbf{M}_O(t) = I_3\boldsymbol{\omega}' \times \boldsymbol{\varpi}, \tag{15.2.17''}$$

a formula useful in various applications; this formula can be used also if $(J - I_3)/I_3 << 1$, becoming an exact one if the ellipsoid of inertia is a sphere.

15.2.1.5 Geometric Representation of the Motion. Jacobi's Theorem

The first integrals (15.2.1'), (15.2.1") which appear in the motion of the rigid solid $\mathscr{S}$ with a fixed point, in the Lagrange-Poisson case, may be written also in the form

$$\omega_1\alpha_1 + \omega_2\alpha_2 + a\omega_3^0 u = \alpha, \quad \omega_1^2 + \omega_2^2 = \beta - bu, \quad \omega_3 = \omega_3^0, \tag{15.2.18}$$

using the notions previously introduced. Let be, as well, a rigid solid Σ which has a motion of constant rotation $(\varpi^0 - \omega_3^0)\mathbf{i}_3$, $\varpi^0 = \text{const}$, about the axis of dynamical symmetry of the rigid solid $\mathscr{S}$; if $\boldsymbol{\varpi}$ is the angular velocity of the rigid solid Σ with respect to the inertial frame of reference $\mathscr{R}'$, then we have $\varpi_1 = \omega_1$, $\varpi_2 = \omega_2$, $\varpi_3 = \omega_3^0 + (\varpi^0 - \omega_3^0) = \varpi^0$. We choose the angular velocity ϖ_3^0 so as to have $I_3\omega_3^0 = J\varpi_3^0$; in these conditions, the first integrals (15.2.18) become

$$\overline{\omega}_1 \alpha_1 + \overline{\omega}_2 \alpha_2 + \overline{\omega}^0 u = \alpha\,, \quad \overline{\omega}_1^2 + \overline{\omega}_2^2 = \overline{\beta} - bu\,, \quad \overline{\omega}_3 = \overline{\omega}^0\,, \tag{15.2.18'}$$

where we have introduced a new constant $\overline{\beta} = \left[2h - I_3 \left(\overline{\omega}_3^0 \right)^2 / a^2 \right] J$. If, in particular, the ellipsoid of inertia at the fixed point, in the Lagrange-Poisson case, is a sphere, then we have $a = 1$, the first integrals (15.2.18), (15.2.18'), corresponding to the motion of the rigid solid $\mathscr{S}$ or of the rigid solid Σ, respectively, having the same form.

The motion of the rigid solid Σ, considered independent of the rigid solid $\mathscr{S}$, takes place as it would be acted upon by its own weight, its ellipsoid of inertia being a sphere. We can thus state that, in general, the motion of a heavy rigid solid with a fixed point, in the Lagrange-Poisson case, may be obtained by the composition of the motion of a heavy rigid solid for which the ellipsoid of inertia is a sphere with a motion of constant rotation about the axis of symmetry of the ellipsoid of inertia of the considered rigid solid. Research concerning the geometric representation of the motion is due to C.G.J. Jacobi, E. Lottner, J.J. Sylvester, N.B. Delone etc.

Using the theory of conjugate motions in the sense of Darboux, one can show that the motion of a heavy rigid solid Σ with a fixed point, for which the ellipsoid of inertia is a sphere, may be obtained by the composition of a motion of Poinsot type with an inverse motion of Poinsot type. The demonstration of this theorem is particularly arduous; one puts in evidence, in a constructive form, the two motions, with arbitrary initial conditions.

We notice that the motion of rotation of the rigid solid $\mathscr{S}$ with respect to the rigid solid Σ may be considered as a motion of rotation about the normal to the fixed rolling plane in the motion of Poinsot type, component of the motion of the rigid solid Σ. Taking into account Sylvester's theorem (see Sect. 15.1.2.10), we can show that, by the composition of the motion of rotation considered above with a motion of Poinsot type, one obtains a motion of Poinsot type too. The previous results allow to state

Theorem 15.2.2 (*Jacobi*) *In general, the motion of a heavy rigid solid with a fixed point, in the Lagrange-Poisson case, can be obtained by the composition of a motion of Poinsot type with an inverse motion of Poinsot type.*

Studies in this direction have been made by E. Padova, G.H. Halphen, G. Darboux, W. Hess, E.J. Routh, R. Marcolongo, A.G. Greenhill and F. Kötter too.

15.2.2 The Sonya Kovalevsky Case

In 1888, a century after Lagrange's researches of 1788, Sonya Kovalevsky has been awarded by the Academy of Sciences of Paris for her studies concerning what we call now *the Sonya Kovalevsky case* (considered as the III*rd case of integrability*). In this case, the ellipsoid of inertia is of rotation, the centre of mass C is situated in its equatorial plane, the squares of the semi-axes in this plane being half of the square of the semi-axis corresponding to the axis of symmetry ($I_1 = I_2 = 2I_3$, $\rho_3 = 0$, the ellipsoid of inertia being a prolate spheroid); without any loss of generality (the axes in the equatorial plane are equivalent from the point of view of the properties of inertia), we can assume that $\rho_2 = 0$, the centre C being thus situated on the Ox_1-axis.

After obtaining the fourth first integral found by Kovalevsky, one passes to the systematic determination of the components of the angular velocity vector $\boldsymbol{\omega}$ and to the specification of the position of the rigid solid with respect to the inertial frame of reference.

15.2.2.1 First Integrals of the Motion

In the Sonya Kovalevsky case, the equations (15.1.21) read

$$2\dot{\omega}_1 - \omega_2\omega_3 = 0\,, \quad 2\dot{\omega}_2 + \omega_3\omega_1 = \gamma\alpha_3\,, \quad \dot{\omega}_3 = -\gamma\alpha_2\,, \tag{15.2.19}$$

with $\gamma = Mg\rho_1 / I_3 = \mathrm{const}$, while the first integrals (15.1.42)–(15.1.44) take the form

$$\begin{gathered} 2(\omega_1\alpha_1 + \omega_2\alpha_2) + \omega_3\alpha_3 = 2\Omega\,, \\ 2(\omega_1^2 + \omega_2^2) + \omega_3^2 = -2(\gamma\alpha_1 - \bar{\gamma})\,, \\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\,, \end{gathered} \tag{15.2.20}$$

where we have introduced the constants $\Omega = K'_{O3'} / I_3$, $\bar{\gamma} = h / I_3$.

Amplifying the second equation (15.2.19) by $\mathrm{i} = \sqrt{-1}$ and summing with the first of these equations, we obtain

$$2\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1 + \mathrm{i}\omega_2) + \mathrm{i}\omega_3(\omega_1 + \mathrm{i}\omega_2) = \mathrm{i}\gamma\alpha_3\,.$$

Proceeding analogously with the equations (14.1.54), we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1 + \mathrm{i}\alpha_2) + \mathrm{i}\omega_3(\alpha_1 + \mathrm{i}\alpha_2) = \mathrm{i}\alpha_3(\omega_1 + \mathrm{i}\omega_2)\,.$$

If we eliminate α_3 between these relations, then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\omega_1 + \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 + \mathrm{i}\alpha_2)\right] = -\mathrm{i}\omega_3\left[(\omega_1 + \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 + \mathrm{i}\alpha_2)\right]$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left[(\omega_1 + \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 + \mathrm{i}\alpha_2)\right] = -\mathrm{i}\omega_3\,.$$

Replacing i by –i, we obtain an analogous relation; eliminating ω_3 between this relation and the previous one, it results

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left[(\omega_1 + \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 + \mathrm{i}\alpha_2)\right]\left[(\omega_1 - \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 - \mathrm{i}\alpha_2)\right] = 0\,.$$

Integrating, we get a fourth algebraic first integral (*the Kovalevsky integral*)

$$\left[(\omega_1 + \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 + \mathrm{i}\alpha_2) \right]\left[(\omega_1 - \mathrm{i}\omega_2)^2 - \gamma(\alpha_1 - \mathrm{i}\alpha_2) \right] = \gamma_0^2 , \quad \gamma_0 = \text{const} , \tag{15.2.21}$$

which can be written also in the form (separating the real parts from the imaginary ones in the right parentheses, one obtains a product of sum by difference)

$$\left(\omega_1^2 - \omega_2^2 - \gamma\alpha_1 \right)^2 + (2\omega_1\omega_2 - \gamma\alpha_2)^2 = \gamma_0^2 . \tag{15.2.21'}$$

The constants γ, $\overline{\gamma}$ and γ_0 introduced above have the dimension of an angular acceleration, while the constant Ω has the dimension of an angular velocity.

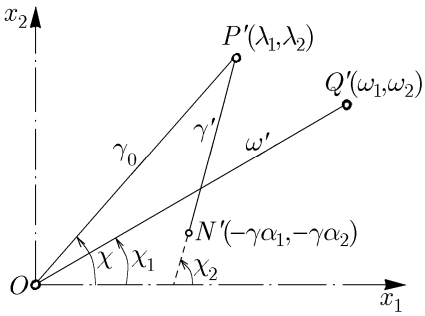


Fig. 15.23 The angle χ in the Sonya Kovalevsky case of integrability

Introducing the notations

$$\lambda_1 = \omega_1^2 - \omega_2^2 - \gamma\alpha_1 , \quad \lambda_2 = 2\omega_1\omega_2 - \gamma\alpha_2 , \quad \lambda_3 = 2\omega_1^2 + \frac{\omega_3^2}{2} , \tag{15.2.22}$$

we can express the first integral of the mechanical energy and the Kovalevsky integral in the form

$$\lambda_3 - \lambda_1 = \overline{\gamma} , \quad \lambda_1^2 + \lambda_2^2 = \gamma_0^2 . \tag{15.2.22'}$$

We notice thus that the trajectory of the representative point P of co-ordinates $\lambda_1 , \lambda_2 , \lambda_3$ is the intersection of a circular cylinder with a plane, hence an ellipse. The components of the velocity of this point are

$$\dot{\lambda}_1 = \omega_3\lambda_2 , \quad \dot{\lambda}_2 = -\omega_3\lambda_1 , \quad \dot{\lambda}_3 = \omega_3\lambda_2 , \tag{15.2.22''}$$

where we took into account the equations (14.1.54) and (15.2.19); therefore, we obtain $\lambda_1\dot{\lambda}_2 - \dot{\lambda}_1\lambda_2 = -\omega_3 \left(\lambda_1^2 + \lambda_2^2 \right) = -\gamma_0^2\omega_3$. Projecting the point P along the generatrix of the cylinder at P', on the plane Ox_1x_2, we introduce the angle χ made by the radius

OP' with the Ox_1-axis (Fig. 15.23); observing that $\tan\chi = \lambda_2 / \lambda_1$ and differentiating with respect to time (we have $\mathrm{d}\tan\chi/\mathrm{d}t = (1 + \tan^2\chi)\dot{\chi}$), we get

$$\dot{\chi} = -\omega_3 \,. \tag{15.2.22'''}$$

Hence, the motion of the point P' takes place as this one would be at the end of a rigid bar of length a, articulated at the fixed point, in the equatorial plane (considered to be perfectly smooth) of the ellipsoid of inertia. The point would participate to the rotations ω_1 and ω_2, without being involved in the rotation ω_3; the angular velocity of the bar with respect to the rigid solid is thus $-\omega_3\mathbf{i}_3$.

Let N be the extremity of the vector $-\gamma\mathbf{i}_3'$ and N' the projection of this point on the Ox_1x_2-plane; in this case, $\lambda_1 + \gamma\alpha_1 = \gamma'\cos\chi_2$, $\lambda_2 + \gamma\alpha_2 = \gamma'\sin\chi_2$, where $\gamma' = |\overrightarrow{N'P'}|$, while $\chi_2 = \sphericalangle\left(\overrightarrow{N'P'}, Ox_1\right)$. As well, let Q be the extremity of the vector $\boldsymbol{\omega}$ applied at O and Q' its projection on the Ox_1x_2-plane; denoting $\omega' = |\overrightarrow{OQ'}|$ and $\chi_1 = \sphericalangle\left(\overrightarrow{OQ'}, Ox_1\right)$, it results $\omega'^2\cos 2\chi_1 = \gamma'\cos\chi_2$, $\omega'^2\sin 2\chi_1 = \gamma'\sin\chi_2$. One obtains thus

$$\omega'^2 = \gamma', \quad 2\chi_1 = \chi_2 \,. \tag{15.2.23}$$

The third relation (15.2.22) and the relations (15.2.23) allow to write

$$\omega_3^2 = 2\lambda_3 - 2\gamma'(1 + \cos\chi_2) \,. \tag{15.2.23'}$$

Using the relations (5.2.35), (5.2.36), we may express the first two first integrals (15.2.20) by means of Euler's angles; we get

$$\begin{gathered} 2\dot{\psi}\sin^2\theta + (\dot{\varphi} + \dot{\psi}\cos\theta)\cos\theta = 2\Omega \,, \\ 2(\dot{\theta} + \dot{\psi}^2\sin^2\theta) + (\dot{\varphi} + \dot{\psi}\cos\theta)^2 = -2(\gamma\sin\theta\sin\varphi - \overline{\gamma}) \,. \end{gathered} \tag{15.2.24}$$

As well, the Kovalevsky first integral may be written in the form

$$\begin{gathered} \left[(\dot{\theta}^2 - \dot{\psi}^2\sin^2\theta)\cos 2\varphi + 2\dot{\theta}\dot{\psi}\sin\theta\sin 2\varphi - \gamma\sin\theta\sin\varphi\right]^2 \\ +\left[2\dot{\theta}\dot{\psi}\sin\theta\cos 2\varphi - (\dot{\theta}^2 - \dot{\psi}^2\sin^2\theta)\sin 2\varphi - \gamma\sin\theta\cos\varphi\right]^2 = \gamma_0^2 \,, \end{gathered}$$

wherefrom

$$\begin{gathered} (\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta)^2 + 2\gamma\left[(\dot{\theta}^2 - \dot{\psi}^2\sin^2\theta)\sin\varphi \right. \\ \left. -2\dot{\theta}\dot{\psi}\sin\theta\cos\varphi\right]\sin\theta + \gamma^2\sin^2\theta = \gamma_0^2 \,. \end{gathered} \tag{15.2.24'}$$

Thus, we have at our disposal three differential equations for the three unknown functions (Euler's angles).

The system of equations (14.1.54), (15.2.19) contains the time only under the differential operator; taking into account the results in Sects. 15.1.1.4 and 15.1.1.5, we can affirm that this system, of the form (15.1.22), may be replaced by a system of only five differential equations, of the form (15.1.24), for which we know the four first integrals (15.2.20), (15.2.21'). Using Jacobi's last multiplier theory, one obtains the fifth first integral. Hence, the complete system of first integrals of the system of equations (14.1.54), (15.2.19) is obtained with the aid of two quadratures. Euler's angles θ and φ are then given by the relations (5.2.36) in the form

$$\theta = \arccos \alpha_3 , \quad \varphi = \arctan \frac{\alpha_1}{\alpha_2} , \tag{15.2.25}$$

while the angle ψ results from the third relation (5.2.35) (we use the equations (14.1.54) and the first first integral (15.2.20))

$$\begin{aligned} \dot{\psi} &= \frac{1}{\cos\theta}(\omega_3 - \dot{\varphi}) = \frac{1}{\alpha_3}\left(\omega_3 - \frac{\alpha_1\dot{\alpha}_2 - \alpha_2\dot{\alpha}_1}{\alpha_1^2 + \alpha_2^2}\right) \\ &= \frac{\alpha_1\omega_1 + \alpha_2\omega_2}{\alpha_1^2 + \alpha_2^2} = \frac{\Omega - \omega_3\alpha_3 / 2}{1 - \alpha_3^2} = \frac{\Omega - (\omega_3 / 2)\cos\theta}{\sin^2\theta} , \end{aligned} \tag{15.2.25'}$$

by a quadrature.

The general problem put by Sonya Kovalevsky has been considered again by N.E. Jukovskiĭ, G.K. Suslov, F. Kötter, G.V. Kolosov and W. von Tannenberg; the particular case $\gamma_0 = 0$ has been studied by N.B. Delone, G.G. Appelrot and B.K. Mlodzevenski. Various researchers hoped that one can use the method introduced by Kovalevsky also for other problems, expressing – conveniently – the given conservative forces by means of certain potential functions; but E. Padova showed in 1895 that this is not possible, the problem being always reduced – from the mathematical point of view – to the problem studied by Kovalevsky.

15.2.2.2 Changes of Variables

The problem is put to give to the first integrals a convenient form, so that the two quadratures which must be effected have a much simpler form. Introducing the complex variables

$$x_1 = \omega_1 + \mathrm{i}\omega_2 , \quad x_2 = \omega_1 - \mathrm{i}\omega_2 , \quad \xi_1 = \lambda_1 + \mathrm{i}\lambda_2 , \quad \xi_2 = \lambda_1 - \mathrm{i}\lambda_2 , \tag{15.2.26}$$

Sonya Kovalevsky succeeds to express all the searched quantities by means of some hyperelliptic functions of the first kind. Therefore, it results

$$2\omega_1 = x_1 + x_2 , \quad 2\omega_2 = \mathrm{i}(x_2 - x_1) , \quad 2\lambda_1 = \xi_1 + \xi_2 , \quad 2\lambda_2 = \mathrm{i}(\xi_2 - \xi_1) ,$$

and the relations (15.2.22) give

$$2\gamma\alpha_1 = x_1^2 + x_2^2 - (\xi_1 + \xi_2), \quad 2\gamma\alpha_2 = \mathrm{i}\left(x_2^2 - x_1^2\right) - \mathrm{i}(\xi_2 - \xi_1).$$

Taking into account the first two equations (15.2.19), we find the differential equations which are verified by the functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$ in the form

$$2\mathrm{i}\dot{x}_1 = \omega_3 x_1 - \gamma\alpha_3, \quad -2\mathrm{i}\dot{x}_2 = \omega_3 x_2 - \gamma\alpha_3. \tag{15.2.26'}$$

The first integrals (15.2.20) take the form

$$\begin{gathered} x_1 x_2 (x_1 + x_2) - (x_1\xi_2 + x_2\xi_1) + \gamma\omega_3\alpha_3 = 2\gamma\Omega, \\ (x_1 + x_2)^2 - (\xi_1 + \xi_2) + \omega_3^2 = 2\gamma, \\ x_1^2 x_2^2 - \left(x_1^2\xi_2 + x_2^2\xi_1\right) + \gamma^2\alpha_3^2 = \gamma^2 - \gamma_0^2, \end{gathered} \tag{15.2.27}$$

where we took into account the Kovalevsky integral

$$\xi_1\xi_2 = \gamma_0^2. \tag{15.2.27'}$$

We amplify successively the first two equations (15.2.27) by $-2x_1$, $-2x_2$, $-(x_1 + x_2)$ and by $x_1^2, x_2^2, x_1 x_2$, respectively; summing then with the third equation (15.2.27), we obtain

$$\begin{gathered} (\omega_3 x_1 - \gamma\alpha_3)^2 = P(x_1) + \xi_1 (x_1 - x_2)^2, \\ (\omega_3 x_2 - \gamma\alpha_3)^2 = P(x_2) + \xi_2 (x_1 - x_2)^2, \\ (\omega_3 x_1 - \gamma\alpha_3)(\omega_3 x_2 - \gamma\alpha_3) = R(x_1, x_2), \end{gathered} \tag{15.2.28}$$

where we have denoted

$$\begin{gathered} P(x) = Ax^2 + 2Bx + C, \quad R(x_1, x_2) = Ax_1 x_2 + B(x_1 + x_2) + C, \\ A = 2\overline{\gamma} - (x_1 + x_2)^2, \quad B = -2\gamma\Omega + x_1 x_2 (x_1 + x_2), \\ C = \gamma^2 - \gamma_0^2 - x_1^2 x_2^2. \end{gathered} \tag{15.2.29}$$

Eliminating ξ_1 and ξ_2 between the first two equations (15.2.28) and the equation (15.2.27'), we find

$$\left[(\omega_3 x_1 - \gamma\alpha_3)^2 - P(x_1)\right]\left[(\omega_3 x_2 - \gamma\alpha_3)^2 - P(x_2)\right] = \gamma_0^2 (x_1 - x_2)^2. \tag{15.2.28'}$$

From this equation and the third equation (15.2.28), we can express the binomials $\omega_3 x_1 - \gamma\alpha_3$ and $\omega_3 x_2 - \gamma\alpha_3$ as functions of x_1 and x_2, finding thus the differential equations of first order verified by the functions $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Eliminating the above mentioned binomials between the relations (15.2.28) and taking into account (15.2.27'), we find the relation

$$\xi_1 P(x_2) + \xi_2 P(x_1) + \gamma_0^2 (x_1 - x_2)^2 = Q(x_1, x_2) = B^2 - AC. \tag{15.2.30}$$

The identity

$$P(x_1)P(x_2) + (x_1 - x_2)^2 Q(x_1, x_2) = R^2(x_1, x_2) \tag{15.2.30'}$$

holds too.

The equations (15.2.27') and (15.2.29) may be written also in the form

$$\left[\sqrt{\xi_1}\sqrt{P(x_2)}\right]^2 + \left[\sqrt{\xi_2}\sqrt{P(x_1)}\right]^2 = Q(x_1, x_2) - \gamma_0^2 (x_1 - x_2)^2, \quad \sqrt{\xi_1}\sqrt{\xi_2} = \gamma_0,$$

wherefrom

$$\begin{aligned}\left[\sqrt{\xi_1}\sqrt{P(x_2)} \pm \sqrt{\xi_2}\sqrt{P(x_1)}\right]^2 &= Q(x_1, x_2) \\ &\pm 2\sqrt{P(x_1)}\sqrt{P(x_2)}\gamma_0 - \gamma_0^2 (x_1 - x_2)^2.\end{aligned}$$

We may write

$$\begin{aligned}\left[\sqrt{\xi_1}\frac{\sqrt{P(x_2)}}{x_1 - x_2} \pm \sqrt{\xi_2}\frac{\sqrt{P(x_1)}}{x_1 - x_2}\right]^2 &= (w_1 \mp \gamma_0)(w_2 \pm \gamma_0) \\ &= w_1 w_2 \pm \gamma_0 (w_1 - w_2) - \gamma_0^2\end{aligned} \tag{15.2.31}$$

too, where

$$w_{1,2} = \frac{1}{(x_1 - x_2)^2}\left[R(x_1, x_2) \pm \sqrt{P(x_1)}\sqrt{P(x_2)}\right] \tag{15.2.31'}$$

are the roots of the equation (we take into account also the identity (15.2.30'))

$$w^2 - 2\frac{R(x_1, x_2)}{(x_1 - x_2)^2} w + \frac{Q(x_1, x_2)}{(x_1 - x_2)^2} = 0, \tag{15.2.31''}$$

which has always real roots for x_1, x_2 complex conjugate numbers.

15.2.2.3 Reduction of the Problem to Ultraelliptic Integrals

We introduce also the polynomial

$$\overline{P}(x) = -x^4 + 2\overline{\gamma}x^2 - 4\gamma\Omega x + \gamma^2 - \gamma_0^2, \tag{15.2.29'}$$

which has the properties $\overline{P}(x_j) = P(x_j)$, $j = 1, 2$. Following Weierstrass's theory concerning the elliptic integrals, one can verify the relations

$$
\begin{aligned}
-\frac{\mathrm{d}x_1}{\sqrt{P(x_1)}}+\frac{\mathrm{d}x_2}{\sqrt{P(x_2)}}&=\frac{\mathrm{d}s_1}{\sqrt{2\varphi(s_1)}},\\
\frac{\mathrm{d}x_1}{\sqrt{P(x_1)}}+\frac{\mathrm{d}x_2}{\sqrt{P(x_2)}}&=\frac{\mathrm{d}s_2}{\sqrt{2\varphi(s_2)}},
\end{aligned}
\tag{15.2.32}
$$

where

$$
\varphi(s)=s\left[(s-\overline{\gamma})^2+\gamma^2-\overline{\gamma}^2\right]-2\gamma^2\Omega^2 \tag{15.2.32'}
$$

is *the Euler resolvent* of the equation $\overline{P}(x)=0$ and where we have introduced *Sonya Kovalevsky's variables*

$$
s_1=w_1+\overline{\gamma}\,,\quad s_2=w_2+\overline{\gamma}\,. \tag{15.2.32''}
$$

The zeros e_1, e_2, e_3 of the polynomial $\varphi(s)$ can be expressed by means of the zeros $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ of the polynomial $P(x)$ in the form

$$
e_1=\frac{1}{2}(\varepsilon_1+\varepsilon_2)^2,\quad e_2=\frac{1}{2}(\varepsilon_1+\varepsilon_3)^2,\quad e_3=\frac{1}{2}(\varepsilon_1+\varepsilon_4)^2. \tag{15.2.33}
$$

We mention that one may use *the Aronhold resolvent* too in the form

$$
\begin{gathered}
\psi(\overline{s})=4\overline{s}^3-g_2\overline{s}-g_3\,,\\
g_2=-\left(\gamma^2-\gamma_0^2\right)+\frac{1}{3}\overline{\gamma}^2,\quad g_3=-\frac{1}{3}\overline{\gamma}\left(\gamma^2-\gamma_0^2\right)+\gamma^2\Omega^2-\frac{1}{27}\overline{\gamma}^3.
\end{gathered}
\tag{15.2.34}
$$

Starting from the equations (15.2.31) and using the Kovalevsky variables and the notations

$$
e_4=\overline{\gamma}+\gamma_0\,,\quad e_5=\overline{\gamma}-\gamma_0\,, \tag{15.2.33'}
$$

we obtain

$$
\begin{aligned}
2\sqrt{\xi_1}\,\frac{\sqrt{P(x_2)}}{x_1-x_2}&=\sqrt{(s_1-e_4)(s_2-e_5)}+\sqrt{(s_1-e_5)(s_2-e_4)}\,,\\
2\sqrt{\xi_2}\,\frac{\sqrt{P(x_1)}}{x_1-x_2}&=\sqrt{(s_1-e_4)(s_2-e_5)}-\sqrt{(s_1-e_5)(s_2-e_4)}\,.
\end{aligned}
\tag{15.2.35}
$$

Noting that

$$
4\frac{P(x_1)P(x_2)}{(x_1-x_2)^2}=(w_1-w_2)^2=(s_1-s_2)^2
$$

and taking into account the relations (15.2.35) and the identities

$$(s_1 - s_2)^2 + \left[\sqrt{(s_1 - e_4)(s_2 - e_5)} \pm \sqrt{(s_1 - e_5)(s_2 - e_4)}\right]^2$$
$$= \left[\sqrt{(s_1 - e_4)(s_1 - e_5)} \pm \sqrt{(s_2 - e_4)(s_2 - e_5)}\right]^2,$$

the relations (15.2.28) allow to write

$$\begin{aligned} \frac{\omega_3 x_1 - \gamma\alpha_3}{\sqrt{P(x_1)}} &= \frac{1}{s_1 - s_2}\left[\sqrt{(s_1 - e_4)(s_1 - e_5)} + \sqrt{(s_2 - e_4)(s_2 - e_5)}\right], \\ \frac{\omega_3 x_2 - \gamma\alpha_3}{\sqrt{P(x_2)}} &= \frac{1}{s_1 - s_2}\left[\sqrt{(s_1 - e_4)(s_1 - e_5)} - \sqrt{(s_2 - e_4)(s_2 - e_5)}\right]. \end{aligned} \tag{15.2.36}$$

Taking into account the relations (15.2.26'), (15.2.32) and (15.2.36), we get

$$\begin{aligned} \frac{\mathrm{d}s_1}{\sqrt{2\varphi(s_1)}} &= \frac{\mathrm{i}}{s_1 - s_2}\sqrt{(s_1 - e_4)(s_1 - e_5)}\mathrm{d}t\,, \\ \frac{\mathrm{d}s_2}{\sqrt{2\varphi(s_2)}} &= -\frac{\mathrm{i}}{s_1 - s_2}\sqrt{(s_2 - e_4)(s_2 - e_5)}\mathrm{d}t\,. \end{aligned}$$

Introducing the polynomial of fifth degree

$$\Phi(s) = (s - e_4)(s - e_5)\varphi(s) = (s - e_1)(s - e_2)(s - e_3)(s - e_4)(s - e_5) \tag{15.2.37}$$

and effecting a linear combination, it results

$$\begin{aligned} \frac{\mathrm{d}s_1}{\sqrt{\Phi(s_1)}} + \frac{\mathrm{d}s_2}{\sqrt{\Phi(s_2)}} &= 0\,, \\ \frac{s_1\mathrm{d}s_1}{\sqrt{\Phi(s_1)}} + \frac{s_2\mathrm{d}s_2}{\sqrt{\Phi(s_2)}} &= \sqrt{2}\mathrm{i}\mathrm{d}t\,. \end{aligned} \tag{15.2.37'}$$

We can apply Jacobi's theory of the last multiplier to this system of differential equations. If

$$F(s) = \int \frac{\mathrm{d}s}{\sqrt{\Phi(s)}}, \quad G(s) = \int \frac{s\mathrm{d}s}{\sqrt{\Phi(s)}}, \tag{15.2.38}$$

then we get

$$F(s_1) + F(s_2) = C_1, \quad G(s_1) + G(s_2) = -\mathrm{i}t + C_2\,, \tag{15.2.38'}$$

where C_1 and C_2 are integration constants. Finally, it results $s_1 = s_1(t; C_1, C_2)$, $s_2 = s_2(t; C_1, C_2)$.

If an integrand is a rational function of the form $S\left(s, \sqrt{\Phi(s)}\right)$, where $\Phi(s)$ is a polynomial of degree greater than four, the corresponding integral is called, in general, *hyperelliptic integral*; if the degree of the polynomial is five or six, the respective integrals are called also *ultraelliptic integrals*. *In the Sonya Kovalevsky case, the* *problem of the rotation of a heavy rigid solid about a fixed point is reduced thus to the inversion of a system of two ultraelliptic integrals.*

If the roots e_1, e_2, e_3 are distinct and if $\varphi'(s)$ is the derivative of the function $\varphi(s)$, then one can calculate the components of the vector $\boldsymbol{\omega}$, contained in the equatorial plane of the ellipsoid of inertia, in the form

$$\omega_1 = -\frac{\displaystyle\sum_{\alpha=1}^{3} \frac{\sqrt{e_\beta e_\gamma}}{\varphi'(e_\alpha)} P_\alpha}{\displaystyle\sqrt{2}\sum_{\alpha=1}^{3} \frac{\sqrt{e_\alpha}}{\varphi'(e_\alpha)} P_\alpha}, \quad \omega_2 = \frac{\mathrm{i}}{\displaystyle\sqrt{2}\sum_{\alpha=1}^{3} \frac{\sqrt{e_\alpha}}{\varphi'(e_\alpha)} P_\alpha},$$
$$\alpha \neq \beta \neq \gamma \neq \alpha, \quad \beta < \gamma, \quad \alpha, \beta, \gamma = 1,2,3, \tag{15.2.39}$$

where we have introduced the functions

$$P_\alpha = \sqrt{(s_1 - e_\alpha)(s_2 - e_\alpha)}, \quad \alpha = 1,2,3, \tag{15.2.39'}$$

symmetric with respect to s_1 and s_2, hence which can be expressed by means of *the theta elliptic functions*. Analogously, we get

$$\omega_3 = \sqrt{2}\frac{\displaystyle\sum_{\alpha=1}^{3} \frac{\sqrt{e_\alpha}}{\varphi'(e_\alpha)} P_{\beta\gamma}}{\displaystyle\sum_{\alpha=1}^{3} \frac{\sqrt{e_\alpha}}{\varphi'(e_\alpha)} P_\alpha}, \quad \alpha_3 = -\frac{1}{\gamma}\frac{\displaystyle\sum_{\alpha=1}^{3} \frac{\sqrt{e_\beta e_\gamma}}{\varphi'(e_\alpha)} P_{\beta\gamma}}{\displaystyle\sum_{\alpha=1}^{3} \frac{\sqrt{e_\alpha}}{\varphi'(e_\alpha)} P_\alpha},$$
$$\alpha \neq \beta \neq \gamma \neq \alpha, \quad \beta < \gamma, \quad \alpha, \beta, \gamma = 1,2,3, \tag{15.2.40}$$

with

$$P_{\beta\gamma} = P_{\gamma\beta} = \frac{P_\beta P_\gamma}{s_1 - s_2}\left[\frac{\sqrt{\Phi(s_1)}}{(s_1 - e_\beta)(s_1 - e_\gamma)} - \frac{\sqrt{\Phi(s_2)}}{(s_2 - e_\beta)(s_2 - e_\gamma)}\right],$$
$$\beta \neq \gamma, \quad \beta, \gamma = 1,2,3. \tag{15.2.40'}$$

The direction cosines are then given by the first two first integrals (15.2.20).

15.2.3 Other Cases of Integrability

The success of Sonya Kovalevsky is due, especially, to a new more general formulation of the problem of motion of the rigid solid with a fixed point by means of the concepts of the theory of analytic functions of complex variable. The general solution can be expressed, in this case, with the aid of the elliptic functions of the time t; the elliptic functions are uniform analytic functions for all finite values of t, excepting some points in the complex plane t, in which these functions have poles of first order. In this order of ideas, Kovalevsky has worked to the problem of finding all the cases in which the general solution, which contains five arbitrary constants, is uniform and has no other singularities, excepting the poles, for all the finite values of t, t being a complex variable; this solution has been searched in the form of the expansion into a power series

$$\omega_i = \frac{1}{t^{n_i}} \sum_{n=0}^{\infty} \omega_n^{(i)} t^n \,, \quad \alpha_i = \frac{1}{t^{m_i}} \sum_{n=0}^{\infty} \alpha_n^{(i)} t^n \,, \quad n_i\,, m_i \in \mathbb{N}\,, \quad i = 1,2,3\,, \tag{15.2.41}$$

where the coefficients $\omega_n^{(i)}$ and $\alpha_n^{(i)}$ must satisfy some conditions. Sonya Kovalevsky has chosen the values $n_i = 1$ and $m_i = 2$, $i = 1,2,3$, the problem of uniqueness of this system of values remaining open.

Kovalevsky said that only the three cases of integrability considered above are possible, as well as the case of kinetic symmetry in which the ellipsoid of inertia is a sphere ($I_1 = I_2 = I_3 = I$). In the latter case, starting from Euler's equation (15.1.21") and by means of a scalar product by $\boldsymbol{\rho}$, we may write $\mathrm{d}[(\mathbf{I}_O \boldsymbol{\omega}) \cdot \boldsymbol{\rho}]/\mathrm{d}t = 0$, wherefrom $(\mathbf{I}_O \boldsymbol{\omega}) \cdot \boldsymbol{\rho} = \mathrm{const}$; the tensor $\mathbf{I}_O$ being spheric, we have

$$\boldsymbol{\omega} \cdot \boldsymbol{\rho} = \omega_j \rho_j = \Omega h\,, \quad \Omega, h = \mathrm{const}\,, \tag{15.2.42}$$

hence a fourth algebraic first integral. Taking into account that any co-ordinate axis is a principal axis of inertia, that case is a Lagrange-Poisson one. These statements have been justified, using other methods than those of Sonya Kovalevsky, by A.M. Lyapunov and G.G. Appelrot, the general theorems concerning the uniformity of the solutions being considered in what follows.

We will present also the most important particular cases of integrability, as well as other cases of loading in the dynamics of the rigid solid with a fixed point.

15.2.3.1 General Theorems Concerning the Uniformity of the Solution

We consider that the four cases of integrability mentioned above (the Euler-Poinsot case, the Lagrange-Poisson case, the Sonya Kovalevsky case and the case of kinetic symmetry) are the classical cases of integrability. In what concerns the uniformity of the solution, we can – firstly – state

Theorem 15.2.3 (*S. Kovalevsky*) *In general, excepting the classical cases of integrability, the equations of Euler and Poisson do not allow uniform solutions which depend on five arbitrary constants and have not other singularities excepting poles in the whole complex plane t.*

We notice that Sonya Kovalevsky did not consider also the case in which the solutions can be uniform, having essential singularities besides the poles. Lyapunov showed that such solutions cannot be uniform on the whole complex plane t (excepting the four mentioned cases) because, by a convenient choice of the initial values, they can become multiform functions of the time t. We can thus state

Theorem 15.2.4 (*A.M. Lyapunov*) *If the principal moments of inertia I_j, $j = 1,2,3$, are real and non-zero quantities and if the co-ordinates ρ_i, $i = 1,2,3$, of the centre of* *mass are real quantities too, then the classical cases of integrability are the only cases in which the functions* $\omega_i(t)$ *and* $\alpha_i(t)$, $i = 1,2,3$, *are uniform functions of the time* t *for arbitrary initial values of these unknowns.*

We consider the particular solutions

$$\omega_i = \frac{a_i}{t}, \quad \alpha_i = \frac{b_i}{t^2}, \quad i = 1,2,3, \tag{15.2.43}$$

where the constants a_i, b_i are not all zero; replacing in the equations of Euler and Poisson, we find the conditions

$$\begin{aligned} I_1 a_1 + (I_2 - I_3) a_2 a_3 &= Mg(b_3\rho_2 - b_2\rho_3), \\ I_2 a_2 + (I_3 - I_1) a_3 a_1 &= Mg(b_1\rho_3 - b_3\rho_1), \\ I_3 a_3 + (I_1 - I_2) a_1 a_2 &= Mg(b_2\rho_1 - b_1\rho_2), \end{aligned} \tag{15.2.43'}$$

$$2b_i + \in_{ijl} b_j a_l = 0, \quad i = 1,2,3. \tag{15.2.43''}$$

If the ellipsoid of inertia relative to the pole O is not of rotation, one uses the particular solution

$$\begin{gathered} a_1 = -\mathrm{i}\sqrt{\frac{I_2 I_3}{(I_1 - I_2)(I_1 - I_3)}}, \quad a_2 = -\sqrt{\frac{I_3 I_1}{(I_1 - I_2)(I_2 - I_3)}}, \\ a_3 = -\mathrm{i}\sqrt{\frac{I_1 I_2}{(I_1 - I_3)(I_2 - I_3)}}, \end{gathered} \tag{15.2.44}$$

$$b_1 = b_2 = b_3 = 0. \tag{15.2.44'}$$

In the case in which the ellipsoid of inertia is of rotation, one of the components ρ_1, ρ_2, ρ_3 vanishes and we can use the solution (for $\rho_2 = 0$)

$$a_1 = a_3 = 0, \quad a_2 = 2\mathrm{i}, \quad b_1 = b_3\mathrm{i} = \frac{2I_2\mathrm{i}}{Mg(\rho_3 + \mathrm{i}\rho_1)}, \quad b_2 = 0. \tag{15.2.45}$$

If $\rho_1 = 0$ or $\rho_3 = 0$, then one uses analogous solutions.

To the initial moment $t = t_0$ correspond the initial conditions $\omega_i = \omega_i^0$ and $\alpha_i = \alpha_i^0$, $i = 1,2,3$; in this case, the general solution of the system (15.1.21), (14.1.54) varies continuously with these parameters. At a variation of at least one of these parameters correspond variations of the solution, which will be of the form $\omega_i + \delta\omega_i$, $\alpha_i + \delta\alpha_i$, $i = 1,2,3$. These new solutions must verify the considered system of equations; subtracting now the latter system thus obtained, we find the conditions which must be verified by the variations $\delta\omega_i$ and $\delta\alpha_i$, $i = 1,2,3$, in the form

$$I_1 \frac{\mathrm{d}(\delta\omega_1)}{\mathrm{d}t} + (I_3 - I_2)(\omega_2\delta\omega_3 + \omega_3\delta\omega_2) = Mg(\rho_3\delta\alpha_2 - \rho_2\delta\alpha_3),$$

$$
\begin{aligned}
&I_2 \frac{\mathrm{d}(\delta\omega_2)}{\mathrm{d}t} + (I_1 - I_3)(\omega_3 \delta\omega_1 + \omega_1 \delta\omega_3) = Mg(\rho_1 \delta\alpha_3 - \rho_3 \delta\alpha_1), \\
&I_3 \frac{\mathrm{d}(\delta\omega_3)}{\mathrm{d}t} + (I_2 - I_1)(\omega_1 \delta\omega_2 + \omega_2 \delta\omega_1) = Mg(\rho_2 \delta\alpha_1 - \rho_1 \delta\alpha_2),
\end{aligned} \tag{15.2.46}
$$

$$
\frac{\mathrm{d}(\delta\alpha_i)}{\mathrm{d}t} + \in_{ijk} \left(\omega_j \delta\alpha_k + \alpha_k \delta\omega_j \right) = 0, \quad i = 1,2,3. \tag{15.2.46'}
$$

Replacing the particular solutions (15.2.43), we obtain the system

$$
\begin{aligned}
&I_1 t \frac{\mathrm{d}(\delta\omega_1)}{\mathrm{d}t} + (I_3 - I_2)(a_2 \delta\omega_3 + a_3 \delta\omega_2) = Mgt(\rho_3 \delta\alpha_2 - \rho_2 \delta\alpha_3), \\
&I_2 t \frac{\mathrm{d}(\delta\omega_2)}{\mathrm{d}t} + (I_1 - I_3)(a_3 \delta\omega_1 + a_1 \delta\omega_3) = Mgt(\rho_1 \delta\alpha_3 - \rho_3 \delta\alpha_1), \\
&I_3 t \frac{\mathrm{d}(\delta\omega_3)}{\mathrm{d}t} + (I_2 - I_1)(a_1 \delta\omega_2 + a_2 \delta\omega_1) = Mgt(\rho_2 \delta\alpha_1 - \rho_1 \delta\alpha_2),
\end{aligned} \tag{15.2.47}
$$

$$
t^2 \frac{\mathrm{d}(\delta\alpha_i)}{\mathrm{d}t} + \in_{ijk} \left(a_j t \delta\alpha_k + b_k \delta\omega_j \right) = 0, \quad i = 1,2,3. \tag{15.2.47'}
$$

This system allows non-zero solutions of the form $\delta\omega_i = k_i t^k$, $\delta\alpha_i = k_{i+3} t^{k-1}$, $i = 1,2,3$ (the system of six linear equations in k_i, k_{i+3}, $i = 1,2,3$, has non-trivial solutions) if the exponent k is a root of the equation of six degree

$$
\varDelta \equiv \begin{vmatrix}
I_1 k & (I_3 - I_2) a_3 & (I_3 - I_2) a_2 & 0 & -Mg\rho_3 & Mg\rho_2 \\
(I_1 - I_3) a_3 & I_2 k & (I_1 - I_3) a_1 & Mg\rho_3 & 0 & -Mg\rho_1 \\
(I_2 - I_1) a_2 & (I_2 - I_1) a_1 & I_3 k & -Mg\rho_2 & Mg\rho_1 & 0 \\
0 & b_3 & -b_2 & k-1 & -a_3 & a_2 \\
-b_3 & 0 & b_1 & a_3 & k-1 & -a_1 \\
b_2 & -b_1 & 0 & -a_2 & a_1 & k-1
\end{vmatrix} = 0. \tag{15.2.48}
$$

We notice that ω_i and α_i are uniform functions if the variations $\delta\omega_i$ and $\delta\alpha_i$ have the same property too; in this case, it is necessary and sufficient that the roots of the equation (15.2.48) be integers ($k \in \mathbb{Z}$) and that a multiple root of mth order equates to zero all the minors of an order greater than m of the determinant $\varDelta$.

We assume firstly that $I_1 > I_2 > I_3 > 0$. Using the solution (15.2.44), (15.2.44'), the equation (15.2.48) takes the form (product of two minors of third order)

$$
k(k-1)^3 \left(k^2 - 4 \right) = 0. \tag{15.2.48'}
$$

The roots of this equation are $0, 1, \pm 2 \in \mathbb{Z}$; on the other hand, all the minors of fourth and fifth order must vanish for the triple root $k = 1$. Thus, the minor of fourth order, obtained by eliminating the second and the third columns and the third and the fourth lines, leads to (for $k = 1$)

$$\begin{vmatrix} I_1 & 0 & -Mg\rho_3 & Mg\rho_2 \\ (I_1 - I_3)a_3 & Mg\rho_3 & 0 & -Mg\rho_1 \\ 0 & a_3 & 0 & -a_1 \\ 0 & -a_2 & a_1 & 0 \end{vmatrix} = 0 .$$

Developing and taking into account (15.2.44), we get

$$\rho_1 \sqrt{I_1 (I_2 - I_3)} + \rho_2 \sqrt{I_2 (I_3 - I_1)} + \rho_3 \sqrt{I_3 (I_1 - I_2)} = 0 . \tag{15.2.49}$$

The middle term being imaginary (we take into account the relation of order of the moments of inertia), it results

$$\rho_2 = 0 , \quad \rho_1 \sqrt{I_1 (I_2 - I_3)} + \rho_3 \sqrt{I_3 (I_1 - I_2)} = 0 . \tag{15.2.49'}$$

This case, which has not been noticed by Kovalevsky and which has been put in evidence by Appelrot in 1892, has been found for the first time by W. Hess in 1890. Afterwards, N.E. Jukovskiĭ, B.K. Mlodzeevski and P.A. Nekrasov have made studies in this direction, the latter one showing that for $\rho_1, \rho_3 \neq 0$ one obtains multiform solutions of the time t if one chooses convenient values for the initial conditions. Indeed, for $\rho_2 = 0$ and for the solutions (15.2.45), the equation (15.2.48) leads to (permuting lines and columns, to can represent Δ as a product of two determinants of third order)

$$\begin{vmatrix} I_2 k & Mg\rho_3 & -Mg\rho_1 \\ \dfrac{2I_2}{Mg(\rho_3 + \mathrm{i}\rho_1)} & k-1 & 2\mathrm{i} \\ -\dfrac{2I_2}{Mg(\rho_3 + \mathrm{i}\rho_1)} & -2\mathrm{i} & k-1 \end{vmatrix} \begin{vmatrix} I_1 k & 2\mathrm{i}(I_3 - I_2) & -Mg\rho_3 \\ 2\mathrm{i}(I_2 - I_1) & I_3 k & Mg\rho_1 \\ -\dfrac{2I_2}{Mg(\rho_3 + \mathrm{i}\rho_1)} & \dfrac{2I_2 \mathrm{i}}{Mg(\rho_3 + \mathrm{i}\rho_1)} & k-1 \end{vmatrix} = 0 .$$

Developing, we obtain

$$[I_2 (k-1)(k+2)(k-3)] \Big\{ (k-2) \Big[I_1 I_3 k(k+1) + 2(I_1 - I_2)(2I_2 - I_3) - 2I_2 (I_1 - I_3) \frac{\rho_1 (\rho_1 + \mathrm{i}\rho_3)}{\rho_1^2 + \rho_3^2} \Big] \Big\} = 0 . \tag{15.2.50}$$

Equating to zero the second right parenthesis, we get $k \in \mathbb{C}$ for $\rho_1, \rho_3 \neq 0$, in the hypothesis made concerning the relation of order of the principal moments of inertia one can have uniform solutions only for $\rho_1 = \rho_2 = \rho_3 = 0$, hence only in the Euler-Poinsot case.

Assuming now that the ellipsoid of inertia relative to the fixed point is of rotation (to fix the ideas, let be $I_1 = I_2 = J$), we can take $\rho_2 = 0$ without any loss of generality; making $\rho_2 = 0$ in (15.2.48), we find again the equation (15.2.50). The equation in k obtained by equating to zero the second right parenthesis has real roots if we have also $I_1 = I_3$ (case of kinetic symmetry) or if we have $\rho_1 = 0$ too (Lagrange-Poisson case) or, finally, if $\rho_3 = 0$; in this last case, it results $k(k+1) = 2(J/I_3 - 1)$, so that we must have $J/I_3 \in \mathbb{N}$. If $\rho_3 = 0$, then the system (15.2.43'), (15.2.43") allows also the solution

$$a_1 = a_2 = 0\,, \quad a_3 = 2\mathrm{i}\,, \quad b_2 = b_1\mathrm{i} = \frac{2I_3\mathrm{i}}{Mg(\rho_1 + \mathrm{i}\rho_2)}\,, \quad b_3 = 0\,, \tag{15.2.45'}$$

corresponding the equation of second degree in k

$$I_1 I_2 k(k+1) - 2(I_2 - I_3)(I_1 - 2I_3) + 2I_3(I_1 - I_2)\frac{\rho_2(\rho_2 + \mathrm{i}\rho_1)}{\rho_1^2 + \rho_2^2} = 0\,. \tag{15.2.50'}$$

If $I_1 = I_2 = J$, then this equation is reduced to

$$\left(k + 2 - \frac{2I_3}{J}\right)\left(k - 1 + \frac{2I_3}{J}\right) = 0\,, \tag{15.2.50"}$$

so that we must have $2I_3/J \in \mathbb{N}$ too. These conditions can be fulfilled simultaneously only if $J/I_3 = 1$ (case of kinetic symmetry) or if $J/I_3 = 2$ (Sonya Kovalevsky case).

From the study made for the three cases of integrability it results that the solutions thus obtained are uniform, the above obtained necessary conditions of uniformity being sufficient too. The Theorems 15.2.3 and 15.2.4 are thus completely justified. Lyapunov showed that these results hold also for arbitrary real initial conditions, as well as in the case of initial conditions of the form (15.1.19'), which verify the condition $\alpha_i^0 \alpha_i^0 = 1$.

A great number of scientists searched for a century (till now) various cases of integrability, corresponding to particular initial conditions (restrictions imposed to the energy constant h, to the constant $K'_{O3'}$ of the moment of momentum etc.). In a unfinished manuscript, S.A. Chaplygin tried to find all these cases of integrability by a unique method; as a matter of fact, this was not possible till now. One can say that the literature in this direction is one of special cases.

15.2.3.2 The Hess's Case. The Loxodromic Pendulum

W. Hess has considered in 1890 the case in which: (i) the mass centre of the rigid solid is on the normal at the fixed point to one of the planes of circular section of the ellipsoid of gyration, assuming that the latter one is not of rotation; (ii) at the initial moment, the moment of momentum is situated in the respective plane of circular section. This case has been studied again by G.G. Appelrot, P.A. Nekrasov, B.K. Mlodzeevski, N.E. Jukovskiĭ, S.A. Chaplygin, R. Liouville and others, due to its importance. T. Manacorda considered in 1950 the case in which the moment of the given external forces with respect to the fixed point is normal to the straight line OC, concluding that the moment of momentum $\mathbf{K}'_O$ must have the same property (verifying the considerations of Hess and Appelrot).

The equations of the planes of circular section are obtained by the intersection of the ellipsoid of gyration (15.1.64) with the sphere $x_1^2 + x_2^2 + x_3^2 = I_2 / M$ (the radius of which is the mean semi-axis of the ellipsoid of gyration), being thus given by

$$x_1^2\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + x_3^2\left(\frac{1}{I_3} - \frac{1}{I_2}\right) = 0\,.$$

Let

$$x_1\sqrt{I_3(I_1 - I_2)} - x_3\sqrt{I_1(I_2 - I_3)} = 0$$

be the equation of one of these planes; putting the condition that the vector $\boldsymbol{\rho}$ be normal to this plane (corresponding to the hypothesis i)), we find again the relations (15.2.49').

Euler's equations (15.1.21) become

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= Mg\rho_3\alpha_2\,, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= Mg(\rho_1\alpha_3 - \rho_3\alpha_1)\,, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= -Mg\rho_1\alpha_2\,. \end{aligned} \tag{15.2.51}$$

Eliminating α_2 between the first and the third equations, it results

$$I_1\rho_1\dot{\omega}_1 + I_3\rho_3\dot{\omega}_3 = \omega_2[(I_2 - I_3)\rho_1\omega_3 + (I_1 - I_2)\rho_3\omega_1]\,.$$

We notice that the second relation (15.2.49') can be decomposed in the form

$$I_1 - I_2 = CI_1\rho_1^2\,,\quad I_2 - I_3 = CI_3\rho_3^2\,,\quad C = \text{const}$$

being thus led to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_1\rho_1\omega_1 + I_3\rho_3\omega_3) = C\rho_1\rho_3\omega_2(I_1\rho_1\omega_1 + I_3\rho_3\omega_3)\,.$$

We obtain a fourth first integral

$$I_1\rho_1\omega_1 + I_3\rho_3\omega_3 = 0 \tag{15.2.52}$$

if we have

$$I_1\rho_1\omega_1^0 + I_3\rho_3\omega_3^0 = 0 \tag{15.2.52'}$$

at the initial moment $t = t_0$. The components of the moment of momentum at the initial moment $\mathbf{K}_O'^0$ being $I_1\omega_1^0, I_2\omega_2^0, I_3\omega_3^0$, we notice that the relation (15.2.52') is equivalent to $\mathbf{K}_O'^0 \cdot \boldsymbol{\rho} = 0$, corresponding to the hypothesis (ii).

By particularizing the conditions (15.2.49'), we obtain the Euler-Poinsot case ($\rho_1 = \rho_2 = \rho_3 = 0$) or the Lagrange-Poisson case ($I_1 = I_2$, $\rho_1 = \rho_2 = 0$); unlike these cases, in which the initial conditions are arbitrary, in *Hess's case* the initial conditions must verify the relation (15.2.52'), so that one does not obtain a general solution for a certain repartition of masses, but only a particular one.

N.E. Jukovskiĭ gave an interesting geometric interpretation to Hess's motion. We notice that, in the frame of reference $\mathscr{R}'$, the velocity $\mathbf{v}_C' = \boldsymbol{\omega}\times\boldsymbol{\rho}$ of the mass centre has the components

$$v_{C1}' = \omega_2\rho_3\,,\quad v_{C2}' = \omega_3\rho_1 - \omega_1\rho_3\,,\quad v_{C3}' = -\omega_2\rho_1\,.$$

If

$$\mathbf{K}_O'^{(C)} = \boldsymbol{\rho}\times(M\mathbf{v}_C')\,,\quad \mathbf{K}_O' = I_1\omega_1\mathbf{i}_1 + I_2\omega_2\mathbf{i}_2 + I_3\omega_3\mathbf{i}_3\,,$$
$$T'^{(C)} = \frac{1}{2}Mv_C^2\,,\quad T' = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right),$$

where $\mathbf{K}_O'^{(C)}$ and $T'^{(C)}$ are the moment of momentum with respect to the pole O and the kinetic energy of the mass centre, at which we suppose that the whole mass of the rigid solid is concentrated, respectively, and if we take into account the second condition (15.2.49') and the first integral (15.2.52), then we find the relation

$$\mathbf{K}_O'^{(C)} = \varepsilon^2\mathbf{K}_O'\,,\quad T'^{(C)} = \varepsilon^2 T'\,,\quad \varepsilon = \frac{\rho}{i_2}\,,\quad i_2 = \sqrt{\frac{I_2}{M}}\,. \tag{15.2.53}$$

Noting that we can write $\rho_{3'} = \mathbf{i}_3' \cdot \boldsymbol{\rho} = \rho_j\alpha_j$ in the frame of reference $\mathscr{R}'$ and introducing the polar co-ordinates $\rho_{C'}, \psi$ for the projection C' of C on the fixed plane $Ox_1'x_2'$, we express the first integrals (15.1.42'), (15.1.43') in the form

$$\rho_{C'}^2\dot\psi = \bar K_{O3'}'\,,\quad \frac{1}{2}Mv_C'^2 = -M\bar g\rho_{3'} + \bar h\,,\quad \bar K_{O3'}' = \varepsilon^2 K_{O3'}'\,,\quad \bar h = \varepsilon^2 h\,,\quad \bar g = \varepsilon^2 g\,. \tag{15.2.53'}$$

Taking into account the results in Chap. 7, Sect. 1.3.7, we see that *the centre of mass C is moving as a spherical pendulum acted upon by a gravitational field of conventional acceleration $\bar g$.*

To put in evidence the rotation of the rigid solid about the straight line OC, we consider the velocity $\mathbf{v}_A$ of the extremity $A(0, i_2, 0)$ of the mean axis of the ellipsoid of gyration, of components $v_{A1} = -i_2 \omega_3$, $v_{A2} = 0$, $v_{A3} = i_2 \omega_1$ and of magnitude given by $v_A^2 = \left(\omega_1^2 + \omega_3^2 \right) i_2$. We obtain

$$v_A = i_2 \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2 + I_3)}{I_3 (I_2 - I_3)}}, \quad \sin\theta = \sqrt{\frac{I_1 I_3}{I_2 (I_1 - I_2 + I_3)}}, \tag{15.2.54}$$

where θ is the angle made by the velocity $\mathbf{v}_A$ with the plane of the circular section (given by $\sin\theta = \cos(\mathbf{v}_A, \operatorname{vers}\boldsymbol{\rho}) = \mathbf{v}_A \cdot \operatorname{vers}\boldsymbol{\rho} / v_A$). We notice that the angle θ is constant. Thus, the considered motion is entirely characterized.

If, in particular, *at the initial moment, the moment of momentum* $\mathbf{K}'_O$ *is horizontal*, then the constant of areas $K'_{O3'}$ vanishes and *the centre of mass* C *moves as a mathematical pendulum*; the plane of the circular section is rotating about a fixed horizontal straight line, normal to the trajectory of the centre C. The trajectory of the point A on the sphere $(0, i_2)$ is a loxodrome (with the characteristic property $\theta = \text{const}$). If the mass centre C has an oscillatory motion, then the point A oscillates on an arc of the loxodrome and if the centre C has an asymptotic motion, then the motion of the rigid solid tends asymptotically to a rotation about the mean axis of inertia. The rigid solid is, in this case, a *loxodromic pendulum*.

15.2.3.3 The Goryachev-Chaplygin case. The Merkalov case

D.N. Goryachev considered in 1900 the case in which $I_1 = I_2 = 4I_3$, the centre of mass being the plane of equal moments of inertia at O (the plane Ox_1x_2, hence $\rho_3 = 0$); without any loss of generality, we take $\rho_2 = 0$ too. One assumes that the moment of momentum $\mathbf{K}'_O$ is situated in the horizontal plane $Ox'_1x'_2$, hence $K'_{O3'} = 0$, which represents a particularization of the initial condition. In 1901, S.A. Chaplygin took back the problem, giving a solution which contains an arbitrary fourth constant. Results in this direction have given L.N. Sretenskiĭ and Yu.A. Arkhangelskiĭ too.

Euler's equations are of the form

$$4\dot{\omega}_1 = 3\omega_2\omega_3, \quad 4\dot{\omega}_2 + 3\omega_3\omega_1 = a\alpha_3, \quad \dot{\omega}_3 = -a\alpha_2, \quad a = \frac{Mg\rho_1}{I_3}. \tag{15.2.55}$$

The first integrals (15.1.42'), (15.1.43') become

$$\begin{gathered} 4(\omega_1\alpha_1 + \omega_2\alpha_2) + \omega_3\alpha_3 = \frac{K'_{O3'}}{I_3} = 0, \\ 4\left(\omega_1^2 + \omega_2^2\right) + \omega_3^2 = -2a\alpha_1 + \frac{2h}{I_3}. \end{gathered} \tag{15.2.55'}$$

Multiplying the first equations (15.2.55) by ω_1 and ω_2, respectively, we get $2\mathrm{d}(\omega_1^2 + \omega_2^2)/\mathrm{d}t = a\omega_2\alpha_3$; taking into account also the third equation (15.2.55), it results

$$\frac{\mathrm{d}}{\mathrm{d}t}[\omega_3(\omega_1^2 + \omega_2^2)] = -a\alpha_2(\omega_1^2 + \omega_2^2) + \frac{1}{2}a\alpha_3\omega_2\omega_3 .$$

Using the equations (15.1.54) and the first equation (15.2.55), we have $\mathrm{d}(\alpha_3\omega_1)/\mathrm{d}t = 3\alpha_3\omega_2\omega_3/4 + \omega_1(\alpha_1\omega_2 - \alpha_2\omega_1)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\omega_3(\omega_1^2 + \omega_2^2) - a\alpha_3\omega_1] = -a\omega_2\left(\omega_1\alpha_1 + \omega_2\alpha_2 + \frac{\omega_3\alpha_3}{4}\right);$$

the first first integral (15.2.55') leads then to the fourth first integral

$$\omega_3(\omega_1^2 + \omega_2^2) - a\alpha_3\omega_1 = C , \quad C = \text{const} . \tag{15.2.55''}$$

By convenient changes of variable, the problem can be reduced to the calculation of some hyperelliptic integrals. In 1902, R. Marcolongo showed that ω_i, α_i, $i = 1,2,3$, can be expressed by means of theta functions of two arguments, each function being linear in the time t.

Later, in 1946, N.I. Merkalov introduced a new variable η, with $\mathrm{d}\eta/\mathrm{d}t = \omega_2$, obtaining a fourth first integral in the form

$$(\omega_1^2 + \omega_2^2)\frac{\mathrm{d}\omega_1}{\mathrm{d}\eta} - \frac{3}{2}\omega_1\frac{\mathrm{d}}{\mathrm{d}\eta}(\omega_1^2 + \omega_2^2) = -\frac{3K'_{O3'}\rho_1}{4I_1I_3}\eta + C , \quad C = \text{const} , \tag{15.2.56}$$

corresponding to the case in which $K'_{O3'} \neq 0$. If $K'_{O3'} = 0$, then we find again the first integral (15.2.55'') (we take into account the first equation (15.2.55) and the relation $2\mathrm{d}(\omega_1^2 + \omega_2^2)/\mathrm{d}t = a\omega_2\omega_3$).

15.2.3.4 The Bobylev-Steklov Case

In this case, which has been – independently – put in evidence by D. Bobylev and V.A. Steklov in 1896, one assumes that $I_1 = 2I_2$, the centre of mass being on the Ox_1-axis ($\rho_2 = \rho_3 = 0$, $\rho_1 > 0$). Euler's system of equations becomes

$$\begin{aligned} 2I_2\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 , \\ I_2\dot{\omega}_2 + (2I_2 - I_3)\omega_3\omega_1 &= Mg\rho_1\alpha_3 , \\ I_3\dot{\omega}_3 - I_2\omega_1\omega_2 &= -Mg\rho_1\alpha_2 . \end{aligned} \tag{15.2.57}$$

We obtain the particular solutions (we take into account the equations (14.1.54))

$$\omega_1 = \omega_1^0 = \text{const} , \quad \omega_2 = k\alpha_2 , \quad \omega_3 = 0 , \quad k = \frac{Mg\rho_1}{I_2\omega_1^0} = \text{const} . \tag{15.2.57'}$$

In this case, the equations (14.1.54) take the form

$$\dot{\alpha}_1 = -k\alpha_2\alpha_3\,, \quad \dot{\alpha}_2 = \omega_1^0\alpha_3\,, \quad \dot{\alpha}_3 = \alpha_2\left(k\alpha_1 - \omega_1^0\right). \tag{15.2.58}$$

We get the first integral

$$2\omega_1^0\alpha_1 + k\alpha_2^2 = C\,, \quad C = \text{const}\,, \tag{15.2.58'}$$

using the first two above equations. Taking into account also (15.1.44), we can write

$$\alpha_3^2 = 1 - \alpha_2^2 - \left(\lambda - \mu\alpha_2^2\right)^2, \quad \lambda = \frac{C}{2\omega_1^0}\,, \quad \mu = \frac{k}{2\omega_1^0}$$

and, replacing in the second equation (15.2.58), we are led to the elliptic integral

$$t = t_0 + \int_{\alpha_2^0}^{\alpha_2} \frac{\mathrm{d}\xi}{\omega_1^0\sqrt{1 - \xi^2 - \left(\lambda - \mu\xi^2\right)^2}}\,, \tag{15.2.58''}$$

the problem being thus completely solved.

15.2.3.5 Other cases of integrability of Goryachev, Steklov and Chaplygin

At the beginning of XXth century, Goryachev, Steklov and Chaplygin have put the problem to find new cases of integrability, where the first integrals, independent of time, be algebraical, not containing arbitrary constants. Thus, assuming that $\rho_2 = \rho_3 = 0$, Chaplygin searched the conditions in which there exist integrals of the form

$$-\rho_1\alpha_2 = \delta\omega_1\omega_2 + \lambda\omega_1^n\omega_2\,, \quad -\rho_1\alpha_3 = \varepsilon\omega_1\omega_3 + \mu\omega_1^n\omega_3 \tag{15.2.59}$$

for the equations of Euler and Poisson; the constants δ, ε, λ and μ remain to be determined.

Goryachev considered in 1898 the case in which $\lambda = 0$ and $n = 3$. Assuming that $I_1 I_3 = 8(I_1 - 2I_2)(I_2 - I_3)$, he put into evidence the existence of the integrals (15.2.59), with

$$\delta = -\frac{3I_1 - 4I_2}{2}\,, \quad \varepsilon = -\frac{2\delta(2I_2 - I_3)(2I_2 - 3I_3)}{I_2 I_3}\,,$$
$$\mu = \frac{\delta\varepsilon I_1(4I_2 - 3I_3)(4I_2 - 5I_3)}{16\rho_1 I_2 I_3(I_2 - I_3)}\,, \tag{15.2.60}$$

which contain only one arbitrary parameter.

In 1899, Steklov studied the case in which $\lambda = \mu = 0$, finding

$$(2I_3 - I_1)\delta = (2I_2 - I_1)\varepsilon = (I_3 - I_1)(I_2 - I_1)\,, \tag{15.2.61}$$

in the hypothesis in which $I_2 > I_1 > 2I_3$. The components $\omega_1, \omega_2, \omega_3$ of the vector $\boldsymbol{\omega}$ are, in this case, in direct proportion to $\operatorname{cn}\kappa t$, $\operatorname{sn}\kappa t$ and $\operatorname{dn}\kappa t$, respectively, where κ depends on I_1, I_2, I_3 and ρ_1.

Chaplygin assumed in 1904 that $n = -1/3$ and that the relation $9(I_1 - 2I_2)(I_1 - 2I_3) = 4I_2 I_3$ takes place, with $0.5965 < I_3 / I_1 < 0.6000$, $1.5000 < I_2 / I_1 < 1.5965$. He showed that the integrals (15.2.59) exist for

$$\begin{gathered}(2I_3 - I_1)\delta = (2I_2 - I_1)\varepsilon = (I_2 - I_1)(I_3 - I_1), \\ \lambda = \frac{I_3(3I_1 - 2I_2)}{2I_3 - I_1}s, \quad \mu = \frac{I_2(3I_1 - 2I_3)}{2I_2 - I_1}s,\end{gathered} \tag{15.2.62}$$

where s verifies the equation

$$I_1^3[2(I_2 + I_3) - I_1]s^3 = \frac{4(2I_2 - I_1)^2(2I_3 - I_1)^2}{9(3I_1 - 2I_2)(3I_1 - 2I_3)}\rho_1^2. \tag{15.2.62'}$$

By a transformation of Hermite type, the solution of the problem can be represented with the aid of the elliptic integrals.

Chaplygin showed also that these are the only cases in which one can find integrals of the form (15.2.59).

15.2.3.6 Other Particular Cases of Integrability

N. Kovalevski considered in 1908 the case in which the mass centre is on the principal axis Ox_1 ($\rho_2 = \rho_3 = 0$), searching all the cases in which ω_2^2 and ω_3^2 can be expressed in the form of polynomials of the third degree in ω_1. Thus, he found again the particular cases studied by Goryachev, Steklov and Chaplygin (see Sect. 15.2.3.5), as well as a new case in which the relation $I_1(9I_2 - 10I_3) = 18I_2(I_2 - I_3)$ takes place. Much later, in 1932–1934, J.J. Corliss assumes also that $\rho_2 = \rho_3 = 0$, but imposes the condition $K'_{O3'} = 0$ too, limiting thus the generality of the initial conditions. A particular case of the above one has been considered by P. Field in the same period of time. He deals also with the interesting case in which I_2 is close to I_1, while I_3 is small, so that the ratio $(I_1 - I_2)/I_3$ be practically indeterminate.

P.V. Myasnikov proposed in 1954 a new method to find cases of integrability in a unitary mode, assuming that the centre of mass C lies in the characteristic plane, determined by the vectors $\boldsymbol{\omega}$ and $\mathbf{K}'_O$; in this case, one obtains a fourth first integral of the form

$$I_1\rho_1\omega_1 + I_2\rho_2\omega_2 + I_3\rho_3\omega_3 = \text{const}. \tag{15.2.63}$$

If the point C is situated at the same time in the characteristic plane and on one of the principal axes of inertia at O, one obtains again various classical cases previously considered (Euler-Poinsot, Lagrange-Poisson, Bobylev-Steklov etc.).

In 1950, T. Manacorda and A. Nadile have considered the planar case of motion of a rigid solid with a fixed point (after P. Stäckel's terminology), hence the case in which the mass centre is situated in one of the principal planes of inertia; there have been thus put in evidence various relations which take place, as well as the cases of integrability which can be obtained.

In 1959, E.I. Harlamova formed a new case of integrability of the equations of Euler and Poisson, assuming that $\rho_2 = 0$, $\rho_1 < 0$,

$$\rho_1 (2I_3 - I_1) \sqrt{I_1 (I_3 - I_2)(2I_3 - I_1)} + \rho_3 (I_3 - 2I_1) \sqrt{I_3 (I_1 - I_2)(I_3 - 2I_1)} = 0$$

and $I_3 > 2I_1 > 2I_2$; but the last hypothesis is not possible from a physical point of view. M.P. Gulyev showed in 1961 that this solution is however possible in the case of a body filled up with an incompressible perfect fluid. This is the only case of regular precession with respect to a vertical line, dynamically possible.

The possibility to find linear integrals has been investigated by P.V. Harlamov in 1962, being stimulated by Chaplygin's affirmation in conformity to which linear integrals other than those known at the respective moment do not exist, but which was afterwards contradicted by the discovery of such integrals.

A.N. Filatov introduced in 1963 the notion of generalized Lie series, which allowed the systematic determination of a fourth first integral in various known cases of integrability.

15.2.3.7 Permanent Axes of Rotation

In 1894, O. Stande and B.K. Mlodzeevski have shown – independently – that the equations of Euler and Poisson allow a simple infinity of solutions if no one restriction concerning the rotations of the rigid solid about fixed axes in the solid and in the space is imposed. Mlodzeevski showed that such permanent axes of rotation can be the principal axes of inertia, if these axes are horizontal (the case of the physical pendulum) or a family of vertical axes; in the latter case, studied in detail by Stande, the magnitude of the instantaneous angular velocity is constant.

In this order of ideas, we search a constant unit vector $\mathbf{i}'_3$ satisfying these equations. We notice that Poisson's equation is reduced to $\boldsymbol{\omega} \times \mathbf{i}'_3 = \mathbf{0}$ (because $\dot{\mathbf{i}}'_3 = \mathbf{0}$; hence,

$$\boldsymbol{\omega} = \pm \omega \mathbf{i}'_3 , \tag{15.2.64}$$

so that $\boldsymbol{\omega}$ is a constant vector too. The equation (15.1.21") becomes

$$\omega^2 \mathbf{i}'_3 \times (\mathbf{I}_O \mathbf{i}'_3) = Mg\mathbf{i}'_3 \times \boldsymbol{\rho} . \tag{15.2.65}$$

Hence, if the constant unit vector $\mathbf{i}'_3$ and the constant scalar ω satisfy the equation (15.2.65) and the equation $\mathbf{i}'^2_3 = 1$, then $\mathbf{i}'_3 = \overrightarrow{\text{const}}$ and (15.2.64) represents a solution of the equations of Euler and Poisson. *If – in the above conditions – a rigid solid with a fixed point O begins to move with an angular velocity of rotation $\boldsymbol{\omega}$ about an axis rigidly linked to the rigid solid, specified by the unit vector $\mathbf{i}'_3$ and if this axis is situated along the ascendent vertical, then the rigid solid remains in a state of* *permanent rotation about the respective axis*. A scalar product of the equation (15.2.65) by $\mathbf{I}_O \mathbf{i}'_3$ allows to write

$$(\mathbf{I}_O \mathbf{i}'_3, \mathbf{i}'_3, \boldsymbol{\rho}) = 0 \tag{15.2.66}$$

or, in a scalar form

$$\begin{vmatrix} I_1\alpha_1 & I_2\alpha_2 & I_3\alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \rho_1 & \rho_2 & \rho_3 \end{vmatrix} = (I_2 - I_3)\rho_1\alpha_2\alpha_3 + (I_3 - I_1)\rho_2\alpha_3\alpha_1 + (I_1 - I_2)\rho_3\alpha_1\alpha_2 = 0; \tag{15.2.66'}$$

this is the condition which must be verified by the unit vector $\mathbf{i}'_3$, hence by the direction cosines α_i, $i = 1,2,3$. We can thus state that the permanent axis of rotation must be on *the cone of the mass centres* relative to the fixed point O, of equation

$$\begin{vmatrix} I_1x_1 & I_2x_2 & I_3x_3 \\ x_1 & x_2 & x_3 \\ \rho_1 & \rho_2 & \rho_3 \end{vmatrix} = (I_2 - I_3)\rho_1x_2x_3 + (I_3 - I_1)\rho_2x_3x_1 + (I_1 - I_2)\rho_3x_1x_2 = 0\,. \tag{15.2.66''}$$

This cone is concentric with the ellipsoid of inertia at O, but is not coaxial with this one. If we take an element of the cone, to which we impose a certain sense, then we obtain the direction cosines α_i, $i = 1,2,3$, while the formula (15.2.65) allows to calculate the quantity ω; we notice that the signs must be chosen (we can have $\pm\alpha_i$) so that the quantity ω^2 be positive. We still observe that the moment of momentum is situated along the axis of direction parameters $I_1\alpha_1$, $I_2\alpha_2$, $I_3\alpha_3$, called *secondary axis* too; hence, the centre of mass lies on a plane determined by the axis of rotation and by the secondary one. It can be easily seen that the three principal axes of inertia, the OC-axis and the OC'-axis, where C' is a *centre associated to the mass centre*, of co-ordinates in direct proportion to ρ_1 / I_1, ρ_2 / I_2, ρ_3 / I_3, are five *exceptional axes* situated on the cone of the mass centres. The intersection of this cone with the unit sphere of centre O is called *the mass centre curve* and plays an important rôle in the study of the motion too; considering that α_i, $i = 1,2,3$, are the co-ordinates of a point on the sphere, it results that the equations (15.2.66), (15.2.66') are just the equations of this curve.

In the case of an ellipsoid of inertia of rotation ($I_1 = I_2 > I_3$ or $I_1 > I_2 = I_3$) one obtains interesting particular results, after the position of the mass centre. If $\boldsymbol{\rho} = \mathbf{0}$, then we are in the Euler-Poinsot case, the corresponding problem being studied in Sect. 15.1.2.7.

15.2.3.8 Other Cases of Loading. The Nadolschi Case

Euler considered in 1758 the most simple case of loading of the rigid solid with a fixed point, i.e. the case in which the resultant moment of the given forces vanishes ($\mathbf{M}_O = \mathbf{0}$). Another case, particular too, is that in which the direction of the moment $\mathbf{M}_O$ is rigidly connected to the rigid solid ($\mathbf{M}_O = M_O(t)\mathbf{u}$, $\mathbf{u} = \overrightarrow{\mathrm{const}}$ in the frame of reference $\mathscr{R}$), but is movable with respect to the fixed frame $\mathscr{R}'$. R. Grammel calls *autoexcited rigid solid* that solid which is acted upon by a moment $\mathbf{M}_O(t)$ resulting from internal actions of the respective solid, which does not change in an appreciable manner the distribution of masses. One can consider also the case in which the vector function $\mathbf{u} = \mathbf{u}(t)$ is given. Firstly was studied the simpler case in which $M_O = \mathrm{const}$ (*stationary autoexcitation*) and then the case $M_O = M_O(t)$ (*non-stationary autoexcitation*). One obtains different results, as the moment $\mathbf{M}_O$ is situated along the minor axis, the major axis or the mean axis of the ellipsoid of inertia. We mention also that in the phase space one obtains diagrams of the type of those in Figs 7.22 and 7.25. In 1952, U.T. Bödewadt considers the case of a symmetric ellipsoid of inertia ($I_1 = I_2$), the rigid solid being stationary autoexcited, with $\mathbf{u} = \overrightarrow{\mathrm{const}}$; realizing a partial decoupling of Euler's equations, he could give a solution by means of some Fresnel's integrals. W. Braunbeck studied in 1953 the case of a symmetric rigid solid hanged up at its mass centre C and acted upon by a moment of the form (external excitation)

$$\mathbf{M}(t) = \mu[\mathbf{u} \times (\mathbf{H}_0 - \mathbf{H}_1)], \tag{15.2.67}$$

generated by a bar of magnetic moment μ, situated along the symmetry axis and subjected to the action of a homogeneous magnetic field $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$, where $\mathbf{H}_0$ is a constant field, while $\mathbf{H}_1$ is a field which varies periodically with t; the unit vector $\mathbf{u}$ is situated along the axis of symmetry. If $\mathbf{H} = \mathbf{H}_0$ ($\mathbf{H}_1 = \mathbf{0}$), then the motion of the rigid solid is identical to that due to the influence of a gravitational field. Interesting results have been obtained for $\mathbf{H}_1 \parallel \mathbf{H}_0$ or $\mathbf{H}_1 \perp \mathbf{H}_0$.

The case of gravity forces, which has been studied at large, is a *conservative case*. We will consider also the case of arbitrary conservative forces, of potential $U = U(\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3)$, where $\mathbf{i}'_j$, $j = 1,2,3$, are the unit vectors of the fixed frame of reference $\mathscr{R}'$; we assume the presence of a *gyrostatic moment* $\mathbf{m}_g$ (of the nature of a moment of momentum) and of some intrinsic cyclic motions (due, e.g., to symmetric rotors or to voids completely filled with an incompressible ideal fluid) too. The equations of Euler and Poisson are then of the form

$$\mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}_O\boldsymbol{\omega}) = -\boldsymbol{\omega} \times \mathbf{m}_g - L\mathbf{i}'_j \times \nabla_j U, \quad L = \mathrm{const}, \tag{15.2.68}$$

$$\dot{\mathbf{i}}'_j + \boldsymbol{\omega} \times \mathbf{i}'_j = \mathbf{0}, \quad j = 1,2,3. \tag{15.2.68'}$$

Excepting the obvious six first integrals

$$\mathbf{i}'_j \cdot \mathbf{i}'_k = \delta_{jk}, \quad j,k = 1,2,3, \tag{15.2.69}$$

we mention the first integral of the mechanical energy

$$\frac{1}{2}(\mathbf{I}_O \boldsymbol{\omega}) \cdot \boldsymbol{\omega} = U + h, \quad h = \text{const}. \tag{15.2.69'}$$

The problem has been dealt with in detail, in 1986, by H.M. Yehia, in the case in which the conservative and the gyroscopic forces allow a common axis of symmetry, which passes through the fixed point O. In the symmetric case, F. Brun, in 1907, O.I. Bogoyavlenski, in 1984, and H.M. Yehia, in 1986, have put in evidence six cases of integrability. Thus, if $I_1 = I_2 = 2I_3$, $\mathbf{m}_g = I_3 \omega_0 \mathbf{i}_3$, $\omega_0 = \text{const}$, $U = I_3 \boldsymbol{\gamma}_j \cdot \mathbf{i}'_j$, $\boldsymbol{\gamma}_j$, $j = 1,2,3$, being constant vectors in the plane Ox_1x_2; the dimensions of all these quantities are chosen so as to verify the dimensional equations (ω_0 is an angular velocity and $\boldsymbol{\gamma}_j$ is an angular acceleration). We find the first integral

$$\begin{aligned} &\left(\omega_1^2 - \omega_2^2 - \gamma_{j1} i'_{j1} + \gamma_{j2} i'_{j2}\right)^2 + \left[2\omega_1\omega_2 - \left(\gamma_{j1} i'_{j2} + \gamma_{j2} i'_{j1}\right)\right]^2 \\ &+ 2\omega_0 \left(\omega_3 - \omega_0\right)\left(\omega_1^2 + \omega_2^2\right) - 4\omega_0 \left(\omega_1 \gamma_{j1} + \omega_2 \gamma_{j2}\right) i'_{j3} = \text{const}, \end{aligned} \tag{15.2.70}$$

where γ_{jk}, i'_{jk}, $j,k = 1,2,3$, are the components of the vectors $\boldsymbol{\gamma}_j$ and $\mathbf{i}'_j$, respectively, along the Ox_k-axis. By particularization, one obtains various interesting results, among them the first integral of Sonya Kovalevsky.

In 1944, V.L. Nadolschi studied the case in which the ellipsoid of inertia is symmetric ($I_1 = I_2 = J$), the moment of the given forces being of the form $\mathbf{M}_O = \mathbf{M}_O(t)$. Euler's equations are of the form

$$\begin{aligned} \dot{\omega}_1 - \frac{1}{\sqrt{f(t)}}\omega_2 &= \frac{1}{J} M_{O1}(t), \\ \dot{\omega}_2 + \frac{1}{\sqrt{f(t)}}\omega_1 &= \frac{1}{J} M_{O2}(t), \\ \dot{\omega}_3 &= \frac{1}{I_3} M_{O3}(t), \end{aligned} \tag{15.2.71}$$

where $f(t) = J^2 / (J - I_3)^2 \omega_3^2(t)$. Eliminating successively ω_2 and ω_1, respectively, we find the equations

$$\begin{aligned} f(t)\ddot{\omega}_1 + \frac{1}{2}\dot{f}(t)\dot{\omega}_1 + \omega_1 = \Phi_1(t),\ \Phi_1(t) = \frac{\sqrt{f(t)}}{J}\left\{M_{O2}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[M_{O1}(t)\sqrt{f(t)}\right]\right\}, \\ f(t)\ddot{\omega}_2 + \frac{1}{2}\dot{f}(t)\dot{\omega}_2 + \omega_2 = \Phi_2(t),\ \Phi_2(t) = \frac{\sqrt{f(t)}}{J}\left\{M_{O1}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[M_{O2}(t)\sqrt{f(t)}\right]\right\}. \end{aligned} \tag{15.2.72}$$

The third component of the angular velocity is given by ($\omega_3^0 = \omega_3(t_0)$)

$$\omega_3(t) = \omega_3^0 + \frac{1}{I_3}\int_{t_0}^{t} M_{O3}(\overline{t})\,\mathrm{d}\overline{t}\,. \tag{15.2.72'}$$

By a change of independent variable

$$\tau = \tau_0 - \frac{J - I_3}{J}\int_{t_0}^{t} \omega_3(\overline{t})\,\mathrm{d}\overline{t}\,, \tag{15.2.73}$$

we get

$$\frac{\mathrm{d}^2\omega_1(\tau)}{\mathrm{d}\tau^2} + \omega_1(\tau) = \Phi_1(\tau),\quad \frac{\mathrm{d}^2\omega_2(\tau)}{\mathrm{d}\tau^2} + \omega_2(\tau) = \Phi_2(\tau)\,,$$

wherefrom (we use the method of variation of constants)

$$\begin{aligned}\omega_1(\tau) &= \int_{\tau_0}^{\tau} \Phi_1(u)\sin(\tau - u)\mathrm{d}u + C_1\cos\tau + C_2\sin\tau\,,\\ \omega_2(\tau) &= \int_{\tau_0}^{\tau} \Phi_2(u)\sin(\tau - u)\mathrm{d}u + C_1\sin\tau - C_2\cos\tau\,.\end{aligned} \tag{15.2.74}$$

We notice also that, using only the function $\Phi_1(\tau)$, we obtain

$$\omega_2(\tau) = -\int_{\tau_0}^{\tau} \Phi_1(u)\cos(\tau - u)\mathrm{d}u + C_1\sin\tau - C_2\cos\tau - \frac{M_{O1}(\tau)}{(J - I_3)\,\omega_3(\tau)}\,, \tag{15.2.74'}$$

C_1 and C_2 being, as above, integration constants. Euler's angles are then obtained by quadratures.

Chapter 16

Other Considerations on the Dynamics of the Rigid Solid

The modelling of a continuum as a rigid solid allows the study of many problems of practical interest. In this order of ideas, we consider, in the frame of this chapter, the motions of the Earth and make a presentation of the theory of the gyroscope; as well, we deal with the model of the rigid solid of variable mass, with applications to the motion of the aircraft.

16.1 Motions of the Earth

The theory developed in the previous chapters concerning the dynamics of the rigid solid may be applied successfully to the study of the motion of the Earth with respect to a heliocentric frame of reference $\mathscr{R}'$, considered to be inertial (fixed) and with respect to a geocentric non-inertial (movable) frame $\mathscr{R}$. Modelling the Earth as a rigid solid, one can put in evidence the motion of revolution about the Sun, the motion of rotation about its axis, as well as the motions of precession and nutation. We mention other motions of the Earth too, as: the displacement of the geographic poles of the Earth on its surface, the tides (studied in Chap. 10, Sect. 2.2.3), the displacement (drift) of the continents etc.

16.1.1 Euler's Cycle. The Regular Precession

In what follows, we give firstly some general results concerning Euler's cycle, passing then to the calculation of the regular precession; we put thus in evidence the corresponding secular variations. By analogy, we consider then the Larmor precession.

16.1.1.1 General Considerations

In a modelling as a particle, the Earth is attracted by the Sun (modelled as a particle too), considered to be fixed, after the law of universal attraction, having a Keplerian motion with respect to the latter one; as a matter of fact, this is the motion of the mass centre C of the Earth, if we assume that the forces of attraction which act upon it have a resultant passing through this point. In reality, the Earth is not a homogeneous sphere or is not formed by homogeneous spherical strata, the above condition concerning the resultant of the attraction forces being fulfilled only approximately (see Chap. 9, Sect. 1.2 too); hence, the trajectory of the point C differs from an ellipse, this one being only a first approximation of the real one. However, we will assume that the mass centre C describes an ellipse, the Sun being situated at one of the foci, as it was shown in Chap. 9,

P.P. Teodorescu, *Mechanical Systems, Classical Models*,

Sect. 2.1.4; this approximation is as more acceptable as the perturbations due to the presence of other celestial bodies are less important. The respective motion of translation is called *motion of revolution* about the Sun during a sidereal year; it has been put in evidence by considerations concerning the aberration of light and the stellar parallax.

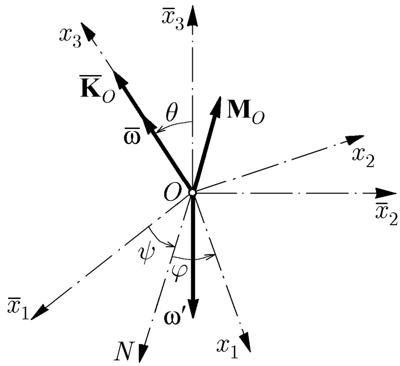


Fig. 16.1 Non-inertial frames of reference in case of the motions of the Earth

The motion of rotation of the Earth about the centre C is independent on its motion of revolution about the Sun, mentioned above. To study this motion, we will consider a non-inertial (movable) frame of reference $\mathscr{R}$, rigidly linked to the Earth and having the pole at its centre ($O \equiv C$); the central principal axes of inertia of the Earth will be taken as axes of the frame $\mathscr{R}$. We will assume, in a first approximation, that the Earth is an oblate spheroid (ellipsoid of rotation with respect to the minor axis), choosing the axis of the poles as Ox_3-axis (directed towards the north pole, putting thus in evidence the sense of the diurnal motion of the Earth); the plane Ox_1x_2 will be the equatorial plane of the Earth and it results $I_3 > I_1 = I_2 = J$ (the hypothesis thus made is acceptable, because $(I_1 - I_2)/I_3 < 10^{-6}/3$). It is convenient to introduce also a geocentric frame of reference $\overline{\mathscr{R}}$ with the axes parallel to the axes of the heliocentric frame $\mathscr{R}'$ and with the pole at the same pole O. We choose as plane $O\overline{x}_1\overline{x}_2$ the plane of the ecliptic (which contains the trajectory of the pole O), the heliocentric frame being, as well, an ecliptic heliocentric frame; the sense of the $O\overline{x}_3$-axis will be chosen so that the motion of the mass centre on its trajectory have a positive sense with respect to this axis. The line of nodes ON will be at the intersection of the ecliptic plane with the equatorial plane of the Earth, being directed towards the first point of Aries, which indicates the vernal equinox (Fig. 16.1).

16.1.1.2 Diurnal Rotation of the Earth. Euler's Cycle

By means of the usual notions in the dynamics of the rigid solid, we can write Euler's equations (15.1.11") in the form,

$$\begin{aligned} J\dot{\omega}_1 + (I_3 - J)\omega_2\omega_3 &= M_{O1}, \\ J\dot{\omega}_2 - (I_3 - J)\omega_3\omega_1 &= M_{O2}, \\ I_3\dot{\omega}_3 &= M_{O3}, \end{aligned} \tag{16.1.1}$$

where $\mathbf{M}_O$ is the moment of the given external forces with respect to the centre of the mass of the Earth.

Assuming, in a first approximation, that $\mathbf{M}_O = \mathbf{0}$ it results the system of equations

$$\dot{\omega}_1 + n\omega_2 = 0, \quad \dot{\omega}_2 - n\omega_1 = 0, \quad n = \frac{I_3 - J}{J}\omega_3^0 > 0, \tag{16.1.2}$$

where we took into account that $\omega_3 = \omega_3^0 = \text{const}$, $\omega_3^0 > 0$. Hence, the Earth has a uniform motion of rotation about the Ox_3-axis, with a constant angular velocity $\omega_3^0 = 0.0000729\,\mathrm{rad/s} = 0.0043753\,\mathrm{rad/min} = 0.2625161\,\mathrm{rad/h} = 6.3003876\,\mathrm{rad/day}$, effecting a complete rotation in a *mean solar time* $T_E = 2\pi/\omega_3^0 = 86164.098s$ $= 23\mathrm{h}\,56\mathrm{min}\,04.098\mathrm{s}$ 0, equal to a *sidereal day*. *The equatorial velocity* of the Earth is, in this case, given by $v_E = \omega_3^0 a_1 = 0.0000729 \cdot 6378246\,\mathrm{m/s} \cong 465.11\,\mathrm{m/s}$, where a_1 is the semi-diameter of the terrestrial spheroid at the equator. This motion of the Earth has been put in evidence by various experiments, e.g.: the deviation towards the east point in the free falling to the surface of the Earth (see Chap. 10, Sect. 2.2.8), Foucault's pendulum (see Chap. 10, Sect. 2.2.10) etc.; as well, in the last time, direct observations in the cosmic space have been made.

The two differential equations can be written also in a unitary form

$$\dot{\tilde{\omega}} - n\mathrm{i}\tilde{\omega} = 0, \quad \tilde{\omega} = \omega_1 + \mathrm{i}\omega_2, \quad \mathrm{i} = \sqrt{-1},$$

wherefrom $\tilde{\omega} = \kappa e^{n\mathrm{i}t}$, $\kappa \in \mathbb{C}$; we get

$$\omega_1(t) = \omega^0 \cos(nt - \gamma), \quad \omega_2(t) = \omega^0 \sin(nt - \gamma), \tag{16.1.2'}$$

where the constants ω^0 and γ may be determined by means of the initial conditions $\omega_1(t_0) = \omega_1^0$, $\omega_2(t_0) = \omega_2^0$, obtaining

$$\omega^0 = \sqrt{(\omega_1^0) + (\omega_2^0)^2}, \quad \gamma = nt_0 - \arctan\frac{\omega_2^0}{\omega_1^0}. \tag{16.1.2''}$$

As it has been shown in Sect. 15.1.2.8 too, it can be seen that the corresponding motion is a *regular precession* with the period $T = 2\pi/n$. The instantaneous axis of rotation describes the polhodic cone around the Ox_3-axis, with the angular velocity n;

taking $(I_3 - J)/I_3 = 1/306$, we find $(I_3 - J)/J \cong 1/305$, so that $n \cong 0.0207\,\mathrm{rad/day}$. We notice that $T = \left(2\pi/\omega_3^0\right)J/(I_3 - J)$ $= T_E/[(I_3 - J)/J] = 305T_E$; hence, the polhodic cone will be entirely described' in 305 sidereal days (hence in 304 days 4h 49.89s *mean solar*). This period is known as *Euler's cycle*.

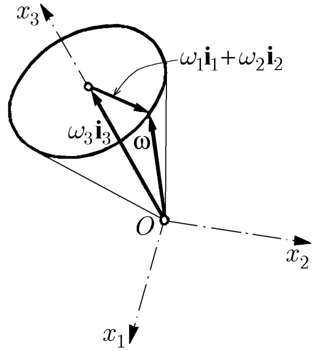


Fig. 16.2 The motion of precession of the extremity of the vector $\boldsymbol{\omega}$

The motion of precession of the extremity of the vector $\boldsymbol{\omega}$ on a director circle of the polhodic cone is put in evidence in Fig. 16.2, taking into account that $|\omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2| = \omega^0$, $|\omega_3\mathbf{i}_3| = \omega_3^0$, hence $|\boldsymbol{\omega}| = \mathrm{const}$. The intersection of the Earth's oblate spheroid with the polhodic cone will be also a circle, called *Euler's circle*.

Introducing a vector $\mathbf{\Omega} = n\mathbf{i}_3$ along the Ox_3-axis, so that $|\mathbf{\Omega}| = n$, the system of equations (16.1.2) may be written in the vector form ($\boldsymbol{\omega} = \omega_j\mathbf{i}_j$)

$$\dot{\boldsymbol{\omega}} = \mathbf{\Omega} \times \boldsymbol{\omega} \tag{16.1.2'''}$$

too. We find thus the equation of motion of a vector $\boldsymbol{\omega}$ of fixed origin and constant modulus, which describes a cone around a direction specified by the vector $\mathbf{\Omega}$, with an angular velocity Ω (in our case $\Omega = n$, corresponding to a regular precession).

Observations made with the greatest precision, corresponding to a much more complex modelling of the Earth (which takes into account the displacement of masses of air, the deformability of the Earth, which assumes that it is elastic etc.), show that the positions of the poles are not fixed on its surface (the axis of rotation of the Earth does not coincide with the axis of the geographic poles). Thus, instead of Euler's period of approximate 10 months, one finds a period of 14 months (*Chandler's period*). One sees that the displacement of a pole on the surface of the Earth can be obtained by the composition of this periodic displacement with another displacement having a period of one year (due to the displacement of the masses of air at the surface of the Earth, to the loading of the continents with masses of snow and to other phenomena, with an annual period); the positions of the pole are contained in the interior of a square of 20 m side.

16.1.1.3 The Mechanical-Magnetic Analogy. The Larmor Precession

Phenomena analogous to those exposed at the preceding subsection take place in the microcosmos too. Let thus be a system $\mathscr{S}$ of particles P loaded with electric charges of the same sign, which have a finite motion in a central field; we assume that at the centre of the field stays a charge considered to be fixed, the motion taking place with respect to it. We suppose also that this system is situated in a homogeneous and constant exterior magnetic field, characterized by the magnetic induction **B**. Such a system is, e.g., an atom placed in a magnetic field; the fixed charge is the nucleus, the electrons being the charges in motion. Assuming that all the particles in motion have the same specific charge e/m, e being the electric charge and m the mass, one obtains a motion quite simple but particularly important in atomic physics.

The magnetic moment of the current distribution can be expressed in the form $(e/2m)\overline{\mathbf{K}}_O$, where $\overline{\mathbf{K}}_O$ is the moment of momentum of the considered motion, of constant modulus, applied at the point O, where we assume that the nucleus of the atom (the fixed charge) stays. The magnetic field is uniform, so that the resultant of the forces which act upon the system $\mathscr{S}$ vanishes, its moment being given, with a good approximation, by

$$\left(\frac{e}{2m}\overline{\mathbf{K}}_O\right)\times\mathbf{B} = \overline{\mathbf{K}}_O\times\left(\frac{e}{2m}\mathbf{B}\right) = \boldsymbol{\Omega}\times\overline{\mathbf{K}}_O, \quad \boldsymbol{\Omega} = -\frac{e}{2m}\mathbf{B}.$$

The theorem of moment of momentum reads

$$\dot{\overline{\mathbf{K}}}_O = \boldsymbol{\Omega}\times\overline{\mathbf{K}}_O, \tag{16.1.2$^{\text{iv}}$}$$

and is a relation of the form (16.1.2'''). We are thus led to a mechanical-magnetic analogy which allows to state that the moment of' momentum $\overline{\mathbf{K}}_O$ describes a cone (analogue to the polhodic cone) with a retrograde regular precession, called *the Larmor precession*; the corresponding angular velocity $|\boldsymbol{\Omega}| = (e/2m)B$ is called *the Larmor pulsation*. Because $e < 0$ for electrons, the motion of rotation takes place counterclockwise about the support of the vector **B**.

We notice that the introduction of the moment $\boldsymbol{\Omega}\times\overline{\mathbf{K}}_O$ is entirely justified in case of a permanent magnetic dipole, the magnitude of which is independent on the orientation of the system $\mathscr{S}$. The approximation of calculation remains very good even in the absence of rigid links, in case of a weak magnetic field ($\mathbf{B}\cong\mathbf{0}$).

The velocity of precession of the vector $\overline{\mathbf{K}}_O$ can be measured experimentally by using the Zeeman effect, resonance methods etc.

16.1.1.4 Calculation of the Regular Precession. The Secular Variation

The hypothesis made in the preceding subsection, in conformity to which $\mathbf{M}_O = \mathbf{0}$ is correct only if the external forces which act upon the Earth are reduced to a resultant which passes through the point O. Taking into account that the Earth is an oblate spheroid with the minor axis along the Ox_3-axis, one sees that $\mathbf{M}_O \neq \mathbf{0}$. Indeed, taking into account only the attraction of the Sun, the two halves in which the Earth is bisected by the ecliptic plane have different contributions in the calculation of this moment (the half over this plane will be acted upon less than the half under it); the resultant moment will be thus non-zero and, qualitatively, will tend to diminish the angle of nutation between Ox_3' and Ox_3. The components M_{Oi}, $i = 1,2,3$, depend on the relative position of the Earth with respect to the Sun, being periodical functions of time; they will play the rôle of perturbing terms in the solution of the system (16.1.1), considered to be homogeneous. The variations of the angles ψ (of precession) and θ (of nutation) due to some mean values of the moments M_{Oi} are *called secular variations*; expanding these moments into power series, one is led to secular variations of superior order. As well, one can obtain variations due to moments M_{Oi} periodic functions of time (expressed by means of Fourier series), the mean values of which vanish.

To establish the secular variations of the trajectory of a planet P_1 due to another neighbouring one P_2, one can assume, after Gauss, that the mass of the second planet is distributed along its trajectory (Kepler's ellipse); more precisely, one assumes that on two arcs of ellipse traveled through are distributed equal masses in equal times. In the case of our interest, we suppose that the mass of the Sun is distributed along its trajectory; for the sake of simplicity, we approximate the ellipse (which has a very small eccentricity) by a circle of radius r_S, the mass m_S of the Sun being uniformly distributed (a linear density $m_S / 2\pi r_S$). In the Euler-Poinsot case concerning the motion of the rigid solid with a fixed point, this one intervenes only by its principal moments of inertia; we can thus vary the distribution of the masses of the solid, if one maintains the quantities J and I_3. This property remains valid also in the case of moments M_{Oi}, $i = 1,2,3$, due to perturbing forces which act at a sufficiently great distance (as it is the distance Earth-Sun with respect to the dimensions of the Earth). Indeed, the potential of these forces of attraction is in direct proportion to $\int_M \mathrm{d}M / r$, $r = \sqrt{(\xi_i - x_i)(\xi_i - x_i)}$, where ξ_i are the co-ordinates of the mass centre of the Sun, while x_i, $i = 1,2,3$, are the co-ordinates of an element of mass $\mathrm{d}M$ of the Earth (all the co-ordinates are considered with respect to the frame of reference $\mathscr{R}$), the integral being extended to the whole mass M of it. Expanding the ratio $1/r$ after the powers of the ratios x_i / r_S, $i = 1,2,3$, and noting that these ratios are very small with respect to unity, one obtains a rapidly convergent series. The terms of first degree disappear, having as factors the integrals $\int_M x_i \mathrm{d}M$, $i = 1,2,3$, while the terms of second degree lead to factors of the nature of principal moments of inertia; neglecting the terms of higher degree, we justify the above statement. In this order of ideas, we replace the Earth by a system formed by a homogeneous sphere of moment of inertia I_Δ with respect to one of its diameters (with a length equal to the mean diameter $2R$ of the Earth) and a homogeneous material equator of mass m_0. In this case, $I_3 = I_\Delta + m_0 R^2$, $J = I_\Delta + m_0 R^2 / 2$, wherefrom

$$I_\Delta = 2J - I_3, \quad m_0 = \frac{2}{R^2}(I_3 - J). \tag{16.1.3}$$

Because of its symmetry, the homogeneous sphere has no one contribution to the perturbations due to the Sun, remaining only the material equator. The resultant couple $\mathbf{M}_O$ tends to diminish the angle θ, hence the inclination of the equatorial plane of the Earth on the plane of the ecliptic, being directed along the axis of nodes, in its opposite sense (towards the autumnal equinox) (Fig. 16.1). Taking into account the diurnal motion of rotation of the Earth, this one behaves as a gyroscope, the effect of the moment $\mathbf{M}_O$ being, in fact, a gyroscopic effect (this effect will be studied in Sect. 16.2.1.4).

Assuming, with a good approximation, that the axis of rotation of the Earth is just the Ox_3-axis, the polhodic cone being reduced to this axis (its vertex angle is very small), we can take the moment of momentum $\overline{\mathbf{K}}_O$ along it, with $\overline{\mathbf{K}}_O = I_3 \overline{\boldsymbol{\omega}} = I_3 \overline{\omega} \mathbf{i}_3$. The rotation angular velocity vector is decomposed in the form (15.2.14) along the axes Ox_3' and Ox_3, the vector components being $\overline{\boldsymbol{\omega}}$ (corresponding to the proper rotation) and $\boldsymbol{\omega}'$ (corresponding to the motion of precession), respectively; supposing that these motions are uniform and that the ratio $(I_3 - J)/I_3$ can be neglected with respect to unity (the factor $(\omega'^0 / \overline{\omega}^0)\cos\theta_0$ being subunitary too), we can assume that the moment $\mathbf{M}_O$ is given by the relation (15.2.17"). Noting that $\omega' = \dot{\psi}$, the vector $\boldsymbol{\omega}'$ being directed towards the negative sense of the Ox_3'- axis, we can write (Fig. 16.1)

$$\dot{\psi} = -\frac{|\mathbf{M}_O|}{I_3 \overline{\omega} \sin\theta}. \tag{16.1.4}$$

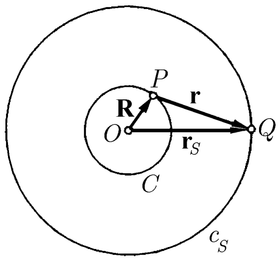


Fig. 16.3 The retrograde annual precession due to the attraction of the Sun

To calculate the modulus of the moment $\mathbf{M}_O$, we will consider the circles (O, R) (in the terrestrial equatorial plane, corresponding to this equator, of radius approximately equal to R, a point P on the circumference being of position vector $\mathbf{R}$) and (O, r_S) (in the plane of the ecliptic, corresponding to the approximate trajectory of the Sun, of radius r_S, a point Q of the trajectory being of position vector $\mathbf{r}_S$, so that $\mathbf{r} = \mathbf{r}_S - \mathbf{R}$ (Fig. 16.3). The potential of the force of reciprocal attraction between the points P and Q of masses $\mathrm{d}m_0$ and $\mathrm{d}m_S$, respectively, is

$$\frac{f\mathrm{d}m_0\mathrm{d}m_S}{r}, \quad r = \sqrt{r_S^2 + R^2 - 2r_S R\cos(\mathbf{r}_S, \mathbf{R})}. \tag{16.1.5}$$

Expanding $1/r$ after the ratio R/r_S, we have (we use the formula of the Newtonian binomial)

$$\frac{1}{r} = \frac{1}{r_S}\left\{1 + \frac{R}{r_S}\cos(\mathbf{r}_S, \mathbf{R}) + \frac{1}{2}\left(\frac{R}{r_S}\right)^2 [3\cos^2(\mathbf{r}_S, \mathbf{R}) - 1] + \ldots\right\}.$$

The ratio R/r_S is very small, so that the power series is rapidly convergent. The searched potential is given by

$$U = f\int_C\int_{c_S}\frac{\mathrm{d}m_0\mathrm{d}m_S}{r}, \tag{16.1.6}$$

where the integral is extended to the circles c_S and C of radii r_S and R, respectively. We notice that

$$\int_C\int_{c_S}\mathrm{d}m_0\mathrm{d}m_S = m_0 m_S, \quad \int_C\int_{c_S}\cos(\mathbf{r}_S, \mathbf{R})\mathrm{d}m_0\mathrm{d}m_S = 0,$$

because of symmetry reasons. To calculate the latter integral, we refer to the frame of reference $\mathscr{R}$ and we suppose that, in the further calculations, the Ox_1-axis coincides with the line of nodes ON. We denote by ε and η the angles made by the position vectors $\mathbf{r}_S$ and $\mathbf{R}$, respectively, with the Ox_1-axis; the co-ordinates of the point Q are $\bar{\xi}_1 = r_S\cos\varepsilon$, $\bar{\xi}_2 = r_S\sin\varepsilon$, $\bar{\xi}_3 = 0$, while the co-ordinates of the point P are written in the form $\bar{x}_1 = R\cos\eta$, $\bar{x}_2 = R\sin\eta\cos\theta$, $\bar{x}_3 = R\sin\eta\sin\theta$. It results

$$\cos(\mathbf{r}_S, \mathbf{R}) = \frac{\bar{x}_j\bar{\xi}_j}{Rr_S} = \cos\varepsilon\cos\eta + \sin\varepsilon\sin\eta\cos\theta,$$
$$\mathrm{d}m_S = \frac{1}{2\pi}m_S\mathrm{d}\varepsilon, \quad dm_0 = \frac{1}{2\pi}m_0\mathrm{d}\eta,$$

so that

$$\int_{c_S}\int_C\cos^2(\mathbf{r}_S, \mathbf{R})\mathrm{d}m_0\mathrm{d}m_S = \frac{m_0 m_S}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(\cos\varepsilon\cos\eta + \sin\varepsilon\sin\eta\cos\theta)^2\,\mathrm{d}\varepsilon\mathrm{d}\eta$$
$$= \frac{m_0 m_S}{4}(1 + \cos^2\theta).$$

In this case,

$$U = \frac{fm_0 m_S}{r_S}\left[1 + \frac{1}{8}\left(\frac{R}{r_S}\right)^2(3\cos^2\theta - 1) + \ldots\right]. \tag{16.1.6'}$$

Restricting us to these terms in the expansion into a series, one observes that the equipotential surfaces are circular cones with the vertex at O, of axis $O\overline{x}_3$. The moment $\mathbf{M}_O$ which tends to diminish the angle $\theta < \pi / 2$ is given by the forces of attraction equal to the gradient of the potential U at an arbitrary point; this gradient is normal to the equipotential surface, so that the modulus $|\mathbf{M}_O|$ is obtained by multiplying the modulus of the gradient by the distance to the above mentioned point. It results

$$|\mathbf{M}_O| = \left|\frac{\partial U}{\partial \theta}\right| = \frac{3}{4} f m_0 m_S \frac{R^2}{r_S^3} \sin\theta \cos\theta .$$

Finally, the formula (16.1.4) allows to write ($\overline{\omega} = \omega_3^0$)

$$\dot{\psi}_S = -\frac{3}{4} f \frac{m_0 m_S R^2}{I_3 \omega_3^0 r_S^3} \cos\theta = -\frac{3}{2} f \frac{I_3 - J}{I_3} \frac{m_S}{\omega_3^0 r_S^3} \cos\theta . \tag{16.1.7}$$

Taking into account that $f = 6.673 \cdot 10^{-8}\,\mathrm{cm}^3 / \mathrm{g} \cdot \mathrm{s}^2$, $(I_3 - J) / I_3 = 1/306$, $\omega_3^0 = 7.29 \cdot 10^{-5}\,\mathrm{rad/s}$, $m_S = 1.989 \cdot 10^{33}\,\mathrm{g}$, $r_S = 1.496 \cdot 10^{13}\,\mathrm{cm}$, $\theta = 23°27'$, we get $\dot{\psi}_S = -2.4454896 \cdot 10^{-12}\,\mathrm{rad/s} = -7.7172197 \cdot 10^{-5}\,\mathrm{rad/year}$, where we took into account that 1 sidereal year has 365.2436 mean solar days, 1 mean solar day having 86400 mean solar seconds; to 1 radian correspond $180 \cdot 3600 / \pi \cong 206264.81''$, so that we obtain $\dot{\psi}_S = -15.917908''$ in a mean solar year. We may thus state that, due to the attraction of the Sun, the Earth has a retrograde annual precession of approximate $16''$.

To put in evidence the influence of the attraction of the Moon, we make an analogous calculation; noting that $m_M = 7.347 \cdot 10^{25}\,\mathrm{g}$ and $r_M = 3.844 \cdot 10^{10}\,\mathrm{cm}$ and taking $\theta = 23°27'$ too (as a matter of fact, the plane of the trajectory of the Moon does not coincide with the plane of the ecliptic, making an angle of $5°09'$ with this plane, the Moon being in an interval of time over this plane and in another interval under it), we find $\dot{\psi}_M = -34.658192''$ in a mean solar year, hence a retrograde annual precession, of approximate $34.7''$. Summing the two effects, we get a precession of $50.5761''$ for a sidereal year. But the above calculations concerning the secular variations have an approximate character, due to the mathematical model used (Gauss's hypothesis, the approximation of the ellipse by a circle, the superposition of the effects, the non-introduction of other influences etc.). If we compose the Moon-Sun precession (the displacement of the equinoctial points along the ecliptic) with the planetary precession (along the celestial equator), then we obtain the general precession (in longitude), which is now of $50.27''/\mathrm{year}$. In this case, the line of nodes ON describes the whole plane of the ecliptic in $360 \cdot 3600 / 50.27 = 2.5780784 \cdot 10^4$ years, hence in approximate 26000 years, while the axis of the Earth poles describes the cone of precession (a circular cone of $O\overline{x}_3$-axis and vertex angle $46°54'$) in the same period of time.

16.1.2 Free Nutation. Pseudoregular Precession

We give some general results and then we calculate the free nutation and the pseudoregular precession in the Lagrange-Poisson case; as well, we make some considerations concerning the general motions of the Earth.

16.1.2.1 General Considerations

We have put in evidence, in the preceding subsection, a polhodic cone, linked to Euler's cycle, which is rolling over a circular herpolhodic cone, the axis of which is the axis of the ecliptic; as well, we have considered the cone of precession, which can play the rôle of a herpolhodic cone. We mention that it does not exist one mechanical link between the polhodic cone with the Eulerian period (for which $\mathbf{M}_O = \mathbf{0}$, corresponding to the Euler-Poinsot case, taking place a rolling of the polhodic cone over the herpolhodic one) and the precession cone with a 26000 years period (for which the constant moment $\mathbf{M}_O \neq \mathbf{0}$ corresponds to the secular variation which leads to the Lagrange-Poisson case, where a precession cone intervenes only in a particular case). In the first case, one obtains an arbitrary nutation (the polhodic cone has a non-determinate vertex angle), while in the second case the nutation vanishes ($\theta = \mathrm{const}$, hence $\dot{\theta} = 0$), because we have taken into account only the secular terms of the perturbing moment $\mathbf{M}_O$; using the periodic terms too, the nutation would be non-zero ($\dot{\theta} \neq 0$), obtaining thus *the nutation of the Earth*.

In the Lagrange-Poisson case, the total absence of nutation ($\theta = \mathrm{const}$) takes place only if a condition of the form (15.2.11) is fulfilled, so that the polynomial $P(u)$ have a multiple solution; if this condition holds only approximately, then the angle θ is no more constant, appearing a nutation ($\dot{\theta} \neq 0$). In case of the Earth (because of the secular terms in the expression of the perturbing moment), this nutation is called *free nutation*. The nutation due to the periodic terms is called *constraint nutation*.

16.1.2.2 Calculation of the Free Nutation

To may calculate the free nutation of the Earth, we model the corresponding problem as a particular Lagrange-Poisson case (a case which differs very little from that of the multiple root of $P(u)$). Let be thus the problem of the rigid solid with a fixed point O upon which acts the moment $\mathbf{M}_O$, of modulus

$$|\mathbf{M}_O| = \frac{3}{4} f m_S \frac{I_3 - J}{r_S^3} \sin 2\theta\,, \tag{16.1.8}$$

along the axis of nodes ON, in its negative sense. As in the Lagrange-Poisson case, we can use the first integral (15.2.1'), which introduces the spin, as the first integral corresponding to the theorem of moment of momentum (15.2.1'''). The theorem of kinetic energy reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[J\left(\omega_1^2 + \omega_2^2\right) + I_3\left(\omega_2^0\right)^2 \right] = \mathbf{M}_O \cdot \boldsymbol{\omega} = |\mathbf{M}_O|\left(-\omega_1 \cos\varphi + \omega_2 \sin\varphi\right),$$

where we took into account that the vector $\mathbf{M}_O$ makes the angles $\pi - \varphi$, $\pi / 2 - \varphi$, $\pi / 2$ with the axes of the frame of reference $\mathscr{R}$. The second relation (14.1.15) allows to write $\mathbf{M}_O \cdot \boldsymbol{\omega} = -|\mathbf{M}_O|\dot{\theta}$ too, an obvious result, corresponding to the rotation in the $Ox_3\overline{x}_3$-plane (Fig. 16.1). Introducing the modulus of the moment $\mathbf{M}_O$ given by (16.1.8) and integrating with respect to time, we find a third first integral

$$J\left(\omega_1^2 + \omega_2^2\right) + I_3\left(\omega_3^0\right)^2 = \frac{3}{2} f m_S \frac{I_3 - J}{r_S^3} \cos^2\theta + 2h\,, \tag{16.1.9}$$

where h is the constant of energy. Using the relations (5.2.35) and eliminating the components ω_1 and ω_2, we find again the first and the last equation (15.2.2), to which we associate

$$\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2 = \gamma + c\cos^2\theta\,, \tag{16.1.10}$$

with $\gamma = \left[2h - I_3\left(\omega_3^0\right)^2\right]/J$, $c = (3/2)f[(I_3 - J)/J]m_S/r_S^3 > 0$; as in the Lagrange-Poisson case, the constants α and β depend on the initial conditions and the constants a and b are functions only of the geometry of the considered mechanical system, as well as of its mechanical properties.

Proceeding as in Sect. 15.2.1.1, we eliminate $\dot{\psi}$ between the first relation (15.2.2) and (16.1.10); thus, we obtain

$$\left(\alpha - a\omega_3^0\cos\theta\right)^2 = \left(\gamma + c\cos^2\theta\right)\sin^2\theta - \dot{\theta}^2\sin^2\theta\,.$$

Denoting $u = \cos\theta$, it results the differential equation

$$\dot{u}^2 = Q(u)\,,\quad Q(u) = \left(\gamma + cu^2\right)\left(1 - u^2\right) - \left(\alpha - a\omega_3^0 u\right)^2\,, \tag{16.1.11}$$

wherefrom

$$t - t_0 = \int_{u_0}^{u} \frac{\mathrm{d}\xi}{\sqrt{Q(\xi)}}\,, \tag{16.1.11'}$$

with $u_0 = \cos\theta_0$, $\theta_0 = \theta(t_0)$; one takes the sign + or – before the radical as, at the initial moment, we have an increasing or a decreasing of the motion. The polynomial $Q(u)$ is of fourth degree, hence the integral (16.1.11') is an elliptic one. As in the Lagrange-Poisson case, $u_0 \in [u_1, u_2]$, $u_1, u_2 \in (-1,1)$ being two roots of the polynomial $Q(u)$. We obtain thus the free nutation $\theta \in [\theta_1, \theta_2]$, the extreme values corresponding to the roots u_1 and u_2.

16.1.2.3 Calculation of the Pseudoregular Precession

Assuming that, at the initial moment $t = t_0 = 0$ (for the sake of simplicity, without any loss of generality), the Ox_3-axis is situated along a generatrix of the cone of precession, we have $\theta_0 = 23°27'$, $\omega_1^0 = \omega_2^0 = 0$, while the first integrals (15.2.1'''), (16.1.9) lead to the relation $K'_{O3'} = I_3 \omega_3^0 \cos\theta_0$, as well as to $\alpha = a\omega_3^0 \cos\theta_0$, $\gamma + c\cos^2\theta_0 = 0$. Let be $\varepsilon = (3/2)f[(I_3 - J)/I_3](m_S / r_S^3)/\omega_3^0$; it results $c = a\varepsilon\omega_3^0$ and $\gamma + a\varepsilon\omega_3^0 \cos^2\theta_0 = 0$. We can write

$$Q(u) = a\omega_3^0 (u - u_0)\left[\varepsilon(u + u_0)(1 - u^2) - a\omega_3^0 (u - u_0)\right], \tag{16.1.12}$$

with the new notations. We notice that $u = u_0$ is a double zero of the polynomial $Q(u)$ if $2a\omega_3^0 \varepsilon u_0 (1 - u_0^2) = 0$, which can be assumed with a good approximation (error of the order of magnitude of ε, which is very small because of the denominator r_S^3).

Denoting by $u_1 = u_0 + 2\bar{u}$ the zero of the square bracket in (16.1.12), we find the condition

$$\begin{aligned} &\varepsilon(u_0 + \bar{u})\left[1 - (u_0 + 2\bar{u})^2\right] - a\omega_3^0 \bar{u} \\ &= a\omega_3^0 \left[\varepsilon \frac{u_0(1 - u_0^2)}{a\omega_3^0} - \bar{u} + \varepsilon \frac{(1 - 5u_0^2)\bar{u}}{a\omega_3^0} + \ldots\right] = 0, \end{aligned}$$

where we have neglected the higher-order powers of $\bar{u}$; we can assume the approximate value

$$\bar{u} = \frac{\varepsilon}{a\omega_3^0} u_0 (1 - u_0^2) = \frac{\varepsilon}{a\omega_3^0} \cos\theta_0 \sin^2\theta_0, \tag{16.1.13}$$

with an error of the order of magnitude $(\varepsilon / a\omega_3^0)^2 u_0 (1 - u_0^2)(1 - 5u_0^2)$. If θ_1 is the value of the angle θ corresponding to u_1, then we have $\cos\theta_1 = \cos\theta_0 + 2\bar{u}$, $\bar{u} > 0$, so that $\theta_1 \leq \theta \leq \theta_0$. Noting that $\sin[(\theta_1 - \theta_0)/2]\sin[(\theta_1 + \theta_0)/2] + \bar{u} = 0$ and assuming that the arc $(\theta_1 - \theta_0)/2$ is sufficient small in modulus to can approximate its sinus, we obtain (as well, we approximate $\sin((\theta_1 + \theta_0)/2) \cong \sin\theta_0$)

$$\theta_1 = \theta_0 - \frac{2\bar{u}}{\sin\theta_0} = \theta_0 - \frac{\varepsilon}{a\omega_3^0} \sin 2\theta_0, \tag{16.1.13'}$$

where we have used the approximate relation (16.1.13) too. Because the angle θ is not constant, having a variation of amplitude $(\varepsilon / a\omega_3^0)\sin\theta_0 \cos\theta_0$, it results a *pseudoregular precession*.

We can calculate the angle of nutation as a function of t, making, after F. Klein, $u = u_0 + v$, $v \in [0, 2\overline{u}]$, in the polynomial $Q(u)$ given by (16.1.12). We find thus

$$Q(u) = a\omega_3^0 v\left[2\varepsilon u_0 \left(1 - u_0^2\right) - 4\varepsilon u_0^2 v + \varepsilon v\left(1 - u_0^2\right) - a\omega_3^0 v\right],$$

where we have neglected the higher-order powers of v, which is of the order of magnitude of ε; taking into account (16.1.13) and neglecting the higher-order powers (the product εv with respect to ε or to v), it results

$$Q(u) = \left(a\omega_3^0\right)^2 v(2\overline{u} - v). \tag{16.1.14}$$

The formula (16.1.11') allows to calculate

$$a\omega_3^0 (t - t_0) = \int_0^v \frac{\mathrm{d}\xi}{\sqrt{\xi(2\overline{u} - \xi)}} = \arccos\frac{\overline{u} - v}{\overline{u}},$$

wherefrom $v = \overline{u}\left[1 - \cos a\omega_3^0 (t - t_0)\right]$; the relation

$$v = u - u_0 = \cos\theta - \cos\theta_0 = -2\sin\frac{\theta - \theta_0}{2}\sin\frac{\theta + \theta_0}{2} \cong -(\theta - \theta_0)\sin\theta_0$$

gives the angle of free nutation in the form

$$\theta(t) = \theta_0 - \frac{\overline{u}}{\sin\theta_0}\left[1 - \cos a\omega_3^0 (t - t_0)\right] = \theta_0 - \varepsilon\frac{\sin 2\theta_0}{2a\omega_3^0}\left[1 - \cos a\omega_3^0 (t - t_0)\right], \tag{16.1.15}$$

by means of the relation (16.1.13). Hence, the variation of the angle of nutation θ is periodical, with the period $2\pi / a\omega_3^0 = \left(2\pi / \omega_3^0\right)(J / I_3) < 2\pi / \omega_3^0 = T_E$, ω_3^0 being the angular velocity in the diurnal rotation of the Earth (which is acceptable in the frame of the above approximation); hence, the period of variation of the nutation angle is a little smaller than a sidereal day ($305/306$ of T_E).

The angle of precession is given by the first relation (15.2.2) in the form $\dot{\psi} = \left(\alpha - a\omega_3^0 \cos\theta\right)/\sin^2\theta$; taking into account (16.1.15), neglecting the higher order powers of ε and noting that $\cos\theta = \cos\theta_0 + v = \cos\theta_0 - (\theta - \theta_0)\sin\theta_0$, we can write (we use the same method as above)

$$\dot{\psi} = -\frac{a\omega_3^0 \overline{u}}{\sin^2\theta_0}\left[1 - \cos a\omega_3^0 (t - t_0)\right] = -\varepsilon\cos\theta_0\left[1 - \cos a\omega_3^0 (t - t_0)\right]$$

too. By integration, one obtains the angle of pseudoregular precession (we put the initial condition $\psi(t_0) = \psi_0$)

$$\psi(t) = \psi_0 - \frac{\varepsilon}{a\omega_3^0} \cos\theta_0 \left[a\omega_3^0 (t - t_0) - \sin a\omega_3^0 (t - t_0) \right], \tag{16.1.16}$$

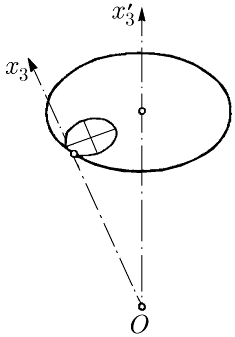


Fig. 16.4 The elliptic cone described by the Ox_3 -axis in the motion of the Earth

which – unlike the regular precession – is a non-linear function of time. Thus, besides the motion of regular precession on the cone of vertex angle $2\theta_0$, intervenes a supplementary motion given by the additive terms in the formulae (16.1.15), (16.1.16); hence, the Ox_3 -axis describes also an elliptic cone (Fig. 16.4) of extreme vertex angles $(\varepsilon / a\omega_3^0)\cos\theta_0$ and $(\varepsilon / a\omega_3^0)\sin 2\theta_0$ about axes specified by the angle of precession $-\varepsilon(t - t_0)\cos\theta_0$ and by the angle of nutation $\theta_0 - (\varepsilon / 2a\omega_3^0)\sin 2\theta_0$, respectively. The ratio of the extreme vertex angles is equal to $2\sin\theta_0 = 2\sin 23°27' \cong 0.7959$; hence, the elliptic cone described by the Ox_3 -axis is, approximately, a circular cone (the ratio of the axes is approximately equal to 0.8), the period of rotation being equal to $2\pi / a\omega_3^0 = (305/306) T_E$, hence somewhat smaller than a sidereal day.

The third equation (15.2.2) leads to $\dot\varphi = \omega_3^0 - \dot\psi\cos\theta$; proceeding as in the case of the precession angle and neglecting afterwards ε^2 with respect to ε, we obtain $\dot\varphi = \omega_3^0 + \varepsilon\left[1 - \cos a\omega_3^0 (t - t_0)\right]\cos^2\theta_0$, wherefrom

$$\varphi(t) = \varphi_0 + \left(\omega_3^0 + \varepsilon\cos^2\theta_0\right)(t - t_0) - \frac{\varepsilon}{a\omega_3^0}\cos^2\theta_0 \sin a\omega_3^0 (t - t_0), \tag{16.1.17}$$

with the condition $\varphi(t_0) = \varphi_0$.

Once Euler's angles determined, we can calculate the components ω_j and $\overline{\omega}_j$, $j = 1,2,3$, of the angular velocity vector $\boldsymbol{\omega}$ in the frames of reference $\mathscr{R}$ and $\overline{\mathscr{R}}$, obtaining then the equations of the polhodic and of the herpolhodic cone, respectively.

16.1.2.4 Other Considerations

To obtain a much more correct image of the motions of the Earth, one must take into account also the periodic terms in the perturbing forces due to the Sun, Moon etc.; one can no more assume that the mass of the Sun, e.g., is uniformly distributed along its trajectory. Modelling the Sun as a particle of mass m_S we can calculate the moment $\mathbf{M}_O$ in the form

$$\mathbf{M}_O = f m_S \int_M \frac{1}{r^3} \mathbf{r}_P \times \mathbf{r} \mathrm{d}M , \tag{16.1.18}$$

where $\mathbf{r}_P$ is the position vector of a particle P of the Earth, of mass $\mathrm{d}M$ (M is the mass of the Earth), while $\mathbf{r} = \overrightarrow{PS}$, S being the mass centre of the Sun (Fig. 16.5); in this case

$$M_{Oi} = f m_S \int_M \in_{ijk} \frac{1}{r^3} x_j \xi_k \mathrm{d}M , \tag{16.1.18'}$$

where x_j are the co-ordinates of the point P, while ξ_k are the co-ordinates of the point S, $j,k = 1,2,3$, with respect to the frame of reference $\mathscr{R}$.

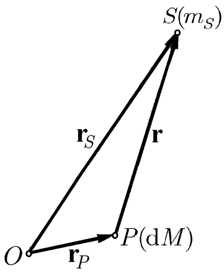


Fig. 16.5 The influence of perturbing terms in the motion of the Earth

Noting that the dimensions of the Earth are small with respect to the distances r and r_S ($\mathbf{r} = \mathbf{r}_S - \mathbf{r}_P$), we can expand the ratio $1/r$ into a series after the powers of the ratios x_i / r_S, $i = 1,2,3$; neglecting the higher-order powers, it results

$$\frac{1}{r^3} = \frac{1}{r_S^3} \left(1 - \frac{2}{r_S^2} \xi_i x_i + \frac{1}{r_S^2} x_i x_i \right)^{-3/2} \cong \frac{1}{r_S^3} \left(1 + \frac{3}{r_S^2} \xi_i x_i \right).$$

Taking into account that O is the centre of the terrestrial oblate spheroid, we can make considerations analogous to those in Sect. 16.1.1.3, so that

$$\int_M x_i \mathrm{d}M = 0 , \quad \int_M x_j x_k \mathrm{d}M = 0 , \quad j \neq k , \quad i,j,k = 1,2,3 ,$$
$$\int_M \left(x_1^2 - x_3^2 \right) \mathrm{d}M = \int_M \left(x_2^2 - x_3^2 \right) \mathrm{d}M$$
$$= \int_M \left(x_1^2 + x_2^2 \right) \mathrm{d}M - \int_M \left(x_2^2 + x_3^2 \right) \mathrm{d}M = I_3 - J .$$

It results

$$M_{O1} = 3f\frac{m_S}{r_S^5}(I_3 - J)x_2x_3, \quad M_{O2} = -3f\frac{m_S}{r_S^5}(I_3 - J)x_1x_3, \quad M_{O3} = 0. \tag{16.1.18"}$$

Euler's equations (16.1.1) read

$$\begin{gathered}\dot{\omega}_1 + n\omega_2 = \frac{M_{O1}}{J} = \frac{n'}{r_S^3}\frac{x_2}{r_S}\frac{x_3}{r_S}, \quad \dot{\omega}_2 - n\omega_1 = \frac{M_{O2}}{J} = -\frac{n'}{r_S^3}\frac{x_1}{r_S}\frac{x_3}{r_S}, \\ n = \frac{I_3 - J}{J}\omega_0^3 > 0, \quad n' = 3fm_S\frac{I_3 - J}{J} > 0.\end{gathered} \tag{16.1.19}$$

These equations may be expressed also by means of Euler's angles and of other angles which specify the position of the Sun with respect to the frame of reference $\mathscr{R}$.

However, neither the results thus obtained do not coincide with those given by the astronomic observations, because of the model of rigid solid assumed for the Earth. In reality, the Earth is a deformable solid or, more correct, a mechanical system formed by solid and fluid parts; some of them can be even rigid. In the hypothesis of rigid of the Earth, its central ellipsoid of inertia being an oblate spheroid, it results that the rotation angular velocity about the axis of the poles is constant; but if we take into account a modelling of the Earth much closer to the reality, one sees that this velocity is varying, resulting difficult problems for the determination of the unit of time (specified by the diurnal rotation of the Earth).

16.2 Theory of the Gyroscope

A heavy rigid solid with a fixed point O for which the ellipsoid of inertia corresponding to this point is of rotation about the principal axis of inertia Ox_3 ($I_1 = I_2 = J$), its initial motion being a rapid motion about this axis, is called *gyroscope*. After some general results, we present various applications with theoretical or technical character.

16.2.1 General Results

We make firstly some general considerations concerning the motion of regular precession of the gyroscope, assuming to be in the Euler-Poisson case; as well, we put in evidence the gyroscopic effect which appears if the gyroscope is acted upon by its own weight or by an arbitrary force. We introduce then the gyroscopic moment and the gyroscopic reactions, calculating also the inertial forces which arise in the motion of regular precession of the gyroscope.

16.2.1.1 The Euler-Poinsot Case. Stability of the Motion

If the moment with respect to the fixed point O, corresponding to all the given external forces which act upon the gyroscope, vanishes ($\mathbf{M}_O = \mathbf{0}$), then we are in the Euler-Poinsot case; we can thus use all the results given in Sect. 15.1.

If we wish that the symmetry axis of the gyroscope, which is a symmetry axis of the motion too, maintains its direction, then we must have $\boldsymbol{\omega} = \overline{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}' = \mathbf{0}$, corresponding to the decomposition (15.2.14) in Sect. 15.2.1.4; we equate thus to zero the motion of regular precession. The formula (15.2.17') shows that this property of the motion takes place if $\mathbf{M}_O = \mathbf{0}$, hence in the considered Euler-Poinsot case (Fig. 16.6). Hence, *a gyroscope to which it was imparted a motion about its axis of symmetry maintains unchanged the direction of this axis if the moment of all the external forces with respect to the fixed point vanishes*. In the case in which the gyroscope is acted upon only by its own weight, we fulfill this condition choosing the centre of gravity as fixed point.

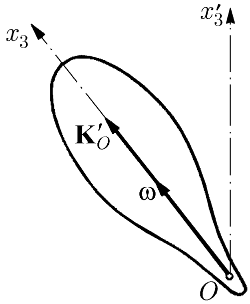


Fig. 16.6 The gyroscope for which the motion of regular precession is equated to zero

As it has been shown in Sect. 15.1.2.7, the symmetry axis of the gyroscope, which is also an extreme principal axis of inertia, is stable during the motion. Indeed, let us suppose that an arbitrary perturbation imparts to the vector $\boldsymbol{\omega}$ a direction somewhat different from Ox_3, having the components $\omega_1, \omega_2 \neq 0$; these components verify the equations (16.1.2), being of the form (16.1.2'), so that $|\omega_1|, |\omega_2| \leq \omega^0$, ω^0 arbitrary. We can state that *the stability of the axis is as greater as the period* $T = 2\pi / n$ (n given by (16.1.2)) *is smaller, hence as the proper velocity of rotation of the gyroscope is greater*. As well, we notice that the stability increases as the moment of inertia I_3 is greater than the moment of inertia J (the ellipsoid of inertia corresponding to the fixed point is a *very oblate spheroid*).

In all these cases, the moment of momentum $\mathbf{K}_O'$ is directed along the Ox_3-axis so that $K_O' = I_3 \omega$.

One can thus explain a great number of mechanical phenomena: (i) *The knife thrower* throws the knife upwards with one hand and catches it with the other hand; to do this, he gives to the knife a motion of rotation about its axis, before throwing it, the axis maintaining thus its direction during the motion. (ii) When he jumps from a certain height, *the skier* rotates both arms stretched laterally in the same sense, about the same horizontal line; in this case, the skier remains in a vertical position, so that he is falling on his feet. (iii) A body with an axis of symmetry becomes a motion of rotation about this axis, in a horizontal position, by means of a thread between two rods, being then thrown upwards; the axis remains horizontal during the motion and the body can be easily caught on the thread from which it was thrown. This is the *diabolo game*. (iv) A *coin* with a vertical diameter or a top with a vertical axis of symmetry Ox_3, staying on a horizontal plane, can maintain for some time the position of its axis if it becomes a motion of rotation about the respective axis $Ox_3' \equiv Ox_3$ (Fig. 16.7); the interval of time is as greater as the rotation angular velocity is greater. (v) The disc of *Gervat's gyroscope* has the horizontal axis maintained in labile equilibrium on one foot (the "equilibrist foot"), formed by a thin metallic tube; giving to the disc a sufficiently rapid motion of rotation, the gyroscope remains in equilibrium.

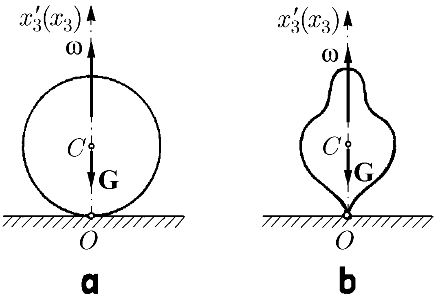


Fig. 16.7 A coin (**a**) or a top (**b**) maintains the position of its axis for some time

16.2.1.2 General Considerations on the Motion of Rotation of the Gyroscope

In general, the moment of the given external forces with respect to the fixed point O is non-zero ($\mathbf{M}_O \neq \mathbf{0}$). If the gyroscope is acted upon only by its own weight, then we are in the Lagrange-Poisson case, so that one can use the results in Sect. 15.2.1.

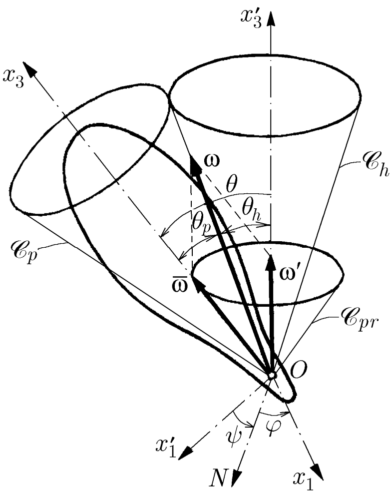


Fig. 16.8 The cone of precession $\mathscr{C}_{pr}$ in the motion of rotation of the gyroscope

Decomposing the rotation angular velocity vector $\boldsymbol{\omega}$ along the fixed Ox_3'-axis, along the movable Ox_3-axis and along the line of nodes ON, respectively, we can write $\boldsymbol{\omega} = \overline{\omega}\mathbf{i}_3 + \omega'\mathbf{i}_3' + \dot{\theta}\mathbf{n}$ with $\overline{\omega} = \dot{\varphi}$, $\omega' = \dot{\psi}$. Assuming that $\dot{\theta} = 0$, hence $\theta = \text{const}$, it results that the vector $\overline{\boldsymbol{\omega}}$ describes the cone of precession $\mathscr{C}_{\text{pr}}$ with the Ox_3'-axis and of vertex angle 2θ (Fig. 16.8). Further, we suppose that $\overline{\omega} = \text{const}$ and $\omega' = \text{const}$; in this case, we have $\omega = \text{const}$ too, the angles θ_h and θ_p made by the vector $\boldsymbol{\omega}$ with the Ox_3'-axis and the Ox_3-axis, respectively, being also constant. We notice that $\theta_h + \theta_p = \theta$ too. In this case, the vector $\boldsymbol{\omega}$ will describe the herpolhodic cone $\mathscr{C}_h$ of axis Ox_3' and vertex angle $2\theta_h$, with respect to the frame of reference $\mathscr{R}'$, and the polhodic cone $\mathscr{C}_p$ of axis Ox_3 and vertex angle $2\theta_p$, with respect to the frame $\mathscr{R}$, respectively. The motion of rotation of the gyroscope will be thus composed by a uniform proper rotation of angle φ and a proper rotation angular velocity $\overline{\omega}$, about the Ox_3-axis and a motion of regular precession of angle ψ and angular velocity of precession ω' about the Ox_3'-axis (Fig. 16.8). By decomposing the vector $\boldsymbol{\omega}$, one obtains easily the relations

$$\begin{aligned} \omega^2 &= \overline{\omega}^2 + \omega'^2 + 2\overline{\omega}\omega'\cos\theta\,, \\ \cos\theta_P &= \frac{1}{\omega}(\overline{\omega} + \omega'\cos\theta)\,, \\ \cos\theta_h &= \frac{1}{\omega}(\omega' + \overline{\omega}\cos\theta)\,. \end{aligned} \tag{16.2.1}$$

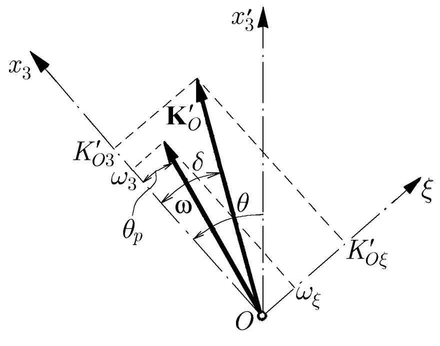


Fig. 16.9 The components of the vectors $\boldsymbol{\omega}$ and $\mathbf{K}_O'$ in the motion of rotation of the gyroscope

Let be the $O\xi$-axis normal to Ox_3, in the Ox_3x_3'-plane. The components of the vectors $\boldsymbol{\omega}$ and $\mathbf{K}_O'$ will be (Fig. 16.9)

$$\begin{aligned} \omega_3 &= \omega\cos\theta_p = \overline{\omega} + \omega'\cos\theta\,, \\ \omega_\xi &= \omega\sin\theta_p = \omega'\sin\theta\,, \\ K_{O3}' &= I_3\omega_3\,, \quad K_{O\xi}' = J\omega_\xi\,. \end{aligned}$$

It results

$$K'_O = \omega\sqrt{I_3^2\cos^2\theta_p + J^2\sin^2\theta_p} = \sqrt{I_3^2(\overline{\omega} + \omega'\cos\theta)^2 + J^2\omega'^2\sin^2\theta}\,,$$
$$\tan\delta = \frac{J}{I_3}\frac{\omega_\xi}{\omega_3} = \frac{J}{I_3}\tan\theta_p = \frac{J}{I_3}\frac{\omega'\sin\theta}{\overline{\omega} + \omega'\cos\theta}\,, \tag{16.2.2}$$

where δ is the angle made by the moment of momentum $\mathbf{K}'_O$ with the Ox_3-axis. This vector is situated in the $Ox_3x'_3$-plane, hence it is rotating together with this plane about the fixed axis Ox'_3 with the angular velocity $\boldsymbol{\omega}'$; this vector describes, as well, a cone of axis Ox'_3 and vertex angle $2(\theta - \delta)$ (we notice that $\delta = \text{const}$ too).

If $\omega'/\overline{\omega} \ll 1$, then we can assume, that $\delta \cong 0$, the moment of momentum $\mathbf{K}'_O$ being directed, with a good approximation, along the Ox_3-axis; hence, $\mathbf{K}'_O = I_3\boldsymbol{\omega} = I_3\omega\mathbf{i}_3$. Noting that, in this case, the velocity of the extremity of the vector $\mathbf{K}'_O$ is given by $\dot{\mathbf{K}}'_O = \boldsymbol{\omega}' \times \mathbf{K}'_O = I_3\boldsymbol{\omega}' \times \overline{\boldsymbol{\omega}}$, we find again the formula (15.2.17") which gives the moment $\mathbf{M}_O$. In a scalar form, we can write

$$M_O = I_3\overline{\omega}\omega'\sin\theta\,, \tag{16.2.3}$$

in the limits of the hypothesis made. In general, we will have $M_O = \omega' K'_O \sin(\theta - \delta)$, wherefrom

$$M_O = \left[I_3 - (J - I_3)\frac{\omega'}{\overline{\omega}}\cos\theta\right]\overline{\omega}\omega'\sin\theta\,, \tag{16.2.3'}$$

corresponding to the formula (15.2.17'). In particular, if $\theta = \pi/2$, hence if $Ox_3 \perp Ox'_3$, then we find again the formula (16.2.3), which is now an exact formula.

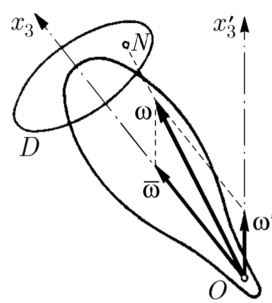


Fig. 16.10 Pohl's experiment

To put in evidence the motion – described above – of the gyroscope, we can make – together with Pohl – a simple experiment. We fix on the gyroscope axis a disc D on which we have put a printed paper (e.g., from a journal) (Fig. 16.10). During the rotation of the gyroscope, one cannot distinguish the letters (one can see only a uniform gray), excepting the piercing point N of the support of the vector $\boldsymbol{\omega}$, on the disc D (the velocity vanishes at the point N, so that the letters in the vicinity of this point are practically at rest and can be read). Pohl says that "the axis of rotation and the axis of the gyroscope rotate one around the other, as a pair of dancers".

If $\omega'/\varpi > 0$ (the positive sense of the components of the vector $\boldsymbol{\omega}$ is the positive sense of the co-ordinate axes Ox_3' and Ox_3 respectively, hence if $0 \leq \theta < \pi/2$, then we obtain

$$\tan\theta_p = \frac{\omega'\sin\theta}{\varpi + \omega'\cos\theta}, \quad \tan\theta_h = \frac{\varpi\sin\theta}{\omega' + \varpi\cos\theta}, \tag{16.2.4}$$

from (16.2.1). The cones $\mathscr{C}_p$ and $\mathscr{C}_h$ are exterior (as in Fig. 16.8); the motion of precession is progressive and the general motion of the gyroscope is *epicycloidal*.

If $\omega'/\varpi < 0$, then we have $\sphericalangle(\boldsymbol{\varpi}, \boldsymbol{\omega}') = \pi - \theta$ too, so that $\theta_h + \theta_p = \pi - \theta$, while

$$\tan\theta_p = -\frac{\omega'\sin\theta}{\varpi + \omega'\cos\theta}, \quad \tan\theta_h = -\frac{\varpi\sin\theta}{\omega' + \varpi\cos\theta}. \tag{16.2.4'}$$

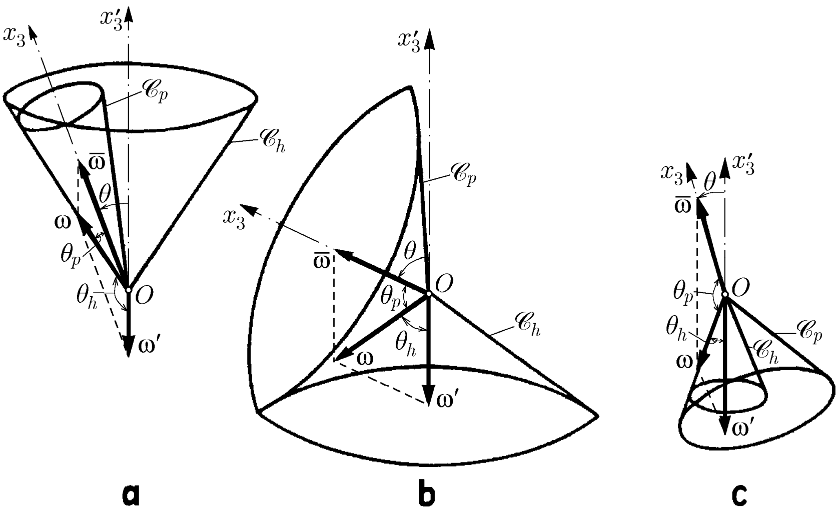


Fig. 16.11 The motion of the gyroscope: hypocycloidal (**a**), inverse epicycloidal (**b**) and inverse hypocycloidal (**c**)

In this case, the motion of precession is *retrograde*. We notice that $\pi/2 < \pi - \theta < \pi$, hence $0 < \pi/2 - \theta < \pi/2$. If $\theta_h > \pi/2$, then it results $\theta_p < \pi/2$ and $\tan\theta_p > 0$,

$\tan\theta_h < 0$, so that $-\cos\theta < \omega'/\varpi < 0$, the cone $\mathscr{C}_p$ being *interior* to the cone $\mathscr{C}_h$ (Fig. 16.11a); the motion of the gyroscope is *hypocycloidal*. If $\theta_p < \pi/2$ and $\theta_h < \pi/2$, then one can show that $\tan\theta_p > 0$ and that $\tan\theta_h > 0$, so that $-1/\cos\theta < \omega'/\varpi < -\cos\theta$, the cone $\mathscr{C}_p$ being *exterior* to the cone $\mathscr{C}_h$ (Fig. 16.11b); the motion of the gyroscope is *inverse epicycloidal*. If $\theta_p > \pi/2$, then it results $\theta_h < \pi/2$ and $\tan\theta_p < 0$, $\tan\theta_h > 0$, so that $\omega'/\varpi < -1/\cos\theta$, the cone $\mathscr{C}_h$ being interior to the cone $\mathscr{C}_p$ (Fig. 16.11c); the motion of the gyroscope is *inverse hypocycloidal* (*pericycloidal*).

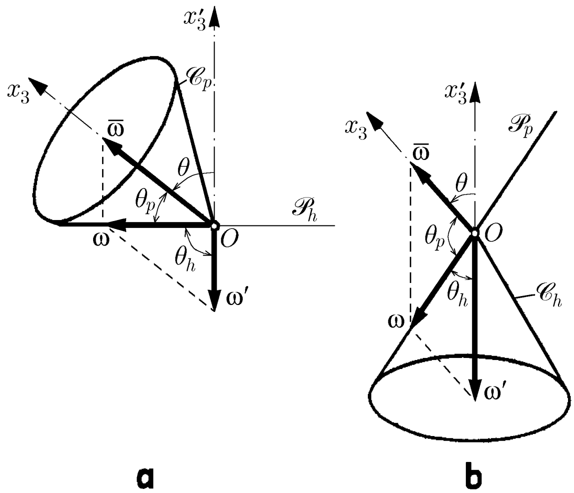


Fig. 16.12 The motion of the gyroscope: the cone $\mathscr{C}_p$ is rolling slidingless over the fixed plane $\mathscr{P}_h$ (**a**); the plane $\mathscr{P}_p$ is rolling slidingless over the fixed cone $\mathscr{C}_h$ (**b**)

The limit case $\omega'/\varpi = 0$ has been considered in Sect. 16.2.1.1. If $\omega'/\varpi = -\cos\theta$, then $\theta_h = \pi/2$ (the component $\boldsymbol{\omega}'$ is normal to the vector $\boldsymbol{\omega}$), while the cone $\mathscr{C}_h$ is reduced to the plane $\mathscr{P}_h$, which passes through the vector $\boldsymbol{\omega}$, being normal to Ox_3' (Fig. 16.12a); the cone $\mathscr{C}_p$ is rolling without sliding over the fixed plane $\mathscr{P}_h$. If $\omega'/\varpi = -1/\cos\theta$, then $\theta_p = \pi/2$ (the component $\boldsymbol{\varpi}$ is normal to the vector $\boldsymbol{\omega}$), while the cone $\mathscr{C}_p$ is reduced to the plane $\mathscr{P}_p$, which passes through the vector $\boldsymbol{\omega}$, being normal to Ox_3 (Fig. 16.12b); this plane is rolling slidingless over the fixed cone $\mathscr{C}_h$.

Let be a gyroscope fixed at its centre of gravity C ($O \equiv C$) and subjected to the action of its own weight **G** (the Euler-Poinsot case); we assume that to this gyroscope is imparted an initial rotation angular velocity ω_0, which makes an angle θ_p with the Ox_3-axis of the gyroscope (Fig. 16.13). If the fixed axis is Ox_3' and $\theta = \sphericalangle(Ox_3', Ox_3)$, then we can write $\omega/\sin\theta = \varpi/\sin(\theta - \theta_p) = \omega'/\sin\theta_p$

(obviously, $\omega = \omega_0$, the magnitude of the angular velocity being constant). Noting that $\mathbf{M}_O = \mathbf{0}$, the formula (15.2.17') leads to the relation $I_3 - (J - I_3)(\omega'/\overline{\omega})\cos\theta = 0$. Eliminating the ratio $\omega'/\overline{\omega}$, these results lead to

$$\tan\theta = \frac{J}{I_3}\tan\theta_p\,, \tag{16.2.5}$$

so that we can determine the position of the fixed axis Ox_3' in the plane formed by the Ox_3-axis and the initial angular velocity $\boldsymbol{\omega}_0$. We obtain, as well,

$$\overline{\omega} = \frac{J - I_3}{J}\omega\cos\theta_p\,, \quad \omega' = \frac{\omega}{J}\left(J^2\sin^2\theta_p + I_3^2\cos^2\theta_p\right)^{1/2}, \tag{16.2.5'}$$

the last relation corresponding to the first relation (16.2.2). Hence, *if to a heavy gyroscope, hanged at its centre of gravity, one gives an initial rotation about an axis inclined with the angle θ_p with respect to its axis, then the gyroscope becomes a motion of precession about a fixed axis situated in the plane determined by its initial position and by the axis of initial rotation and which makes the angle θ* (specified by the formula (16.2.5)) *with the initial position.* We must notice that the fixed axis (the Ox_3'-axis) is not necessarily vertical.

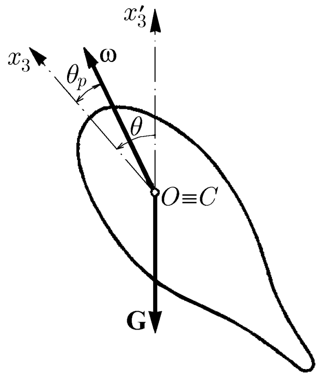


Fig. 16.13 The motion of the gyroscope acted upon by its own weight **G** and for which $O \equiv C$

The first formula (16.2.2) gives K_O', while the second formula leads to $\delta = 0$; hence, the moment of momentum $\mathbf{K}_O'$ is directed along the fixed axis Ox_3'. Indeed, from the theorem of moment of momentum it results $\dot{\mathbf{K}}_O' = \mathbf{0}$, hence $\mathbf{K}_O'$ has a fixed direction in space; but this vector is contained in the Ox_3x_3'-plane, which has only one fixed direction, i.e. Ox_3'.

In the particular case in which $\theta_p = 0$ it results $\theta = 0$, $\omega' = 0$, $\overline{\omega} = \omega$, while the Ox_3'-axis will coincide with the Ox_3-axis; we are in the case considered in Sect. 16.2.1.1.

16.2.1.3 The Motion of Regular Precession of a Heavy Gyroscope

Let be a gyroscope of weight **G**, having the fixed point O situated on the symmetry axis Ox_3, subjected to a proper rotation of angular velocity $\overline{\boldsymbol{\omega}}$. The moment of the given external forces is $\mathbf{M}_O = \overrightarrow{OC} \times \mathbf{G}$, of magnitude $M_O = Gl \sin\alpha$, where $l = \overline{OC}$, while α is the angle made by the Ox_3-axis with the vertical line (Fig. 16.14) (we are in the Lagrange-Poisson case). To obtain a motion of regular precession ($\omega' = \text{const}$ and $\theta = \text{const}$), together with a uniform proper rotation ($\overline{\omega} = \text{const}$), we must have $M_O = \text{const}$, in conformity to the formula (16.2.3'), wherefrom it results $\alpha = \text{const}$. The support of the vector $\mathbf{M}_O$ must be, during the motion, normal to the Ox_3x_3'-plane (along the line of nodes), corresponding to the formula (15.2.17'); it describes a horizontal plane, the Ox_3'-axis being thus vertical and $\alpha = \theta$. At the initial moment, the angular velocities $\overline{\omega}$ and ω' must verify the condition

$$I_3 \overline{\omega}\omega' + (I_3 - J)\omega'^2 \cos\theta = Gl, \tag{16.2.6}$$

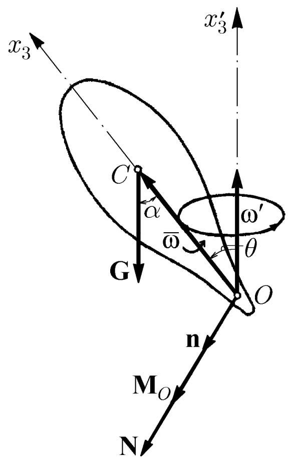


Fig. 16.14 The motion of the gyroscope with a progressive precession

corresponding to the condition (15.2.11), to can obtain the wanted motion. In the case in which $\omega'/\varpi \ll 1$ or in the case in which $(I_3 - J)/I_3 \ll 1$, we get the approximate formula

$$\omega' = \frac{Gl}{I_3 \varpi}, \tag{16.2.6'}$$

which specifies the angular velocity ω'; this formula is exact if $\theta = \pi/2$. If these initial conditions do not hold, then a supplementary motion of nutation ($\theta \neq \text{const}$) appears. In general,

$$\omega' = \frac{1}{2(J - I_3)\cos\theta}\left[I_3 \varpi \pm \sqrt{I_3^2 \varpi^2 + 4Gl(I_3 - J)\cos\theta} \right], \tag{16.2.6''}$$

obtaining two values for the angular velocity ω', with the condition $\varpi > (2/I_3)\sqrt{(J - I_3)Gl\cos\theta}$, if $J > I_3$; if $J < I_3$, then the quantity under the radical is always positive (we assume that $\cos\theta > 0$).

Practically, it is difficult to obtain angular velocities which fulfil the above conditions; as well, the frictions and the resistance of the air have an influence on the angular velocities $\boldsymbol{\varpi}$ and $\boldsymbol{\omega}'$, which can remain constant only with a certain approximation.

In the case considered in Fig. 16.14 ($\overrightarrow{OC}$ has the same sense as $\boldsymbol{\varpi}$), the moment $\mathbf{M}_O$ is directed in the positive sense of the ON-axis, the precession being progressive (such a gyroscope can be, e.g., a top, as that in Fig. 16.7b, but with an axis inclined with respect to the vertical line). If the fixed point O is on the other part of the mass centre C, so that $\boldsymbol{\varpi}$ has an inverse sense with respect to $\overrightarrow{OC}$, then the angular velocity $\boldsymbol{\omega}'$ is directed in a opposite sense with respect to the Ox_3'-axis and the moment $\mathbf{M}_O$ has a sense opposite to that of the ON-axis; thus the precession is retrograde. *The gyroscopic balance* allows to put in evidence this phenomenon. If, on the Ox_3-axis, on one part of the fixed point O we have a gyroscope of weight $\mathbf{G}$ (the symmetry axis of the gyroscope is along Ox_3) and if on the other part acts a weight $\mathbf{P}$ (which can glide along the axle), then the centre of gravity C, were acts the force $\mathbf{G} + \mathbf{P}$, will be on one part or on the other one with respect to the point O; thus, the motion of precession about the fixed vertical axis Ox_3' can change its sense.

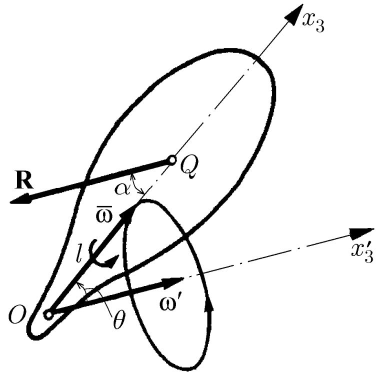


Fig. 16.15 The motion of the gyroscope for which the resultant **R** is applied at Q on the symmetry axis

More general, if the resultant of the given external forces to which is subjected the gyroscope is a force **R** applied at the point Q on the symmetry axis, inclined by the angle α with respect to the Ox_3-axis (Fig. 16.15), then we obtain $\mathbf{M}_O = \overrightarrow{OQ} \times \mathbf{R}$ and $M_O = Rl\sin\alpha = \text{const}$, where $l = \overline{OQ}$. Making the same considerations as in the preceding case, we obtain analogous results; thus $\alpha = \theta$, with the only difference that the Ox_3'-axis is not necessarily vertical, but is parallel to the resultant **R**. If the angular velocity $\overline{\boldsymbol{\omega}}$ is sufficiently great, the velocity $\boldsymbol{\omega}'$ is given, with a good approximation, by the formula (16.2.6'), the angle being practically constant. For an illustration of these results, we present the experiment of Charron. In this case, the Ox_3-axis of the gyroscope is formed by a magnet NS, the point O being the pole S (Fig. 16.16). We bring close to the pole N a horizontal magnet m with a pole P; if the gyroscope is immobile, then to can attract the pole N it is necessary that the pole P be a south pole. If we impart to the gyroscope an angular velocity $\overline{\boldsymbol{\omega}}$ and if we bring near the pole N the magnet with the pole P as north pole, then we see that the point N of the gyroscope comes close to this one, instead to move away; it seems that, paradoxically, the two poles are attracted instead to be repulsed. As a matter of fact, the pole N is repulsed with a force **F** which is composed with the weight **G** of the gyroscope, giving the resultant **R**, which pierces the Ox_3-axis at the point Q; in conformity with the results obtained above, the gyroscope will have a motion of precession about the Ox_3-axis, parallel to the resultant **R**, coming near to the magnet m. As we come closer to the gyroscope with the magnet m, as the force **F** grows in intensity, while the resultant **R** and the Ox_3'-axis are inclined more, the point N coming closer to the magnet. If the pole P is a south pole, then the gyroscope is moving away from the magnet, giving the paradoxical impression that poles of opposite sense are repulsive.

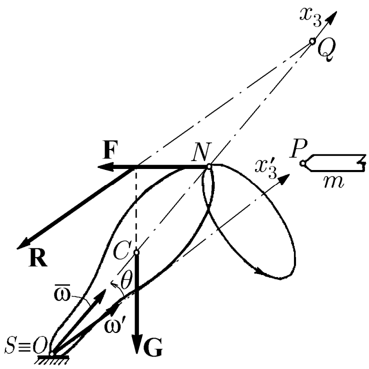


Fig. 16.16 Charron's experiment

One can establish an interesting analogy between the motion of regular precession of the gyroscope and the uniform circular motion of a particle. Thus, to the constant velocity $\mathbf{v}$ of the particle corresponds a constant angular velocity $\boldsymbol{\omega}$ of the gyroscope, to the constant force $\mathbf{F}$ which acts upon the particle corresponds the constant moment $\mathbf{M}_O$ which acts upon the gyroscope and to $\mathbf{F} \perp \mathbf{v}$ corresponds $\mathbf{M}_O \perp \boldsymbol{\omega}$, hence the work of the force $\mathbf{F}$ vanishes ($\mathbf{F} \cdot \mathbf{v} = 0$), corresponding $\mathbf{M}_O \cdot \boldsymbol{\omega} = 0$ (the work of the moment $\mathbf{M}_O$ is, as well, zero). The momentum vector $m\mathbf{v}$, of constant magnitude, is rotating about the fixed point with a constant velocity, while the moment of momentum vector $\mathbf{K}_O'$, constant in magnitude, rotates with a constant velocity about the fixed axis Ox_3'. If the force $\mathbf{F}$ is no more acting, then $m\mathbf{v}$ maintains a fixed direction and the particle has a uniform and rectilinear motion; if the moment $\mathbf{M}_O$ does no more act, then the vector $\mathbf{K}_O'$ maintains a fixed direction, while the gyroscope has a natural regular precession about it.

16.2.1.4 The Gyroscopic Effect in Case of the Gyroscope Acted Upon by its Own Weight

Let us consider a heavy gyroscope fixed at the point O on its axis of symmetry (the Lagrange-Poisson case) (Fig. 16.14). We assume that, in the initial position, determined by Euler's angles $\psi_0, \theta_0, \varphi_0$, $0 < \theta_0 < \pi$, a rapid initial rotation is imparted to the gyroscope about its axis ($\dot{\psi}_0 = \dot{\theta}_0 = 0$, hence $\omega_1^0 = \omega_2^0 = 0$, and $\dot{\varphi}_0 = \omega_3^0 = \text{const}$, the spin ω_3^0 being non-zero); the first integrals (15.2.2) lead to the relations $\alpha = a\omega_3^0 \cos\theta_0$, $\beta = b\cos\theta_0$. In this case, the relations (15.2.3), (15.2.8) lead to

$$P(u) = (u_0 - u)\left[b(1 - u^2) - a^2 (\omega_3^0)^2 (u_0 - u)\right], \tag{16.2.7}$$

$$\dot{\psi} = a\omega_3^0 \frac{u_0 - u}{1 - u^2}, \quad \dot{\varphi} = \omega_3^0 - a\omega_3^0 \frac{(u_0 - u)u}{1 - u^2}, \tag{16.2.7'}$$

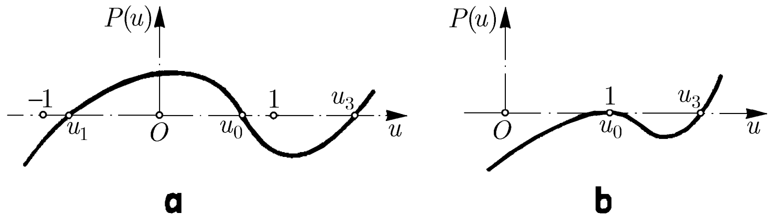


Fig. 16.17 The zeros of the polynomial $P(u)$: general case (**a**); particular case (**b**)

where $u_0 = \cos\theta_0$ and where we use the notations in Sect. 15.2.1.1. Following the general theory exposed in Sects. 15.2.1.1 and 15.2.1.2, in conformity to which $u_1 \leq u_0 \leq u_2$, $u_1 < u' \leq u_2$, one can have only $u_0 = u_2 = u'$, so that $u_1 \leq u \leq u_0$ (Fig. 16.17a). Hence, the zeros u_1 and u_3 of the polynomial $P(u)$ will be given by the equation of the second degree

$$u_0 - u = \frac{b}{a^2 \left(\omega_3^0\right)^2}\left(1 - u^2\right). \tag{16.2.7''}$$

But their calculation is not very useful, because we are interested in the behaviour of the solution for a great spin ω_3^0. From (16.2.7') it is seen that $\operatorname{sign}\dot{\psi} = \operatorname{sign}\omega_3^0$, so that the Ox_3-axis is rotating about the Ox_3'-axis (motion of precession) in the same sense as that of the initial rotation imparted to the gyroscope (about the Ox_3-axis) (Fig. 16.15).

From (16.2.7") one observes that $0 < u_0 - u_1 < b / a^2 (\omega_3^0)^2$, so that, for a great spin ω_3^0 or for a very small b (hence, for a fixed point O close to the mass centre C), the interval of variation of u is very small; practically, $u \cong u_0$, wherefrom $\theta \cong \theta_0$, the motion of nutation being, with a good approximation, negligible. For a more exact calculation, we put $\theta = \theta_0 + \varepsilon / (\omega_3^0)^2$, where $\varepsilon = \varepsilon(t)$ is a small parameter; it results thus that

$$\cos\theta = \cos\theta_0 \cos\frac{\varepsilon}{\left(\omega_3^0\right)^2} - \sin\theta_0 \sin\frac{\varepsilon}{\left(\omega_3^0\right)^2} \cong \cos\theta_0 - \frac{\varepsilon\sin\theta_0}{\left(\omega_3^0\right)^2},$$

which corresponds to an expansion into a Taylor series. Hence, $u = u_0 - \varepsilon\sin\theta_0 / (\omega_3^0)^2$, $\dot{u} = -\dot{\varepsilon}\sin\theta_0 / (\omega_3^0)^2$; taking into account (16.2.7) and neglecting the terms of higher order, the differential equation (15.2.3) with (16.2.7), reads

$$\dot{\varepsilon}^2 = \varepsilon\left(\omega_3^0\right)^2\left(b\sin\theta_0 - a^2\varepsilon\right). \tag{16.2.8}$$

Noting that $\varepsilon(t_0) = 0$, we find the solution

$$\varepsilon(t) = \frac{b}{2a^2} \sin\theta_0 \left[1 - \cos a\omega_3^0 (t - t_0)\right], \tag{16.2.8'}$$

wherefrom

$$\theta(t) = \theta_0 + \frac{b}{2a^2 \left(\omega_3^0\right)^2} \sin\theta_0 \left[1 - \cos a\omega_3^0 (t - t_0)\right]. \tag{16.2.9}$$

Hence, the nutation is periodical, of period $T = 2\pi / a\omega_3^0 = (J / I_3) 2\pi / \omega_3^0$, which decreases at the same time with the increasing of the spin ω_3^0 (the nutation is small in amplitude, but very rapid). The mean value of the angle of nutation will be

$$\theta_{\mathrm{med}} = \theta_0 + \frac{b}{2a^2 \left(\omega_3^0\right)^2} = \theta_0 + \frac{J}{I_3} \frac{g\rho_3}{i_3^2 \left(\omega_3^0\right)^2}, \tag{16.2.9'}$$

where we have introduced the radius of gyration $i_3 = \sqrt{I_3 / M}$.

From the first formula (16.2.7') it results

$$0 < \left|\dot\psi\right| < a\left|\omega_3^0\right| \frac{u_0 - u_1}{1 - u^2} = \frac{b}{a\left|\omega_3^0\right|} \frac{1 - u_1^2}{1 - u^2} < \frac{b}{a\left|\omega_3^0\right|} \frac{1}{1 - u^2} < \frac{b}{a\left|\omega_3^0\right|} \frac{1}{1 - u_*^2},$$

with $u_* = \max(|u_0|, |u_1|)$; hence, the angular velocity $\left|\dot\psi\right|$ is very small (of the order of magnitude of $1/\left|\omega_3^0\right|$ or of b, if the fixed point O is very close to the centre of gravity C), the motion of precession being very slow. The piercing point Q of the Ox_3-axis on the unit sphere describes a series of cycloids (we are in the case of Fig. 15.21c). Neglecting, as above, the powers of higher order of the small parameter ε, we find, analogously,

$$\dot\psi = \frac{a\varepsilon}{\omega_3^0 \sin\theta_0} = \frac{b}{2a\omega_3^0}\left[1 - \cos a\omega_3^0 (t - t_0)\right],$$

wherefrom

$$\psi(t) = \psi_0 + \frac{b}{2a^2 \left(\omega_3^0\right)^2}\left[a\omega_3^0 (t - t_0) - \sin a\omega_3^0 (t - t_0)\right]. \tag{16.2.10}$$

The velocity of precession oscillates about the mean value

$$\dot\psi_{\mathrm{med}} = \frac{b}{2a\omega_3^0} = \frac{g\rho_3}{i_3^2 \omega_3^0}. \tag{16.2.10'}$$

We notice also that the symmetry axis of the gyroscope does not shift in the vertical plane (as it would be expected, due to the initial conditions and because the rigid solid is acted upon only by its own weight – a vertical force **G**), but is shifted very slowly in a direction normal to this plane (normal to the force **G**); this effect, which is in contradictions with the direct intuition, being due to the spin ω_3^0, is called *gyroscopic effect.*

The second formula (16.2.7') allows to write

$$0 < \left|\dot{\varphi} - \omega_3^0\right| < a\left|\omega_3^0\right| \frac{(u_0 - u_1)|u|}{1 - u^2} = \frac{b}{a\left|\omega_3^0\right|} \frac{(1 - u_1^2)|u|}{1 - u^2}$$

$$< \frac{b}{a\left|\omega_3^0\right|} \frac{|u|}{1 - u^2} < \frac{b}{a\left|\omega_3^0\right|} \frac{|u_*|}{1 - u_*^2},$$

where u_* has the same significance as above; hence, the angular velocity $\dot{\varphi}$ differs very little from ω_3^0 (with a magnitude of the order of $1/\left|\omega_3^0\right|$), so that, from a practical point of view, the gyroscope is rotating about its axis with the initial angular velocity. Proceeding as above, we find $\dot{\varphi} = \omega_3^0 - (b/2a\omega_3^0)\cos\theta_0\left[1 - \cos a\omega_3^0 (t - t_0)\right]$, wherefrom

$$\varphi(t) = \varphi_0 + \left(\omega_3^0 - \frac{b\cos\theta_0}{2a\omega_3^0}\right)(t - t_0) + \frac{b\cos\theta_0}{2a^2\left(\omega_3^0\right)^2}\sin a\omega_3^0 (t - t_0), \tag{16.2.11}$$

the mean value of the velocity of proper rotation being

$$\dot{\varphi}_{\mathrm{med}} = \omega_3^0 - \frac{b\cos\theta_0}{2a\omega_3^0} = \omega_3^0 - \frac{g\rho_3\cos\theta_0}{i_3^2\omega_3^0}. \tag{16.2.11'}$$

These results are analogue to those in Sect. 16.1.2.3, corresponding to the same mechanical phenomenon.

16.2.1.5 The Sleeping Gyroscope

In the particular case in which $\theta_0 = 0$, hence $u_0 = 1$, we are led to $P(u) = (1 - u)^2\left[b(1 + u) - a^2\left(\omega_3^0\right)^2\right]$; the differential equation (15.2.3) reads

$$\dot{u} = (1 - u)\sqrt{b(1 + u) - a^2\left(\omega_3^0\right)^2}. \tag{16.2.12}$$

The Lipschitz condition is fulfilled by the second member in the neighbourhood of $u = 1$; on the basis of the theorem of existence and uniqueness, it results that the only solution which satisfies the initial condition $u_0 = 1$ is $u(t) = 1$ (hence, $\cos\theta = 1$ and $\theta = 0$). As a matter of fact, this result is illustrated in the Fig. 16.17b. The Ox_3-axis of the gyroscope remains all the time vertical, while the gyroscope "sleeps"! We obtain thus a configuration called the *sleeping gyroscope* (*the sleeping top*). From (16.2.7') one obtains

$$\dot{\psi} = \frac{1}{2} a\omega_3^0 , \quad \dot{\varphi} = \omega_3^0 - \frac{1}{2} a\omega_3^0 , \quad \dot{\psi} + \dot{\varphi} = \omega_3^0 , \tag{16.2.13}$$

the last relation corresponding to the third kinematic equation (5.2.35) too.

Putting $u_0 = 1 - \varepsilon$, $u = 1 - \eta$ and $\eta = \eta(t)$ being small parameters, hence giving a small perturbation to the vertical position, we obtain

$$P(u) = (\eta - \varepsilon)\left[b\eta(2 - \eta) - a^2 \left(\omega_3^0\right)^2 (\eta - \varepsilon)\right];$$

neglecting the powers higher to the second one, the equation (15.2.3) reads

$$\dot{\eta}^2 = \left[2b - a^2 \left(\omega_3^0\right)^2 \right]\eta^2 - 2\left[b - a^2 \left(\omega_3^0\right)^2 \right]\varepsilon\eta - a^2 \left(\omega_3^0\right)^2 \varepsilon^2 .$$

Differentiating with respect to time and simplifying by $2\dot{\eta}$ (if $\dot{\eta} = 0$, then we have $\eta = \mathrm{const}$ and we are led to a stable position of equilibrium), we can write

$$\ddot{\eta} + \left[a^2 \left(\omega_3^0\right)^2 - 2b \right]\eta = \left[a^2 \left(\omega_3^0\right)^2 - b \right]\varepsilon . \tag{16.2.14}$$

If $a^2 \left(\omega_3^0\right)^2 > 2b$, then one obtains a periodic motion of very small amplitude around the initial position, the motion being *stable*; if $a^2 \left(\omega_3^0\right)^2 \le 2b$, then it results an aperiodic motion, the amplitude of which increases in time, so that the motion is *labile*.

Using the notations in Sect. 15.2.1.1, we can state that the motion is stable if $\rho_3 < (I_3^2 / 4JMg)(\omega_3^0)^2$; hence, if the centre of gravity of the gyroscope is under the fixed point $(\rho_3 < 0)$, then the motion is stable. If the centre of gravity C is situated over the fixed point O (see Fig. 16.7b), then the motion of the gyroscope is stable only if we impart to it an initial rotation

$$\omega_3^0 > 2\sqrt{\frac{J}{I_3}\frac{1}{i_3}}\sqrt{g\rho_3} . \tag{16.2.15}$$

If, practically, we would have $\omega_3(t) = \omega_3^0 = \mathrm{const}$, then the gyroscope would continue to "sleep" for ever. But, because of the resistance of the air and of the friction at O, the velocity of proper rotation ω_3^0 decreases in time; the condition of stability (16.2.15) does no more hold and the axis of the gyroscope is inclined more and more, till this one falls.

16.2.1.6 The Gyroscopic Effect in Case of a Gyroscope Acted Upon by a Concentrated Force. The Gyroscopic Moment

In the case considered in Sect. 16.2.1.4, the moment of momentum $\mathbf{K}_O'$ is directed, at the initial moment along the Ox_3-axis of the gyroscope (Fig. 16.18a); we assume that

$\omega'/\overline{\omega} \ll 1$, to can be in the case of the regular precession, so that $\mathbf{K}'_O = \mathbf{K}'_O(t)$ be at any moment along the Ox_3-axis. The action of the own weight **G** leads to the moment $\mathbf{M}_O$ applied at O, while the theorem of moment of momentum reads $\dot{\mathbf{K}}'_O = \mathbf{M}_O$; for an interval of time Δt we have $\Delta\mathbf{K}'_O = \mathbf{M}_O\Delta t$. The extremity N of the vector $\mathbf{K}'_O$ will describe a director circle of the precession cone till the point $\widetilde{N}$ in the time interval Δt. Thus, the vector $\overrightarrow{N\widetilde{N}} = \Delta\mathbf{K}'_O = \tilde{\mathbf{K}}'_O - \mathbf{K}'_O$ will tend at the limit to $\mathrm{d}\mathbf{K}'_O$, being normal to the $Ox_3x'_3$-plane, which contains also the force **G**. The moment $\mathbf{M}_O$ will be along the same direction, the gyroscopic effect being thus put in evidence; this moment is given by the formula (15.2.17") or, more exactly, by the formula (15.2.17'). We notice also that, on the basis of the principle of action and reaction, the axis of the gyroscope will exert upon the exterior with which it comes in contact a gyroscopic moment $\mathbf{M}_g = -\mathbf{M}_O$. In this case

$$\mathbf{M}_g(t) = \left[I_3 - (J - I_3)\frac{\omega'}{\overline{\omega}}\cos\theta \right] \boldsymbol{\varpi} \times \boldsymbol{\omega}' \tag{16.2.16}$$

or, with a good approximation,

$$\mathbf{M}_g(t) = I_3\,\boldsymbol{\varpi} \times \boldsymbol{\omega}'. \tag{16.2.16'}$$

These two formulae are particularly useful in practice.

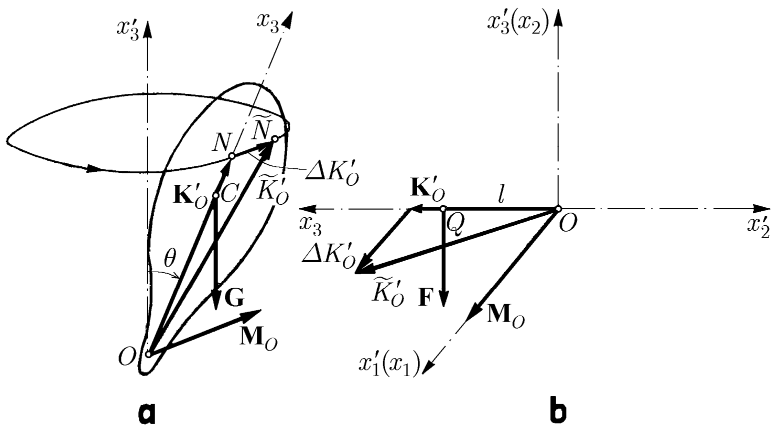


Fig. 16.18 The gyroscopic effect: the case $\mathbf{K}'_O \parallel Ox_3$ (**a**) and the case $\mathbf{F} \parallel Ox'_3$ (**b**)

Let us assume now that the moment $\mathbf{M}_O$ is due to a concentrated force **F**, which acts normal to the axis of rotation Ox'_3, at the point Q of it; intuitively, we would expect that, under the axis of this force, the axis does oscillate in the plane determined by its initial position and the force $\mathbf{F}$. But one sees that, due to the gyroscopic effect, the axis is rotating in the sense of the vector product $\boldsymbol{\omega}_3^0 \times \mathbf{F}$. Obviously, also in this case arises the gyroscopic moment $\mathbf{M}_g = -\overrightarrow{OQ} \times \mathbf{F}$.

To can study the problem, it is convenient to take – at the initial moment – the Ox_3-axis in the direction of the Ox_2'-axis in its negative sense, the Ox_1-axis along the Ox_1'-axis and the Ox_2-axis along the Ox_3'-axis (Fig. 16.18b). If the force $\mathbf{F} = -F\mathbf{i}_3'$ is in the negative sense of the Ox_3'-axis (we suppose that it maintains all the time its direction and its sense) and acts at the point Q on the axis of rotation, then the moment $\mathbf{M}_O = \overrightarrow{OQ} \times \mathbf{F} = (-l\mathbf{i}_2') \times (-F\mathbf{i}_3') = Fl\mathbf{i}_2' \times \mathbf{i}_3' = Fl\mathbf{i}_1'$ is directed towards the positive sense of the Ox_1'-axis. If the moment of momentum $\mathbf{K}_O'$, situated along the Ox_3-axis has a variation $\Delta\mathbf{K}_O'$ parallel to $\mathbf{M}_O$ (corresponding to the relation $\Delta\mathbf{K}_O' = \mathbf{M}_O \Delta t$), in the $Ox_1'x_2'$-plane, then one obtains a moment of momentum $\tilde{\mathbf{K}}_O' = \mathbf{K}_O' + \Delta\mathbf{K}_O'$ along the new position of the axis of the gyroscope. The gyroscopic effect (the displacement of the Ox_3-axis in the horizontal plane $Ox_1'x_2'$ towards the Ox_1'-axis) is thus justified. The initial conditions (at the moment $t = t_0$) are of the form $\psi = 0$, $\theta_0 = \pi/2$, $\varphi_0 = 0$, $\dot{\psi}_0 = 0$, $\dot{\theta}_0 = 0$, wherefrom $\omega_1^0 = \omega_2^0 = 0$ and $\omega_3^0 = \dot{\varphi}_0$. The equations of motion lead to first integrals of the form (15.2.1'''), wherefrom, taking into account the initial conditions, it results $K_{O3'}' = 0$, $I_3 \left(\omega_3^0\right)^2 = 2h$, obtaining thus $\alpha = 0$, $\beta = 0$. If the spin vanishes ($\omega_3^0 = 0$), then the first integrals (15.2.2) have only the solution (we replace Mg by F and ρ_3 by l)

$$\psi(t) = 0\,, \quad \varphi(t) = 0\,, \quad J\dot{\theta}^2 + 2Fl\cos\theta = 0\,. \tag{16.2.17}$$

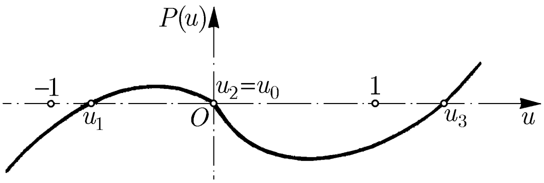


Fig. 16.19 The zeros of the polynomial $P(u)$: the case $u_2 = u_0 = 0$

Hence, the axis of the gyroscope has a pendular oscillation about the fixed point O, in a vertical plane. If $\omega_3^0 \neq 0$, then we are led to

$$\dot{\psi} = -a\omega_3^0 \frac{u}{1-u^2}\,, \quad a = \frac{I_3}{J}\,, \quad \dot{u}^2 = -bu(u-u_1)(u_3-u)\,,$$
$$u_1 = \lambda - \sqrt{1+\lambda^2} < 0\,, \quad u_3 = \lambda + \sqrt{1+\lambda^2} > 1\,, \tag{16.2.18}$$

$$\lambda = \frac{a^2 \left(\omega_3^0 \right)^2}{2b} = \frac{I_3^2 \left(\omega_3^0 \right)^2}{4JFl} .$$

It results that $u_1 \leq u \leq 0$, hence $\theta(t) \geq \pi / 2$ (Fig. 16.19); one obtains $\dot{\psi} > 0$, for $t > t_0$, the gyroscopic phenomenon being thus put in evidence. By means of Jacobi's elliptic functions, the formulae (15.2.5'), (15.2.6') allow to express u in the form

$$u = u_1 \operatorname{cn}^2 p(t - t_0), \quad p = \frac{1}{2}\sqrt{2\lambda b}\sqrt[4]{1 + \frac{1}{\lambda^2}} = \frac{1}{2} a\omega_3^0 \left[1 + \frac{b^2}{a^4 \left(\omega_3^0 \right)^4} - \ldots \right] . \tag{16.2.19}$$

We can use the results obtained concerning the gyroscopic effect to show the stability character of the motion of a gyroscope subjected to an initial rotation about its axis of symmetry (corresponding to the initial conditions assumed above). Choosing the co-ordinate axes as above, the moment of momentum $\mathbf{K}'_O$ will be directed along the Ox_3-axis (Fig. 16.20). We assume a deviation of angle $\mathrm{d}\psi$ of the axis of the gyroscope in the horizontal plane $Ox'_1x'_2$; in conformity to the previous considerations, intervenes the differential $\mathrm{d}\mathbf{K}'_O$ of the moment of momentum, parallel to the Ox_1-axis, corresponding to the moment $\mathbf{M}_O$, along this axis. Taking into account the theorem of moment of momentum written in a scalar form ($\mathrm{d}K'_O = M_O \mathrm{d}t$), we notice that $\mathrm{d}\psi = \tan(\mathrm{d}\psi) = \mathrm{d}K'_O / K'_O = (M_O / K'_O)\mathrm{d}t$, so that $M_O = K'_O \dot{\psi}$. The moment $\mathbf{M}_O$ has the tendency to impart to the gyroscope a motion of rotation of angle θ about the Ox_1-axis, governed by the equation $I_1 \ddot{\theta} = J\ddot{\theta} = K'_O \dot{\psi}$; integrating this equation, we get $J\dot{\theta} = K'_O \psi$, the initial conditions vanishing. To a displacement of angle $\mathrm{d}\theta$ of the gyroscope axis in a vertical plane one obtains a differential $\mathrm{d}\mathbf{K}'_O$ of the moment of momentum, parallel to the Ox_2-axis; it corresponds a moment $\mathbf{M}'_O$ of modulus $K'_O \dot{\theta}$ along this axis, in its negative sense. This moment has the tendency to impart a motion of rotation of angle ψ about the Ox_3-axis, in conformity to the equation $I_2 \ddot{\psi} = J\ddot{\psi} = -K'_O \dot{\theta}$. Eliminating the angle θ, we obtain the equation

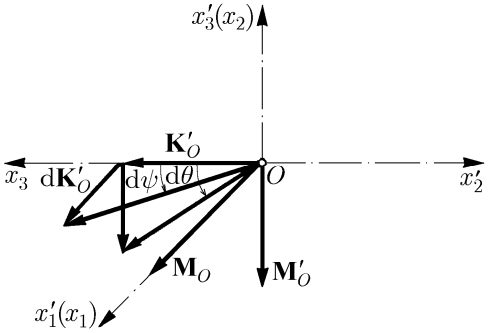


Fig. 16.20 The gyroscope effect: the case $\mathbf{K}_O' \parallel Ox_3$

$$\ddot{\psi} + \frac{(K_O')^2}{J^2}\psi = 0\,, \tag{16.2.20}$$

which leads to an angle ψ as a harmonic function of time; it results that, in the given initial conditions, the axis of the gyroscope is a stable axis of rotation.

16.2.1.7 Prandtl's Wheel. Tendency of Parallelism of the Axes of Rotation

The gyroscopic effect can be put in evidence by means of *Prandtl's wheel.* This is a homogeneous wheel, having a suspension which diminishes the frictions at the centre O; through this point passes a horizontal axle (along the symmetry axis of the gyroscope), hanged up so as to make possible the rotation about a horizontal axis, normal to that one, as well as about a vertical axis (Fig. 16.21). We take the Ox_3-axis along the symmetry axis of the gyroscope. Euler's equations of motion are of the form (16.1.1). We give to the wheel a rapid motion of rotation at the initial moment, taking the horizontal axle fixed ($\omega_1^0 = \omega_2^0 = 0$, $\omega_3^0 \neq 0$); then we let free the axle, so that the wheel does effect a motion about it.

If the gyroscope is acted upon only by its own weight, the centre of gravity being at O, then we have $M_{O1} = M_{O2} = M_{O3} = 0$. We get $\omega_1^2 + \omega_2^2 = \left(\omega_1^0\right)^2 + \left(\omega_2^0\right)^2 = 0$, from the first two equations (16.1.1), so that $\omega_1(t) = \omega_2(t) = 0$; the third equation leads to $\omega_3(t) = \omega_3^0$, the gyroscope continuing its initial motion of rotation about the Ox_3-axis. Noting that $\rho_3 = 0$, from (15.2.1") we get $I_3\left(\omega_3^0\right)^2 = 2h$, so that $\beta = 0$ and $b = 0$; the second first integral (15.2.2) gives $\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2 = 0$, wherefrom $\psi = \psi_0$, $\theta = \theta_0$. The third relation (15.2.2) leads then to $\dot{\varphi} = \omega_3^0$. Hence, we see that the axle of rotation Ox_3 remains fixed during the motion (the Ox_3-axis has the director cosines $\sin\theta_0 \sin\psi_0$, $-\sin\theta_0\cos\psi_0$, $\cos\theta_0$ with respect to the fixed frame of reference; we can take $\theta_0 = \psi_0 = 0$ too, because $Ox_3 \equiv Ox_3'$ as in Fig. 16.20), the wheel having a uniform rotation (we neglect the frictions and the resistance of the air) with respect to this axis; as much the velocity of rotation is greater, as much the parasite phenomena can be neglected.

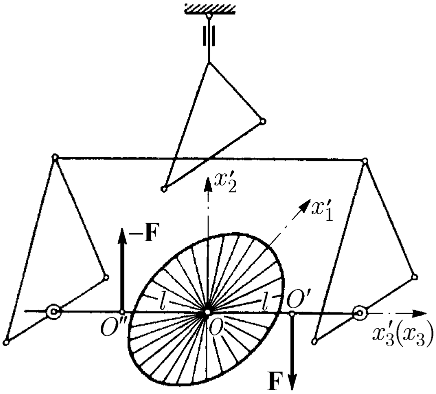


Fig. 16.21 Prandtl's wheel

If at a point O', at the distance l from the fixed point O, acts a force $\mathbf{F} = F_1\mathbf{i}_1 + F_2\mathbf{i}_2$, normal to the symmetry axis of the gyroscope (Fig. 16.21), then we get the moment $\mathbf{M}_O = -lF_2\mathbf{i}_1 + lF_1\mathbf{i}_2$. Because the initial angular velocity ω_3^0 is great, we can assume, in a first approximation, that $\dot{\omega}_1 \cong 0$, $\dot{\omega}_2 \cong 0$, remaining with the equations

$$(J - I_3)\omega_2\omega_3 = lF_2, \quad (J - I_3)\omega_1\omega_3 = lF_1, \tag{16.2.21}$$

which determine ω_1 and ω_2. The velocity v of the point O' is given by

$$\begin{aligned}\mathbf{v} = \boldsymbol{\omega}\times(l\mathbf{i}_3) &= l(\omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3)\times\mathbf{i}_3 = l(\omega_2\mathbf{i}_1 - \omega_1\mathbf{i}_2)\\ &= \frac{l^2}{(J - I_3)\omega_3^0}(F_2\mathbf{i}_1 - F_1\mathbf{i}_2)\end{aligned}$$

or by (we have $I_3 > J$)

$$\mathbf{v} = \frac{l}{(I_3 - J)\omega_3^0}\overrightarrow{OO'}\times\mathbf{F} = \frac{l}{(I_3 - J)\omega_3^0}\mathbf{M}_O. \tag{16.2.22}$$

Hence, after the application of the external force $\mathbf{F}$, normal to the gyroscope axis, the point O' will be displaced along a direction normal to the plane formed by the force and by this axis, in the sense of the vector product $\mathbf{M}_O = \overrightarrow{OO'}\times\mathbf{F}$ (the sense of the Ox_1'-axis). This gyroscopic effect is as much greater as the initial angular velocity ω_3^0 is smaller (formula (16.2.22)); on the contrary, if we maintain fixed the axle Ox_3, then its reaction will be given by the force $\mathbf{F}$, being as much intense as ω_3^0 is greater (formula (16.2.21)).

If, initially, the wheel is at rest, being applied only the force **F** at the point O of the axle Ox_3, to the latter one will be imparted a motion of rotation about the Ox_1'-axis (normal to Ox_3' and to **F**, in the sense of the vector product $\overrightarrow{OO'} \times \mathbf{F}$); hence, the gyroscopic effect, put in evidence by the formula (16.2.22), tends to bring the symmetry axis of the gyroscope (the Ox_3-axis), on the shortest way, in the direction and the sense of the rotation axis of the rigid solid (the Ox_1'-axis), if this one would be acted upon, from its state of rest, by the force **F**. This represents *the parallelism tendency of the rotation axes*, with multiple applications in technics.

Let us suppose that at the point O'' (specified by $\overrightarrow{OO''} = -l\mathbf{i}_3$) acts a force $-\mathbf{F}$ too; in this case, the velocity **v** given by (16.2.22) doubles its intensity and we will have $v = 2l^2 F / (I_3 - J)\omega_3^0$. If, independently, we apply to the mechanical system a rotation of angular velocity $\boldsymbol{\omega}_2' = \omega_2' \mathbf{i}_2'$ about the vertical axis Ox_2', then the motion which takes place (the velocity of the point O' is of magnitude $v = l\omega_2'$) is identical to that corresponding to the gyroscopic effect considered above; it results thus

$$F = \frac{\omega_3^0 \omega_2'}{2l}(I_3 - J). \tag{16.2.23}$$

If $I_3 \gg J$, then one obtains

$$F = \frac{I_3 \omega_3^0 \omega_2'}{2l}, \tag{16.2.23'}$$

a formula particularly useful in applications. The considered forces, of magnitude F, are called *gyroscopic reactions*; corresponding to the principle of action and reaction, the forces which have the same direction and the same magnitude, but opposite sense, are called *gyroscopic pressures*.

16.2.1.8 Inertial Forces in the Motion of Regular Precession of the Gyroscope

The motion of the gyroscope with respect to the fixed frame of reference (the absolute motion) can be obtained by composing the proper rotation of it about the symmetry axis Ox_3, with the constant angular velocity $\boldsymbol{\varpi}$ (the relative motion), with the motion of precession about the fixed axis Ox_3', with the constant angular velocity $\boldsymbol{\omega}'$ (the motion of transportation); we assume that the angle θ between the two axes is constant (Fig. 16.22). An element $\mathrm{d}m$ of the gyroscope, situated at the point P, is subjected to the centrifugal forces (forces of transportation) due to the proper rotation and to the motion of precession, as well as to the Coriolis force.

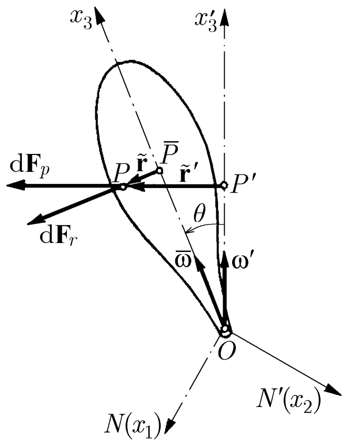


Fig. 16.22 Inertial forces in the motion of regular precession of the gyroscope

The centrifugal force $\mathrm{d}\mathbf{F}_r = \bar{\mathbf{r}}\overline{\omega}^2 \mathrm{d}m$ ($\bar{\mathbf{r}} = \overrightarrow{\overline{P}P}$, where $\overline{P}$ is the projection of P on the Ox_3-axis), due to the proper rotation, leads to a zero torsor with respect to the point O on the Ox_3-axis, because of the properties of symmetry with respect to this axis ($\mathbf{F}_r = \mathbf{0}$, $\mathbf{M}'_{Or} = \mathbf{0}$).

The centrifugal force $\mathrm{d}\mathbf{F}_p = \bar{\mathbf{r}}'\omega'^2 \mathrm{d}m$ ($\bar{\mathbf{r}}' = \overrightarrow{P'P}$, where P' is the projection of P on the Ox_3'-axis) due to the motion of precession, leads to the resultant $\mathbf{F}_p = \int_M \bar{\mathbf{r}}'\omega'^2 \mathrm{d}m$ and to the resultant moment $\mathbf{M}_{Op} = \int_M \overrightarrow{OP} \times \bar{\mathbf{r}}'\omega'^2 \mathrm{d}m$. To simplify the calculation, we choose the Ox_1-axis along the line of nodes ON and the Ox_2-axis along the transverse line ON'. Corresponding to the point $P(x_1, x_2, x_3)$, we obtain $\overline{OP'} = \overrightarrow{OP} \cdot \mathbf{i}_3' = (x_j \mathbf{i}_j) \cdot (\mathbf{i}_3 \cos\theta + \mathbf{i}_2 \sin\theta) = x_2 \sin\theta + x_3 \cos\theta$, so that the co-ordinates of the point P' on the Ox_3'-axis are

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = (x_2 \sin\theta + x_3 \cos\theta)\sin\theta, \quad \tilde{x}_3 = (x_2 \sin\theta + x_3 \cos\theta)\cos\theta.$$

We are thus led to

$$\mathrm{d}F_{p1} = \omega'^2 x_1 \mathrm{d}m, \quad \mathrm{d}F_{p2} = \omega'^2 [x_2 - (x_2 \sin\theta + x_3 \cos\theta)\sin\theta]\mathrm{d}m,$$
$$\mathrm{d}F_{p3} = \omega'^2 [x_3 - (x_2 \sin\theta + x_3 \cos\theta)\cos\theta]\mathrm{d}m.$$

Taking into account the relations which give the static moments and the position of the mass centre $C(0,0,\rho_3)$ on the symmetry axis of the gyroscope, we get

$$F_{p1} = 0, \quad F_{p2} = -M\omega'^2 \rho_3 \sin\theta\cos\theta, \quad F_{p3} = -M\omega'^2 \rho_3 \sin^2\theta. \tag{16.2.24}$$

The resultant $\mathbf{F}_p$ is applied at O being contained in the plane Ox_3x_3' and normal to the Ox_3'-axis and having the magnitude

$$F_P = M\omega'^2\rho_3 \sin\theta = Ma_C', \tag{16.2.24'}$$

where $\mathbf{a}_C'$ is the acceleration of the mass centre with respect to the inertial frame of reference. If $O \equiv C$, then we have $\mathbf{F}_p = \mathbf{0}$.

Analogously,

$$\begin{aligned} \mathrm{d}M_{Op1} &= x_2\mathrm{d}F_{p3} - x_3\mathrm{d}F_{p2} = \omega'^2\{x_2[x_3 - (x_2\sin\theta + x_3\cos\theta)\cos\theta] \\ &\quad - x_3[x_2 - (x_2\sin\theta + x_3\cos\theta)\sin\theta]\}\mathrm{d}m, \\ \mathrm{d}M_{Op2} &= \omega'^2\{x_3x_1 - x_1[x_3 - (x_2\sin\theta + x_3\cos\theta)\cos\theta]\}\mathrm{d}m, \\ \mathrm{d}M_{Op3} &= \omega'^2\{x_1[x_2 - (x_2\sin\theta + x_3\cos\theta)\sin\theta - x_2x_1]\}\mathrm{d}m. \end{aligned}$$

By means of the relations which give the principal moments of inertia (the centrifugal moments of inertia vanish), we obtain

$$M_{Op1} = (J - I_3)\omega'^2\sin\theta\cos\theta, \quad M_{Op2} = M_{Op3} = 0; \tag{16.2.25}$$

hence, the moment

$$\mathbf{M}_{Op} = (J - I_3)\omega'^2\cos\theta\mathbf{i}_3' \times \mathbf{i}_3 = (J - I_3)\frac{\omega'}{\varpi}\cos\theta\boldsymbol{\omega}' \times \boldsymbol{\varpi} \tag{16.2.25'}$$

is directed along the line of nodes.

To calculate the Coriolis force $\mathrm{d}\mathbf{F}_C = 2\mathbf{v}_r \times \boldsymbol{\omega}'\mathrm{d}m$, we notice that the relative velocity is given by $\mathbf{v}_r = \boldsymbol{\varpi} \times \mathbf{r}$; by considerations analogous to those above, we get a vanishing resultant

$$\mathbf{F}_C = 2\int_M (\boldsymbol{\varpi} \times \mathbf{r}) \times \boldsymbol{\omega}'\mathrm{d}m = -2\boldsymbol{\omega}' \times \left(\boldsymbol{\varpi} \times \int_M \mathbf{r}\mathrm{d}m\right) = \mathbf{0}.$$

The resultant moment is given by

$$\mathbf{M}_{OC} = 2\int_M \overrightarrow{OP} \times [(\boldsymbol{\varpi} \times \mathbf{r}) \times \boldsymbol{\omega}']\mathrm{d}m$$

and has the differential components

$$\begin{aligned} \mathrm{d}M_{OC1} &= x_2\mathrm{d}F_{C3} - x_3\mathrm{d}F_{C2} = [x_2(-2\varpi\omega' x_2\sin\theta) - x_3(2x_2\varpi\omega'\cos\theta)]\mathrm{d}m, \\ \mathrm{d}M_{OC2} &= [x_3(2x_1\varpi\omega'\cos\theta) - x_1(-2\varpi\omega' x_2\sin\theta)]\mathrm{d}m, \\ \mathrm{d}M_{OC3} &= [x_1(2x_2\varpi\omega'\cos\theta) - x_2(2x_1\varpi\omega'\cos\theta)]\mathrm{d}m = 0, \end{aligned}$$

where we took into account the relations

$$
\begin{gathered}
(\boldsymbol{\varpi} \times \mathbf{r}) \times \boldsymbol{\omega}' = -(\boldsymbol{\omega}' \cdot \mathbf{r})\boldsymbol{\varpi} + (\boldsymbol{\omega}' \cdot \boldsymbol{\varpi})\mathbf{r}\,, \\
\boldsymbol{\omega}' \cdot \mathbf{r} = (\omega' \sin\theta)(r \sin\varphi) = \omega' x_2 \sin\theta\,, \\
\boldsymbol{\omega}' \cdot \boldsymbol{\varpi} = \omega' \varpi \cos\theta\,,
\end{gathered}
$$

with $\varphi = \sphericalangle(Ox_1, \mathbf{r})$. Using the known results concerning the tensor of inertia, we obtain

$$
M_{OC1} = -2\varpi\omega' \sin\theta \int_M x_2^2 \mathrm{d}m = -\varpi\omega' \sin\theta \int_M (x_1^2 + x_2^2)\,\mathrm{d}m\,,
$$

so that

$$
M_{OC1} = -I_3 \varpi\omega' \sin\theta\,, \quad M_{OC2} = M_{OC3} = 0\,, \tag{16.2.26}
$$

the moment

$$
M_{OC} = -I_3 \varpi\omega' \mathbf{i}_3' \times \mathbf{i}_3 = -I_3 \boldsymbol{\omega}' \times \boldsymbol{\varpi}, \tag{16.2.26'}
$$

being directed along the line of nodes too.

Comparing with the relation (16.2.16), we get

$$
\mathbf{M}_g = -\mathbf{M}_O = \mathbf{M}_{Op} + \mathbf{M}_{OC}\,. \tag{16.2.27}
$$

Hence, the gyroscopic moment corresponds to the influence of the inertial forces. We notice that, *by passing from the formula* (16.2.16) *to the approximate formula* (16.2.16'),we neglect $\mathbf{M}_{Op}$, hence *the effect of the centrifugal forces in the motion of precession*, remaining only the effect of the Coriolis forces.

We can replace the torsor $\{\mathbf{F}_p, \mathbf{M}_g\}$ by a resultant $\mathbf{F}_p$ (gyroscopic pressure) in the Ox_3x_3'-plane at a distance M_g / F_p from the fixed point *O*.

16.2.2 Applications

In what follows, we present firstly some applications with a theoretical character, as well as the Cardanic suspension and the gyroscopic pendulum, introducing the influence of the friction forces too. We deal then with some technical applications of the theory of the gyroscope.

16.2.2.1 The Gyroscope with a Cardanic Suspension

The centre of gravity of a gyroscope may be practically fixed with the aid of the suspension discovered in 1545 by Gerolamo Cardano. The gyroscope has the Ox_3-axis hinged in the interior annulus a_i, which is hinged in the exterior annulus a_e by the horizontal axis ON, which can rotate about the fixed vertical axis Ox_3'; because the Ox_3-axis can rotate about both the horizontal and the vertical axes, while the gyroscope can have a proper rotation about this axis, it results that the gyroscope can take any position around its centre of gravity (Fig. 16.23). As a matter of fact, by the rotation of the annulus a_e about the Ox_3'-axis one determines the angle ψ, while by the rotation of the annulus a_i about the ON-axis is specified the angle θ; as well, by the rotation of the gyroscope about the Ox_3-axis one obtains the angle φ. Thus, the position of the gyroscope is given by Euler's angles, which can be measured. To obtain a *Cardanic suspension* of good quality, it is necessary an as good as possible centring of the rotation axes (to eliminate the parasite moments which may appear), eventually with the aid of some adjustable wedging screws; as well, it is necessary to have minimal frictions in the hinges (to eliminate the effect of the moments of friction), while the weights of all accessories be negligible with respect to the own weight of the gyroscope (for analogous reasons).

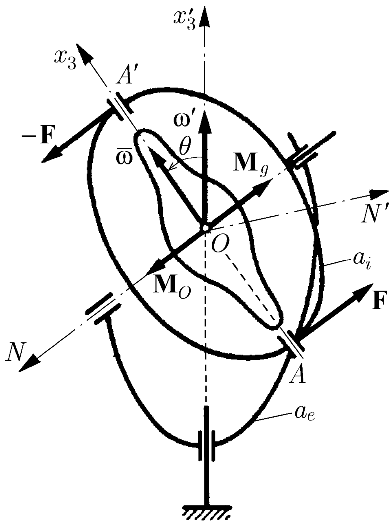


Fig. 16.23 The Gyroscope with a Cardanic Suspension

Let us assume that the gyroscope with Cardanic suspension is acted upon by external forces for which $\mathbf{M}_O = \mathbf{0}$; but if to this gyroscope is imparted, in a certain manner, a proper angular velocity $\bar{\boldsymbol{\omega}}$, about the Ox_3-axis, and then an angular velocity of precession $\boldsymbol{\omega}'$, about the Ox_3'-axis (which must not be necessarily vertical), then arises a gyroscopic moment $\mathbf{M}_g$, given by (16.2.16) or – approximately – by (16.2.16').

Taking into account the sense of this moment, it results that the angle of nutation cannot remain constant, becoming smaller. The Ox_3-axis of the gyroscope tends to the fixed axis Ox_3' on the shortest way, so that the senses of the vectors $\boldsymbol{\varpi}$ and $\boldsymbol{\omega}'$ do coincide; this is, after F. Klein and A. Sommerfeld, *the parallelism tendency of the rotation axes of the gyroscope*, put in evidence by Prandtl's wheel too (see Sect. 16.2.1.7).

If, in the preceding case, we wish to maintain the angle θ constant, then we must annihilate the effect of the gyroscopic moment $\mathbf{M}_g$, by introducing a moment $\mathbf{M}_O = -\mathbf{M}_g$, given by the external forces, along the line of nodes ON. To do this, we act upon the Ox_3-axis of the gyroscope with the force $\mathbf{F}$ at A and with the force $-\mathbf{F}$ at A', with the lever arm $\overline{AA'} = d$; these forces are *gyroscopic reactions* (the reactions of the annulus a_i on the Ox_3-axis). Noting that $M_O = Fd$, we obtain

$$F = \frac{I_3}{d}\varpi\omega'\sin\theta\,, \tag{16.2.28}$$

where we have used the approximate formula (16.2.16'). The force $-\mathbf{F}$ applied at A and the force $\mathbf{F}$ applied at A' represent *the gyroscopic pressures* (the pressures of the gyroscopic couple upon the annulus a_i); these pressures induce the tendency of parallelism mentioned above (see Sect. 16.2.1.7 too).

Let be a gyroscope with a Cardanic suspension at the centre of gravity, subjected to a proper rotation about its axis; we assume that upon this gyroscope acts also a moment $\mathbf{M}_O$, given by external forces. If this moment is directed along the Ox_3-axis of the gyroscope, then it will lead to the increasing or to the decreasing of the proper rotation velocity; if the respective moment is directed along the ON-axis of the interior annulus a_i, then its effect will be a precession about the fixed axis Ox_3', situated in the normal to ON plane, which passes through Ox_3. If the moment $\mathbf{M}_O$ is directed along the Ox_3'-axis of the exterior annulus a_e, then we make a decomposition of it along the Ox_3-axis (the respective effect has been already mentioned) and along the transverse axis ON'; this second component leads to a motion of precession of the Ox_3-axis of the gyroscope about the ON-axis. If the moment $\mathbf{M}_O$ directed along the Ox_3'-axis is constant, then one has a tendency of parallelism of the axes (the Ox_3-axis tends, on the shortest way, to the Ox_3'-axis). These considerations are particularly useful in various technical applications of the gyroscope.

16.2.2.2 The Influence of the Friction Forces on the Gyroscope

It is quite difficult to introduce the influence of the forces of friction in a mathematical model of the gyroscope. Let us consider, e.g., a heavy gyroscope Cardanically suspended at its centre of gravity. If we assume that the proper velocity of rotation $\boldsymbol{\varpi}$ is maintained constant by external means, then the effect of friction of the axle AA' of the gyroscope on the interior annulus a_i is annihilated (Fig. 16.23); because of the friction due to the motion of precession, the velocity ω' decreases. Because $\mathbf{M}_g = \mathbf{0}$, from (16.2.16) it results

$$I_3 \overline{\omega} + (I_3 - J)\,\omega' \cos\theta = 0\,. \qquad (16.2.29)$$

In this case, $\cos\theta$ must increase till 1 for $I_3 < J$ (prolate spheroid) or to decrease till -1 (oblate spheroid). It results thus $\theta = 0$ and $\theta = \pi$, respectively, the axis of the gyroscope taking the place of the axis of precession; the motion of the gyroscope is thus reduced to the proper rotation about its axis.

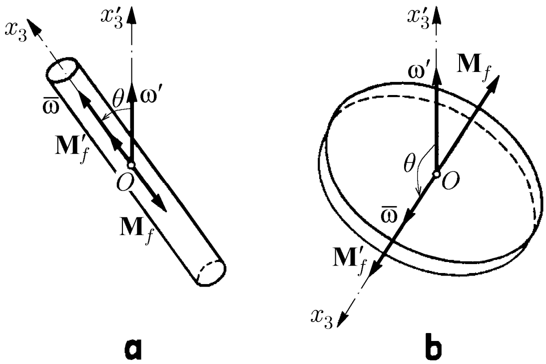


Fig. 16.24 The influence of a moment of friction $\mathbf{M}_f$ on the gyroscope: case of a prolate cylinder (**a**); case of an oblate cylinder (**b**)

If the velocity $\boldsymbol{\overline{\omega}}$ is no more maintained constant, then appears a moment of friction $\mathbf{M}_f$ along the axis of the gyroscope and of sense opposite to $\boldsymbol{\overline{\omega}}$, which leads to a decrease of $|\boldsymbol{\overline{\omega}}|$; the action of this couple on the annulus a_i is given by the moment $\mathbf{M}'_f$ ($\mathbf{M}_f + \mathbf{M}'_f = \mathbf{0}$), the projection of which on the axis of precession is $M'_f \cos\theta$. In the case of a prolate spheroid we have $0 < \theta < \pi/2$, the moment $M'_f \cos\theta$ tending to increase ω' as in Fig. 16.24a – prolate cylinder); $\cos\theta$ must decrease, hence θ must reach the value $\pi/2$, the axis of the gyroscope being normal to the axis of precession (labile position), so that the relation (16.2.29) be verified. If the gyroscope is an oblate spheroid, then it results $\pi/2 < \theta < \pi$, the moment $M'_f \cos\theta$ having the tendency to decrease ω' (Fig. 16.24b – oblate cylinder); $\cos\theta$ must increase, hence θ must decrease, tending to zero, the axis of the gyroscope coinciding thus with the axis of precession (stable position).

Let be now a heavy gyroscope, which is rotating with the proper velocity of rotation $\boldsymbol{\overline{\omega}}$ about the fixed point O, situated under the centre of gravity C (Fig. 16.25). If the initial velocity is great, then we can assume – with a good approximation – that the moment of momentum $\mathbf{K}'_O = I_3 \boldsymbol{\overline{\omega}}$ is directed along the Ox_3-axis, the velocity of the extremity Q of this vector being given by $\mathbf{v}_Q = \mathbf{M}_O$, where $\mathbf{M}_O$ is the moment of the own weight **G** with respect to the fixed point. Neglecting the pivoting friction around the fixed axis Ox_3', as well as the friction of nutation, we take into account the moment of sliding friction $\mathbf{M}_f$ at the fixed bearing support O; we assume that the moment $\mathbf{M}_f$ is constant and of horizontal direction, being contained in the Ox_3x_3'-plane. Decomposing $\mathbf{M}_f$ along the axes Ox_3 and Ox_3', we see that the velocity of proper rotation $\overline{\omega}$ decreases, while the velocity of precession ω' increases. In this case, the velocity of the point Q will be $\mathbf{v}_Q' = \mathbf{v}_Q + \mathbf{v}_Q^f = \mathbf{M}_O + \mathbf{M}_f$, so that $\mathbf{v}_Q^f = \mathbf{M}_f$. We notice that, under the action of the moment of friction $\mathbf{M}_f$, the axis of proper rotation of the gyroscope is brought on the shortest way to the vertical line (it is straightened);

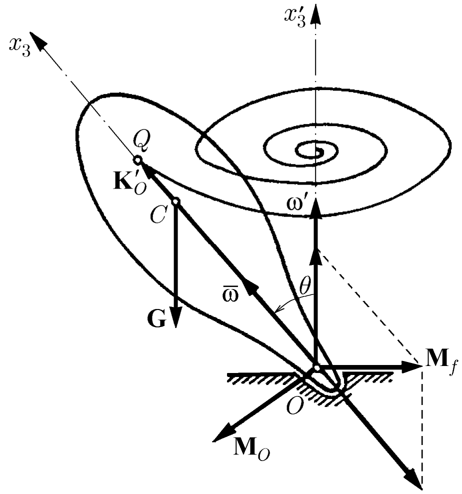


Fig. 16.25 The decreasing of the motion of nutation of a heavy gyroscope, due only to the moment of sliding friction

the point Q describes, in a horizontal plane, a spiral, attaining the vertical line after a finite number of nutations in a finite interval of time. Afterwards, the moment $\mathbf{M}_f$ disappears, remaining the action of the pivoting friction, which leads to a decreasing of the angular velocity $\overline{\omega}$ and of the moment of momentum K_O', till the gyroscope is falling. The velocity of nutation $\dot{\theta}$ changes always and fast its sense, the friction which arises changing its sense too; the effect of this friction consists in the decreasing of the velocity $\dot{\theta}$, hence in *the decreasing of the motion of nutation*.

16.2.2.3 The Influence of the Rotation of the Earth on the Gyroscope

Let be a Cardanically suspended centred gyroscope, to which was imparted an angular velocity of proper rotation $\overline{\boldsymbol{\omega}}$ about its Px_3-axis, situated in the horizontal plane Π of the position (the Px_3x_3'-plane), at the point P on the surface of the Earth, at the latitude λ (Fig. 16.26a). If the velocity of rotation of the Earth is $\boldsymbol{\omega}$, then we obtain the component $\boldsymbol{\omega}'' = \omega \sin \lambda \mathbf{n}$ along the local vertical (the line of nodes PN) and the component $\boldsymbol{\omega}' = \omega \cos \lambda \mathbf{i}'_3$ along the tangent Px'_3 to the local meridian, in the plane Π. Because of the rotations of velocities $\overline{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}'$ arises a gyroscopic moment $\mathbf{M}''_g$ of magnitude $M''_g = I_3 \overline{\omega} \omega'' = I_3 \omega \overline{\omega} \sin \lambda$, contained in the plane Π ($\overline{\boldsymbol{\omega}} \perp \boldsymbol{\omega}''$); imposing, by construction, that the symmetry axis of the gyroscope does oscillate only in the plane Π, the moment $\mathbf{M}''_g$ has not one effect on it (it cannot take it out of this plane). The rotations of angular velocities $\overline{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}'$ lead to a gyroscopic moment $\mathbf{M}'_g$, normal to the plane Π, in the negative sense of the ON-axis, its magnitude being $M'_g = I_3 \overline{\omega} \omega' \sin \theta = I_3 \omega \overline{\omega} \cos \lambda \sin \theta$; this moment leads to a decreasing of the angle θ between the axes Ox'_3 and Ox_3, hence to a rotation of the Ox_3-axis in the Π-plane (Fig. 16.26b), in conformity to the equation of motion

$$J \ddot{\varphi} + I_3 \omega \overline{\omega} \cos \lambda \sin \theta = 0 . \qquad (16.2.30)$$

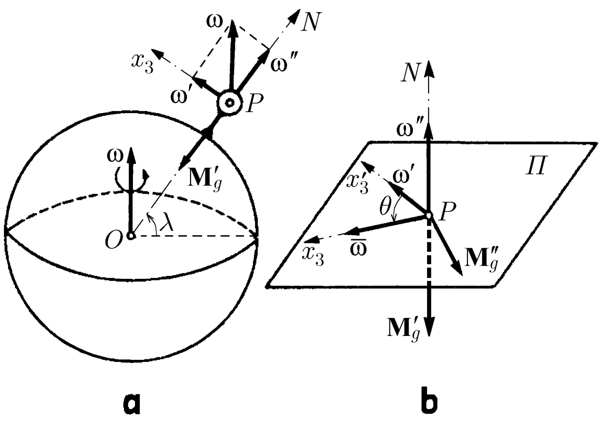


Fig. 16.26 The gyroscopic compass: position (**a**); rotation of the Ox_3-axis in the Π-plane (**b**)

If we can take $\sin \theta \cong \theta$ then the axis of the gyroscope will perform a harmonic oscillation about the fixed axis Ox'_3, of period

$$T = 2\pi \sqrt{\frac{J}{I_3 \omega \overline{\omega} \cos \lambda}}, \qquad (16.2.30')$$

the position of equilibrium (stable) corresponding to $\theta = 0$, hence to the situation in which the axis of the gyroscope is directed along the local meridian. Hence, by a convenient system of damping, the Px_3-axis will stop on the direction Px'_3, property put in evidence by Foucault's experiments and used to build up *the gyroscopic compass* for the determination of the north pole.

Let us suppose now that the gyroscope is built up so as to allow the rotation of the

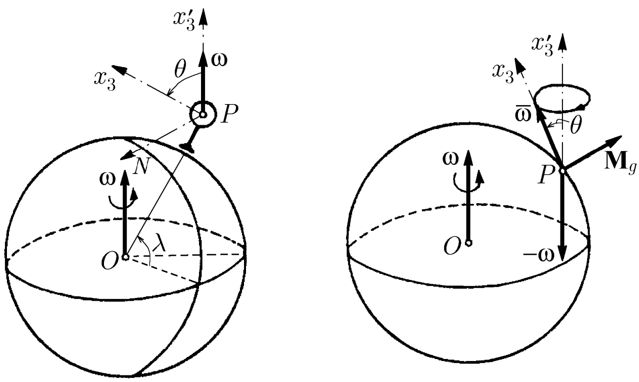
Ox_3-axis of it only in the meridian plane which passes through P (Fig. 16.27a). In this case, the rotation $\boldsymbol{\omega}$ of the Earth leads to a rotation of the same angular velocity about the Ox_3'-axis. The angular velocities $\boldsymbol{\omega}$ and $\overline{\boldsymbol{\omega}}$ determine a gyroscopic moment $\mathbf{M}_g$ in the negative sense of the PN-axis, its magnitude being $I_3\omega\overline{\omega}\sin\theta$; the equation of motion

$$J\ddot{\theta} + I_3\omega\overline{\omega}\sin\theta = 0 \tag{16.2.31}$$

puts in evidence a harmonic oscillation of the Px_3-axis about the Px_3'-axis with the period

$$T = 2\pi\sqrt{\frac{J}{I_3\omega\overline{\omega}}}\,. \tag{16.2.31'}$$

The position of equilibrium (stable) of the gyroscope takes place for $\theta = 0$, corresponding to the situation in which its axis of symmetry is along the direction of the Earth's poles. By a convenient system of damping, the axis of the gyroscope will stop on the same direction (so as, Foucault too, showed experimentally); one can thus build up *the azimuthal gyroscope*, which allows to determine the latitude of the point P at the surface of the Earth (one determines the azimuth, that is the angle $\pi/2 - \lambda$).



a **b**

Fig. 16.27 The gyroscope the axis of which rotates only in the meridian plane (**a**); the free gyroscope (**b**)

Let be, in general, a free gyroscope for which the motion of rotation is not at all hindered; the gyroscope has a motion of rotation of angular velocity $\overline{\boldsymbol{\omega}}$ about the Ox_3-axis, which makes an angle θ with the Ox_3'-axis, parallel to the velocity of rotation $\boldsymbol{\omega}$ of the Earth (Fig. 16.27b). Supposing the Earth fixed, we must introduce the gyroscopic moment $\mathbf{M}_g = I_3\overline{\boldsymbol{\omega}} \times \boldsymbol{\omega}$, of magnitude $M_g = I_3\omega\overline{\omega}\sin\theta$, as an external loading (corresponding to the Coriolis force in the relative motion); the gyroscope has thus a motion of precession of angular velocity $-\boldsymbol{\omega}$. The symmetry axis of the gyroscope describes thus a cone of precession in 24 sidereal hours, having a sense of rotation opposite to the sense of rotation of the Earth. Thus, the effect of rotation of the Earth is nullified by the rotation of precession of the gyroscope; in a fixed frame of reference, the axis of the free gyroscope remains fixed in space and the rotation of the Earth has not one influence on its motion.

16.2.2.4 The Gyroscopic Pendulum

The *gyroscopic pendulum* is a heavy gyroscope for which the fixed point O (situated on the symmetry axis of the gyroscope) is over the centre of gravity C. If the Ox_3-axis about which the gyroscope has a rapid rotation is along the local vertical Ox_3', then we are in the case of the sleeping gyroscope, while the Ox_3-axis is a stable axis of rotation (see Sect. 16.2.1.5); thus, the gyroscopic pendulum allows to determine the local vertical. Besides its own weight, this pendulum is subjected, in general, also to the action of perturbing vibrations; because of its importance, many researchers dealt with this problem (e.g., M. Schuler, R. Wieblitz and Y.A. Ishlinskiĭ), the gyroscopic pendulum being, thus, an self-excited rigid solid (see Sect. 15.2.3.8).

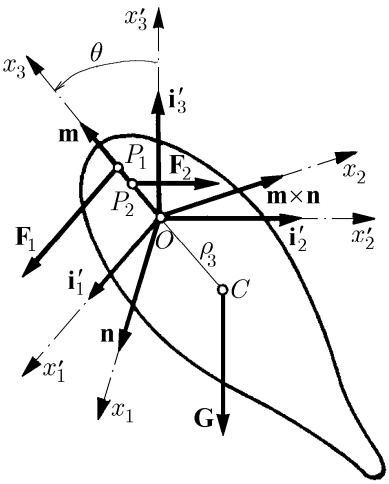


Fig. 16.28 The gyroscopic pendulum

We represent the motion with respect to the inertial frame of reference $Ox_1'x_2'x_3'$ of unit vectors $\mathbf{i}_j'$, the gyroscopic pendulum being rigidly linked to the non-inertial frame $Ox_1x_2x_3$ of unit vectors $\mathbf{i}_j$, $j = 1,2,3$; for the sake of simplicity, we denote $\mathbf{i}_3 = \mathbf{m}$, $\mathbf{i}_1 = \mathbf{n}$, $\mathbf{i}_2 = \mathbf{i}_3 \times \mathbf{i}_1 = \mathbf{m} \times \mathbf{n}$, these axes being principal axes of inertia of the rigid solid (Fig. 16.28). The moment of momentum vector with respect to the fixed frame is thus expressed in the form $\mathbf{K}_O' = \mathbf{I}_O\boldsymbol{\omega} = I_3\omega_3\mathbf{m} + J\omega_1\mathbf{n}$; indeed, because of the symmetry with respect to the Ox_3-axis, we can make the decomposition of this vector in the plane determined by the vector $\boldsymbol{\omega}$ and by the axis of the gyroscope. But

$\mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times (\boldsymbol{\omega} \times \mathbf{m}) = \boldsymbol{\omega} - \omega_3 \mathbf{m} = \omega_1 \mathbf{n}$, so that $\mathbf{K}'_O = I_3 \omega_3 \mathbf{m} + J(\mathbf{m} \times \dot{\mathbf{m}})$, intervening thus only one unit vector. Let us assume that upon the solid act the own weight $-Mg\mathbf{i}'_3$, at the centre of gravity $C(0,0,\rho_3)$, $\rho_3 < 0$, and the perturbing elliptic forces $\mathbf{F}_1 = \pm k_1 \mathbf{i}'_1 \cos \nu t$ (the sense is chosen corresponding to the sense of rotation with respect to the axis of the gyroscope), at the point $P_1(0,0,l_1)$, and $\mathbf{F}_2 = k_2 \mathbf{i}'_2 \sin \nu t$, at the point $P_2(0,0,l_2)$, $l_1, l_2, k_1, k_2, \nu = \text{const}$; if $k_1 = k_2$, then the force is circular, while if $k_1 = 0$ (or $k_2 = 0$), then the force is linear. The theorem of moment of momentum leads to

$$I_3 \omega_3 \dot{\mathbf{m}} + J\mathbf{m} \times \ddot{\mathbf{m}} = \pm k_1 l_1 (\mathbf{m} \times \mathbf{i}'_1) \cos \nu t + k_2 l_2 (\mathbf{m} \times \mathbf{i}'_2) \sin \nu t - Mg\rho_3 (\mathbf{m} \times \mathbf{i}'_3), \tag{16.2.32}$$

wherefrom, in components with respect to the inertial frame of reference, we obtain the non-linear system

$$\begin{aligned} \overline{\Omega} \dot{m}'_1 - m_3 \ddot{m}'_2 + m_2 \ddot{m}'_3 &= \Omega^2 m'_2 - \gamma_2 m'_3 \sin \nu t, \\ \overline{\Omega} \dot{m}'_2 - m_1 \ddot{m}'_3 + m_3 \ddot{m}'_1 &= -\Omega^2 m'_1 \pm \gamma_1 m'_3 \cos \nu t, \\ \overline{\Omega} \dot{m}'_3 - m_2 \ddot{m}'_1 + m_1 \ddot{m}'_2 &= \gamma_2 m'_1 \sin \nu t \mp \gamma_1 m'_2 \cos \nu t, \end{aligned} \tag{16.2.32'}$$

where the constants $\Omega = \sqrt{-M_g \rho_3 / J}$ and $\overline{\Omega} = I_3 \omega_3 / J$ are of the nature of an angular velocity, while the constants $\gamma_1 = k_1 l_1 / J$, $\gamma_2 = k_2 l_2 / J$ are of the nature of an angular acceleration.

In the case of the circular force ($k_1 = k_2 = k$, $l_1 = l_2 = l$, hence $\gamma_1 = \gamma_2 = \gamma$), Wieblitz obtains

$$m_1 = \pm \overline{C} \cos \nu t, \quad m_2 = \overline{C} \sin \nu t, \quad m_3 = C, \quad \overline{C} = \frac{\gamma}{\Omega^2 - C\nu^2 \pm \gamma \nu}, \tag{16.2.33}$$

where the constants C and $\overline{C}$ are determined by the relation $m'_j m'_j = 1$.

If the spin is great, then we can write, approximately, $\mathbf{K}'_{O3} \cong I_3 \omega_3 \mathbf{m}$, and the differential system (16.2.32') reads

$$\begin{aligned} \dot{K}'_{O1'} &= \Omega' (K'_{O2'} - \lambda \overline{\lambda} K'_{O3'} \sin \nu t), \\ \dot{K}'_{O2'} &= -\Omega' (K'_{O1'} \mp \lambda K'_{O3'} \cos \nu t), \\ \dot{K}'_{O3'} &= \lambda \Omega' (\overline{\lambda} K'_{O1'} \sin \nu t \mp K'_{O2'} \cos \nu t), \end{aligned} \tag{16.2.34}$$

in the normal form, where the constant $\Omega' = \Omega^2 / \overline{\Omega} = -Mg\rho_3 / I_3 \omega_3$ is an angular velocity, while the constants $\lambda = \gamma_1 / \Omega^2 = -k_1 l_1 / Mg\rho_3$ and $\overline{\lambda} = \gamma_2 / \gamma_1 = k_2 l_2 / k_1 l_1$ are numbers; the same Wieblitz solves this system by successive approximations, Ω' being a small parameter.

In the case in which the axis of the gyroscope is not very far from the vertical line, the differential system (16.2.32') can be linearized and we can write $m_1' \cong \beta_2$, $m_2' \cong -\beta_1$, $m_3' \cong 1$, where β_1 and β_2 correspond to the rotations by which the fixed frame of reference can attain the position of the movable frame. Indeed, starting from the frame $Ox_1'x_2'x_3'$, by a rotation of angle β_1 about the Ox_1'-axis, we obtain a frame $Ox_1'\xi_2\xi_3$, then, by a rotation of angle β_2 about the $O\xi_2$-axis, we obtain a frame $O\eta_1\xi_2\eta_3$ and, finally, by a rotation of angle β_3 about the $O\eta_3$-axis, we get the frame $Ox_1x_2x_3$; we obtain

$$[\mathbf{i}_j \cdot \mathbf{i}_k'] = \begin{bmatrix} \cos\beta_3 & \sin\beta_3 & 0 \\ -\sin\beta_3 & \cos\beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_2 & 0 & -\sin\beta_2 \\ 0 & 1 & 0 \\ \sin\beta_2 & 0 & \cos\beta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & \sin\beta_1 \\ 0 & -\sin\beta_1 & \cos\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta_2\cos\beta_3 & \sin\beta_1\sin\beta_2\cos\beta_3 + \cos\beta_1\sin\beta_3 & -\cos\beta_1\sin\beta_2\cos\beta_3 + \sin\beta_1\sin\beta_3 \\ -\cos\beta_2\sin\beta_3 & -\sin\beta_1\sin\beta_2\sin\beta_3 + \cos\beta_1\cos\beta_3 & \cos\beta_1\sin\beta_2\sin\beta_3 + \sin\beta_1\cos\beta_3 \\ \sin\beta_2 & -\sin\beta_1\cos\beta_2 & \cos\beta_1\cos\beta_2 \end{bmatrix},$$

wherefrom $m_1' = \sin\beta_2$, $m_2' = -\sin\beta_1\cos\beta_2$, $m_3' = \cos\beta_1\cos\beta_2$, the approximation made being thus justified. The first two equations (16.2.32') read

$$\begin{aligned} \ddot{\beta}_1 + \overline{\Omega}\dot{\beta}_2 &= -\Omega^2\beta_1 - \gamma_2\sin\nu t, \\ \ddot{\beta}_2 - \overline{\Omega}\dot{\beta}_1 &= -\Omega^2\beta_2 \pm \gamma_1\cos\nu t, \end{aligned} \tag{16.2.35}$$

the third equation (16.2.32') becoming a linear consequence of the system (16.2.35), in the frame of the approximation thus made; Wieblitz has given the solution of this system in case of a linear force.

Because of the properties with respect to the gyroscope axis, the Ox_1-axis can be chosen arbitrarily in the plane normal to Ox_3; in particular, the Ox_1-axis can be even the line of nodes ON. The respective frame of reference, called the frame of Résal, is not involved in the proper rotation of the rigid solid, so that the matric relation (3.2.11''') reads (we make $\theta = 0$).

$$[\mathbf{i}_j \cdot \mathbf{i}_k'] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\cos\theta\sin\psi & \cos\theta\cos\psi & \sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{bmatrix}.$$

Hence, $\boldsymbol{\omega} = \dot{\psi}\mathbf{i}_3' + \dot{\theta}\mathbf{n} = \dot{\theta}\mathbf{i}_1 + \dot{\psi}\sin\theta\mathbf{i}_2 + \dot{\psi}\cos\theta\mathbf{i}_3$ and $\mathbf{K}_O' = J(\dot{\theta}\mathbf{i}_1 + \dot{\psi}\sin\theta\mathbf{i}_2) + I_3(\dot{\psi}\cos\theta + \dot{\varphi})\mathbf{i}_3$. With these data, we can write the theorem of moment of momentum ($\partial\mathbf{K}_O'/\partial t + \boldsymbol{\omega}\times\mathbf{K}_O' = \mathbf{M}_O$, where we have introduced the derivative with respect to time in the non-inertial frame of reference) in the form (in the frame

$Ox_1x_2x_3$)

$$\begin{gathered}\ddot{\theta} + (a-1)\dot{\psi}^2 \sin\theta\cos\theta + a\dot{\psi}\dot{\varphi}\sin\theta = -\Omega^2\sin\theta \\ \pm\gamma_1\cos\theta\sin\psi\cos\nu t - \gamma_2\cos\theta\cos\psi\sin\nu t\,, \\ \ddot{\psi}\sin\theta - (a-2)\dot{\theta}\dot{\psi}\cos\theta - a\dot{\theta}\dot{\varphi} = \pm\gamma_1\cos\psi\cos\nu t + \gamma_2\sin\psi\sin\nu t\,, \\ \dot{\psi}\cos\theta + \dot{\varphi} = \mathrm{const}\,,\end{gathered} \tag{16.2.36}$$

where $a = I_3 / J$. This differential system is sensibly simplified if $\dot{\varphi}$ is great, while θ is small (the gyroscope axis, which is rotating rapidly, is close to the vertical line).

16.2.2.5 Technical Applications

The considerations made till now can be applied to a great number of problems with a technical character, as well as to the construction of many apparatuses useful in various domains. Let be thus a rigid solid with the symmetry axis Ox_3 inclined by an angle θ with respect to the $O'O''$-axis, about which it rotates with a constant angular velocity $\boldsymbol{\omega}'$ (Fig. 16.29); one obtains thus a motion of regular precession, arising the gyroscopic moment (we make $\overline{\omega} = 0$ in the formula (16.2.16))

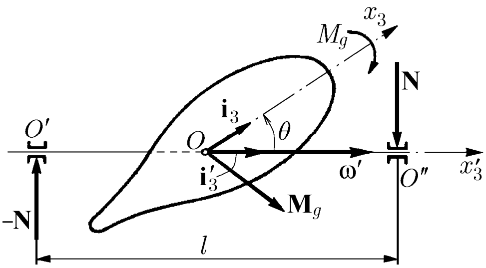


Fig. 16.29 A regular motion of precession – the gyroscopic moment

$$\mathbf{M}_g = (I_3 - J)\omega'^2\cos\theta\,\mathbf{i}_3 \times \mathbf{i}_3', \tag{16.2.37}$$

of magnitude $M_g = (|I_3 - J|/2)\omega'^2\sin 2\theta$. On the bearings at O' and O'' act the gyroscopic forces –**N** and **N**, respectively, of magnitude

$$N = \frac{|I_3 - J|}{2l}\omega'^2\sin 2\theta\,, \tag{16.2.37'}$$

where $l = \overline{O'O''}$; if $I_3 > J$, then the sense of these forces is that in Fig. 16.29 and opposite if $I_3 < J$. The forces –**N** and **N** are contained in the movable plane Ox_3x_3' and rotate together with the rigid solid with the period $T = 2\pi / \omega'$; thus, they load and unload successively the bearings O' and O'', which leads to their wear. Hence, the assembling of wheels and fly-wheels on axles must be made very carefully, so that their axis of symmetry coincide with their axis of rotation (to have $\theta = 0$, hence $N = 0$). In particular, in the case of a full wheel of radius R and weight **G**, which is assembled with the axle inclined by the angle θ, obtaining thus $I_3 = 2J = GR^2/2g$, so that $N = \left(GR^2/8gl\right)\omega'^2 \sin 2\theta$, where g is the gravity acceleration. Numerically, for $R = l = 16\ \mathrm{cm}$, a revolution of $6000\ \mathrm{rot/min}$ (hence, $\omega' = 2\pi \cdot 6000/60$ $= 200\pi\ \mathrm{rad/s}$) and $\theta = 1°$ we get $N = 28.1G$; we see thus that for an assembling deviation of only $1°$ one obtains a significant gyroscopic effect, which cannot be neglected.

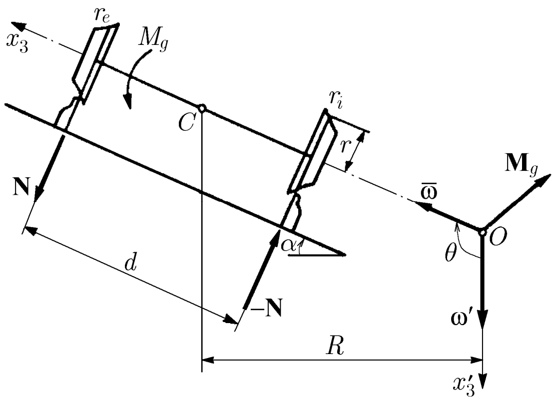


Fig. 16.30 The motion of a railway car in a curve

An analogous study can be made for the motion of vehicles in a curve. Let be thus a pair of wheels of radius r, which form a gyroscope of symmetry axis Ox_3, inclined with the angle α with respect to the horizontal line, and which rotate with an angular velocity $\overline{\omega} = v/r$ (v is the linear velocity); the velocity of precession about the fixed axis Ox_3' is $\omega' = v/R$ where R is the radius of curvature (Fig. 16.30). Because $R \gg r$, it results $\overline{\omega} \gg \omega'$. The system formed by the two wheels acts (e.g., in case of a railway car) on the rails with the gyroscopic moment (16.2.16); in the given conditions, $\theta = \pi/2 + \alpha$ and we can write, with a good approximation,

$$M_g = I_3 \frac{v^2}{rR} \cos\alpha . \tag{16.2.38}$$

Thus, arise the gyroscopic forces **N** and –**N**, of magnitude

$$N = I_3 \frac{v^2}{rRd} \cos\alpha , \tag{16.2.38'}$$

where d is the *rail gauge* (the distance between the rails); these forces load the external rail r_e and unload the internal rail r_i.

Let us consider now a steamer; the axle of rotation Ox_3 (with the angular velocity vector $\boldsymbol{\varpi}$) of the turbine is along the ship and has the bearings O' and O'' as supports (Fig. 16.31a). If, at a certain moment, the steamer has a motion of rotation about the vertical axis Ox_3' with the angular velocity $\boldsymbol{\omega}'$, then appears the gyroscopic moment (16.2.16') (approximate value), which leads to the gyroscopic reactions $\mathbf{N}$ and $-\mathbf{N}$ of magnitude ($\overline{O'O''} = l$)

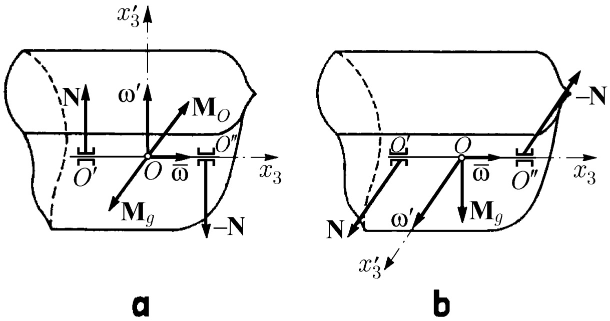


Fig. 16.31 The motion of a steamer. The Ox_3-axis along the axis of the ship (**a**) or the Ox_3'-axis normal to this axis (**b**)

$$N = \frac{I_3}{l}\varpi\omega', \tag{16.2.39}$$

acting on the axle of the turbine; hence, it results a loading of the bearing at O' and an unloading of that at O'' (the gyroscopic forces are of sense opposite to the sense of the gyroscopic reactions; in Fig. 16.31a are specified the gyroscopic forces). Besides this phenomenon, takes place a rotation about the horizontal axis in the transverse plane of the ship (*motion-of pitching*). To put in evidence the effect of this motion (e.g., due to the waves), we assume that the fixed axis Ox_3' is horizontal and normal to the axis of the ship (Fig. 16.31b); the gyroscopic moment $\mathbf{M}_g$ is directed along the descendent vertical at O, while the gyroscopic reactions are given by the same formula (16.2.39), the gyroscopic forces leading to loadings and unloadings too. The tendency of the steamer to rotate about the vertical line is thus explained. The cases in which the axle of rotation of the turbine is vertical or horizontal, in a transverse plane of the ship, can be studied analogously.

These phenomena can be put in evidence, in the same way, also in the case of aircraft. But we mention that, in this case, the ratio of the weight of the propeller or of the rotary engine to the whole weight of the aircraft (built of a material as light as possible) is much more greater; because of this, by a sharp turning takes place a pitch of the aircraft which can lead to unexpected damages. This effect can be eliminated in case of aircraft with two airscrews, the rotations of which are of opposite sense.

Besides the indirect actions mentioned above, the gyroscope can have also a direct action of stabilizing the ships. Such a gyroscope, conceived by Schlick, is formed by a flywheel F which rotates with a very great angular velocity $\boldsymbol{\varpi}$ about the Ox_3-axis. The ends of the axle of the flywheel are sustained by a reinforcement which is rotating about the axle AOA', which is transversal to the ship (Fig. 16.32); at the bottom of the reinforcement is put a weight **G**, so that – in the normal position – the Ox_3-axis be vertical. If, under the action of the waves, the steamer is inclined and rotates about the longitudinal axis Ox_3' with the angular velocity $\boldsymbol{\omega}'$ (*motion of rolling*), then arises a gyroscopic moment $\mathbf{M}_g$, of magnitude $I_3 \varpi \omega'$, which imparts to the flywheel a rotation about the axle AOA' (*motion* of *pitching*); this leads to a new gyroscopic moment $\mathbf{M}_g'$ which, by means of the bearings A and A', is in opposition to the motion of pitching, diminishing thus the inclination of the ship about its longitudinal axis. This gyroscope proves its utility in the rectilinear motion of the steamer, as well as in the case of the rotations about the vertical line in the same sense as its rotation; in the case of a rotation in an opposite sense, take place supplementary motions of pitching, which can be dangerous. For this reason, there have been invented and built also other *anti-pitching gyroscopes* much more perfected, used to stabilize the steamers (e.g., *the Sperry gyroscope*); in this order of ideas, we mention the pilot gyroscope too, which determines the motion of precession.

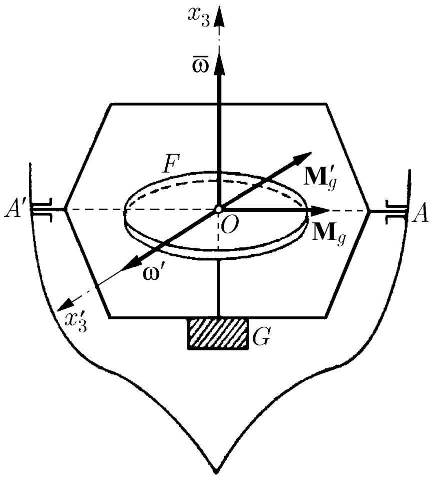


Fig. 16.32 Schlick’s gyroscope

The stabilization of the cars with only one rail (monorail) is analogously realized. *The gyroscope with a vertical axle* considered by Scherl to this purpose is analogous to *Schlick's gyroscope.* Brennan has introduced a gyroscope with a horizontal axle, useful in case of a rectilinear motion. To increase the precession, other auxiliary apparatuses are used. We mention also the realization of *the monorail coach,* of *the gyroscopic motorcar on two wheels* and of *the gyroscopic bicycle with* only *one wheel.*

The motion of *the bicycle with two wheels* is also stabilized by the gyroscopic moments which appear. If the linear velocity is **v**, then the angular velocity is given by $\varpi = v/r$, where r is the radius of the wheel. Assuming that, for some reason, the bicycle is inclined with an angular velocity $\boldsymbol{\omega}'$, arise the gyroscopic moments

$\mathbf{M}_g = I_3 \overline{\boldsymbol{\omega}} \times \boldsymbol{\omega}'$, to which correspond the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, which lead to the new gyroscopic moments $\mathbf{M}'_g = I_3 \overline{\boldsymbol{\omega}} \times \boldsymbol{\omega}_1$, $\mathbf{M}''_g = I_3 \overline{\boldsymbol{\omega}} \times \boldsymbol{\omega}_2$, of a sense opposite to that of $\boldsymbol{\omega}'$ (Fig. 16.33); these moments tend to bring back the bicycle in the vertical position, stabilizing its motion. Obviously, this modelling of the phenomenon is approximate; the problems which arise are multiple and much more difficult.

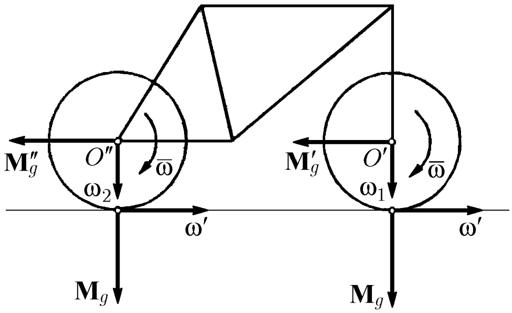


Fig. 16.33 The bicycle with two wheels

We meet applications of the gyroscopic effect also in case of *mills with movable pulleys,* of *pendulary mills* or in case of other mills used to grind cereals, grains or other materials.

As we have seen in Sect. 16.2.2.3, the gyroscope can be used successfully also as an *orientation device.* Thus, the *gyroscopic compass,* with the mobile axle in the local horizontal plane is used to determine the direction north-south in the motion of ships and, especially, of submarines. *The azimuth gyroscope* (with the mobile axle in the local meridian plane) is used to determine the latitude at the respective position on the surface of the Earth. As well, a heavy gyroscope, with the fixed point over the centre of gravity, allows to determine *the gyroscopic horizon* (*the artificial horizon),* even when – on the sea – this one is in a thick fog; the gyroscopic inclination compass determines the inclination with respect to the horizon of the longitudinal axis of an aircraft, as well as of its transverse axis.

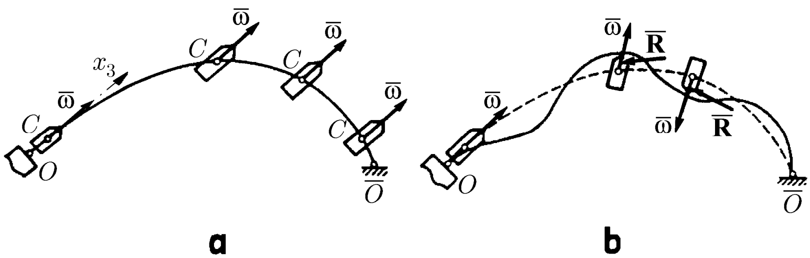


Fig. 16.34 Motion of a projectile. Constant direction of it (**a**). Screw motion of its vertex (**b**)

We meet gyroscopic effects which must be taken into consideration in *ballistics* too; thus, a projectile, besides the motion of its centre of gravity C, has also a motion of rotation of angular velocity $\overline{\boldsymbol{\omega}}$ about its axis of symmetry Ox_3. If the motion takes place in vacuum or in the air of resistance $\overline{\mathbf{R}}$, passing always through C, we have $\mathbf{M}_O = \mathbf{0}$, so that the Ox_3-axis does not change its direction (Fig. 16.34a); the axis of the projectile moves away from the tangent to the trajectory and comes up against a resistance of the air more and more greater, while the target is reached with the inferior part of the projectile. In reality, the resistance of the air $\mathbf{R}$ does not pass through the centre C, but through a point P at the distance l from C; we assume that this resistance is approximately parallel to the velocity $\mathbf{v}$ of the centre of gravity. There appears a gyroscopic moment $\mathbf{M}_g$ applied at C, normal to the plane of the trajectory, which leads to a motion of precession of the projectile about the vector $\mathbf{v}$ with an angular velocity of magnitude

$$\omega' = \frac{M_g}{I_3 \overline{\omega} \sin\theta} = \frac{\overline{R}l}{I_3 \overline{\omega}}. \qquad (16.2.40)$$

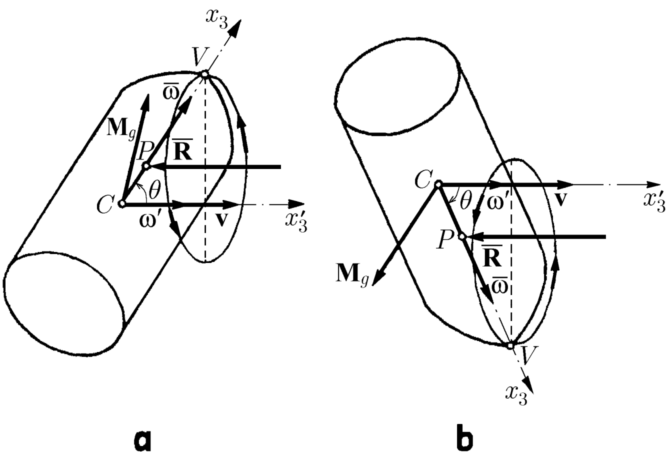


Fig. 16.35 The direction of the vertex of a projectile: upwards (**a**) or downwards (**b**)

Thus, the vertex V of the projectile is directed upwards (Fig. 16.35a) and then downwards (Fig. 16.35b); the vertex of the projectile tends to the tangent to the trajectory *(the projectile runs after its tangent*). In the real motion, the projectile vertex describes a screw motion around the trajectory of its centre of mass (Fig. 16.34b). By a convenient choice of the form and of the dimensions of the projectile, the point P will be situated between the points C and V, the angle remaining sufficiently small, so that the projectile be falling with its vertex.

An important rôle is played by the *directional gyroscopes,* e.g., by those used to maintain the direction of *the automobile torpedo*; we mention also *the automatic directional gyroscopes* (e.g., the *gyropilot*). We put in evidence the *navigation by inertia* and *the gyroscopic directing of the rockets*, useful for *the guided rockets, the intercontinental rockets* and the *interplanetary rockets*.

16.3 Dynamics of the Rigid Solid of Variable Mass

We take again the problem of dynamics of the mechanical systems of variable mass by some considerations concerning the application of variational methods of calculation; we apply then these methods to some particular cases of motion. As well, we consider the case of the aircraft fitted with jet propulsion units.

16.3.1 Variational Methods of Calculation

In the following we present firstly the approximate variational methods of R. Goddard and H. Oberth, passing then to considerations concerning the general methods of calculation with a variational character. We will assume a particle model of the rigid solid, using the results obtained in Chap. 10, Sect. 3.

16.3.1.1 Goddard's Approximate Method

Let us consider the rectilinear motion of a particle of variable mass in a gravitational field, with a resistance of the medium $\mathbf{Q} = \mathbf{Q}(v,x)$, where v is the velocity along the Ox-axis; the equation of motion, corresponding to the equations obtained in Chap. 10, Sects. 3.1.2 and 3.2.3, reads

$$m\dot{v} = -mg\sin\theta - Q(v,x) - \dot{m}w\,, \tag{16.3.1}$$

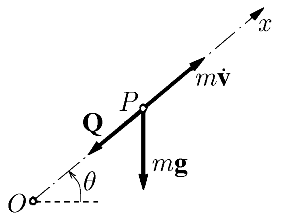


Fig. 16.36 Goddard's approximate method of calculation

where w is the relative velocity of the emitted mass with respect to a non-inertial frame of reference rigidly linked to the particle in motion while θ is the angle made by the trajectory with the horizontal line (Fig. 16.36). Taking the mass of the form (10.3.11') and putting $Q(v,x) = m_0\varphi(v,x)$, we may write

$$f\dot{v} = -fg\sin\theta - g\varphi - \dot{f}w\,, \tag{16.3.1'}$$

finding thus solutions of the form $v = v(t; f, \dot{f}; C_1)$, $x = x(t; f, \dot{f}; C_1, C_2)$, where C_1, C_2 are integration constants; these solutions can be taken as functional equations of some problems of optimization, where various characteristics of the motion (the distance travelled through, the time in which a certain position can be reached, the work and the resistance of the medium) take extreme values. In 1919, R. Goddard proposed an approximate solution to determine the function $v = v(x)$, so as to reach a given height with a minimal mass; the total height is divided in n parts and on each part the resistance of the medium is considered to be constant ($\varphi = \mathrm{const}$). As well, one assumes that $\theta = \pi/2$, $\dot{v} = a = \mathrm{const}$, obtaining an equation of the form (10.3.13') with constant coefficients

$$\dot{f} + \frac{a+g}{w} f + \frac{g\varphi}{w} = 0\,. \tag{16.3.2}$$

The solution (10.3.13") of this equation leads to

$$f(t) = \frac{m}{m_0} = \mathrm{e}^{-[(a+g)/w]t}\left\{C - \frac{g\varphi}{a+g}\mathrm{e}^{[(a+g)/w]t}\right\}, \tag{16.3.2'}$$

with $C = 1 + g\varphi/(a+g)$ (we notice that $f(0) = 1$). If the mass used up at a given moment is $m_0 - m$, hence if $f = 1 - (m_0 - m)/m_0$, then it results

$$\frac{m_0 - m}{m_0} = \left(1 + \frac{g\varphi}{a+g}\right)\left\{1 - \mathrm{e}^{-[(a+g)/w]t}\right\}, \tag{16.3.2"}$$

Imposing, together with Goddard, that that the final mass be equal to unity ($m = 1$), we get

$$m_0 = \mathrm{e}^{[(a+g)/w]t} - \frac{Q}{a+g}\left\{1 - \mathrm{e}^{[(a+g)/w]t}\right\}. \tag{16.3.3}$$

To $Q = 0$ and $g = 0$ corresponds $m_0^* = \mathrm{e}^{(a/w)t}$, so that the problem of extremum of the mass, in the given conditions, consists in minimizing the ratio $m_0 / m_0^* = m_0 \mathrm{e}^{-(a/w)t}$.

The calculation is made successively for each of the n intervals, till the final mass equates the unity. Thus, we obtain also the time in which the displacement takes place and which corresponds to the minimum of the ratio of the masses.

16.3.1.2 Oberth's Approximate Method

For the same problem, H. Oberth assumes in 1929 that the velocity of the particle is negligible with respect to the relative velocity of emission of the mass of combustible ($v \ll w$); denoting $\mathbf{R} = \mathbf{Q} + m\mathbf{g}$, it results the equation of motion along the ascendent vertical (Fig. 16.37)

$$m\dot{v} = -R - \dot{m}w\,. \tag{16.3.4}$$

We notice that $\dot{v} = (\mathrm{d}v/\mathrm{d}x)v$, $\dot{m} = (\mathrm{d}m/\mathrm{d}x)v$, so that we can also write

$$m\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{R}{v} + \frac{\mathrm{d}m}{\mathrm{d}x}w = 0\,. \tag{16.3.4'}$$

If at the height x we impose a condition of minimal consumption of combustible ($\mathrm{d}m/\mathrm{d}x = 0$) and if we differentiate the equation (16.3.4') with respect to v, then we get

$$\frac{\partial}{\partial v}\left(m\frac{\mathrm{d}v}{\mathrm{d}x}\right) + \frac{\partial}{\partial v}\left(\frac{R}{v}\right) = 0 .$$

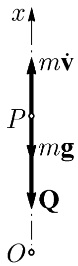


Fig. 16.37 Oberth's approximate method of calculation

In the hypothesis $m\mathrm{d}v = \mathrm{const}$, it results

$$\frac{\partial}{\partial v}\left(\frac{R}{v}\right) = \frac{\partial}{\partial v}\left(\frac{mg + Q}{v}\right) = 0 , \tag{16.3.5}$$

where the force $\mathbf{R}$ is opposed to the motion (R/v is the resistance on unit of path and time). One obtains thus the optimal velocity to raise a particle of variable mass (for which the unit loss caused by the force $\mathbf{R}$ is minimal). We have seen in Chap. 10, Sect. 3.2.5 that the resistance can be taken in the form $Q(v) = b\rho A v^2/2$, where $b = b(v)$ is the aerodynamic coefficient, ρ is the density of the air, while A is an area characteristic for the rigid solid modelled as a particle (in case of the aircraft, the area of the wing). If we assume that $m = \mathrm{const}$ on a stratum of width $\mathrm{d}x$, then the relation (16.3.5) leads to

$$-\frac{mg}{v^2} + \frac{1}{2}b\rho A + \frac{1}{2}\rho A v\frac{\mathrm{d}b}{\mathrm{d}v} = 0 ,$$

wherefrom

$$v^2 = \frac{2mg}{\rho A[b + v(db/dv)]} . \tag{16.3.6}$$

Taking $b = \mathrm{const}$, it results

$$v = \sqrt{\frac{2mg}{b\rho A}} . \tag{16.3.6'}$$

Hence, if the resistance of the medium $Q(v)$ is in direct proportion to the square of the velocity, then it will be equal to the weight of the particle ($Q = mg$); in this case, the optimal velocity of the rigid solid of variable mass is the velocity which must have the particle of mass m in free falling in a homogeneous medium of given density ρ. From the above results, one obtains easily all the kinematic and dynamic characteristics of the motion.

16.3.1.3 General Considerations on the Application of Variational Methods of Calculation

The methods of calculation of Goddard and Oberth have an approximate character; we give a formulation of the same problem, in what follows, in a rigorous variational calculus. We have seen in Chap. 7, Sect. 2.1.4 that, in case of a functional $I(y_1, y_2, ..., y_n)$ of the form (7.2.13) for the functions $y_k(x)$, $k = 1,2,...,n$, of the same independent variable x, we are led to the Euler-Lagrange equations (7.2.13'). Imposing the supplementary conditions

$$f_j(x; y_1, y_2, ..., y_n; y_1', y_2', ..., y_n') = 0, \quad j = 1,2,...,h, \tag{16.3.7}$$

too, we introduce the auxiliary function

$$F^* = F + \sum_{j=1}^{h} \lambda_j f_j, \tag{16.3.7'}$$

where $F = F(x; y_1, y_2, ..., y_n; y_1', y_2', ..., y_n')$ is the function under the integral operator in the functional (7.2.13), while $\lambda_j = \lambda_j(x)$, $j = 1,2,...,h$, $h < n$, are Lagrange's multipliers which must be determined; the equations (7.2.13') lead to

$$F_{y_k}^* - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y_k'}^* \right) = 0, \quad k = 1,2,...,n, \tag{16.3.7''}$$

$F_{y_k}, F_{y_k'}$ being the partial derivatives with respect to the corresponding arguments (y_k and $y_k' = \partial y_k / \partial x$). We obtain thus a system of n differential equations (16.3.7'') and a system of h non-holonomic constraint relations (16.3.7) for the n functions $y_1, y_2, ..., y_n$ and for the h parameters $\lambda_1, \lambda_2, ..., \lambda_h$.

In our case, the problem of determination of the law of variation of the particle mass is put, so that the path travelled through $\int_0^x \mathrm{d}x = \int_0^T v\mathrm{d}t$ have a maximum, T being the time in which the motion of the particle takes place, while X is the space travelled through. In this variational problem the function v under the integral must verify the differential equation of the particle of variable mass (16.3.1') on the active segment $[0, t_1]$ (till the end of the process of emission, hence of combustion of a rocket) and the differential equation of the particle of constant mass ($m_1 = m(t_1)$, $f_1 = f(t_1)$)

$$f_1 \dot{v} = -f_1 g \sin\theta - g\varphi , \tag{16.3.8}$$

on the passive segment $[t_1, T]$. To apply the general method of calculation, we assume, after A.A. Kosmodemyanski, some simplifying hypotheses. Thus, we consider that, on the active segment, the consumption of mass on a second is a finite magnitude, being negligible on the passive segment; thus, the function *f*, partially continuous on the interval $[0, T]$, is replaced by a continuous function, sufficiently close to that one. In what concerns the condition (16.3.8), we can have any velocity at the initial moment, as well as any value for $f_1 > 0$; hence, this condition may be overlooked in the variational problem which was put, remaining with the condition (16.3.1'), written in the form ($\dot{v} = v\mathrm{d}v / \mathrm{d}x$, $\dot{f} = v\mathrm{d}f / \mathrm{d}x$)

$$f v v'_x + f g \sin\theta + g\varphi(v, x) + v w f'_x = 0 . \tag{16.3.8'}$$

In our case, $F = 1$, while $y_1 = f(x)$, $y_2 = v(x)$, so that

$$F^* = 1 + \lambda [f v v' + f g \sin\theta + g\varphi(v, x) + v w f'] .$$

The equations (16.3.7") lead to

$$\lambda (f v' + g\varphi'_v + w f') - \frac{\mathrm{d}}{\mathrm{d}x} (\lambda v w) = 0 ,$$
$$\lambda (v v' + g \sin\theta) - \frac{\mathrm{d}}{\mathrm{d}x} (\lambda v w) = 0 .$$

Developing and eliminating λ and λ'_x between these equations, we get

$$\left(\frac{f'}{f} + \frac{v'}{w} \right) (v - w) - \frac{g}{f} \varphi'_v + \frac{g}{w} \sin\theta = 0 .$$

Eliminating the sum $f' / f + v' / w$ between this relation and (16.3.8'), we obtain

$$f = \frac{1}{w \sin\theta} [(v - w)\varphi + v w \varphi'_v] . \tag{16.3.9}$$

Taking $\varphi(v) = b(v) \rho A v^2 / 2 m_0 g$, as in aerodynamics, we can write

$$m = m_0 f = \frac{\rho A v^2}{2 g w \sin\theta} [(v + w) b + v w b'_v] , \tag{16.3.9'}$$

where we assume, usually, that the gravity acceleration does not vary with the altitude; we have thus determined the relation between the mass and the velocity of the particle in case of an optimal regime. In the hypothesis in which $b = \mathrm{const}$, the relation (16.3.9') is reduced to

$$m = \frac{\rho A b v^2}{2 g \sin\theta}\left(1 + \frac{v}{w}\right), \tag{16.3.10}$$

while, in case of the motion on the vertical, it results

$$mg = \frac{1}{2}\rho A b v^2 \left(1 + \frac{v}{w}\right). \tag{16.3.10'}$$

If $v \ll w$, then we find again Oberth's formula.

16.3.2 Applications to Dynamics of the Rigid Solid of Variable Mass

Using again the modelling as a particle of the rigid solid of variable mass, we apply the above results to the study of the motion in a homogeneous atmosphere and of the motion by simultaneous capture and emission; as well, we make some considerations concerning the motion of the rocket.

16.3.2.1 Motion in a Homogeneous Atmosphere

In 1946, A.A. Kosmodemyanski dealt with the motion of the particle of variable mass in a homogeneous atmosphere ($\rho = \text{const}$), using the above considerations. Noting that $\dot{f} = v\mathrm{d}f / \mathrm{d}x$, the equation (16.3.1') reads

$$(f + w f_v')\dot{v} = -g[f \sin\theta + \varphi(v)]. \tag{16.3.11}$$

In this case, the length of the path travelled through the rectilinear trajectory is given by

$$\int_0^T v\mathrm{d}t = \int_0^V F(v; f, f_v')\mathrm{d}v = -\int_{v_0}^V \frac{(f + w f_v')v\mathrm{d}v}{g[f \sin\theta + \varphi(v)]}, \tag{16.3.12}$$

where v_0 is the initial velocity.

Writing the Euler-Lagrange equation corresponding to this functional, we get

$$\frac{v(f\sin\theta + \varphi) - (f + w f_v')v\sin\theta}{(f\sin\theta + \varphi)^2} - \frac{\mathrm{d}}{\mathrm{d}v}\frac{vw}{f\sin\theta + \varphi} = 0,$$

being thus led to the relation (16.3.9) in the particular case in which $g = \text{const}$ and $\varphi = \varphi(v)$.

The relation (16.3.11) allows to write

$$t = \int_v^{v_0} \frac{f + w f_v'}{g[f\sin\theta + \varphi(v)]}\mathrm{d}v, \tag{16.3.13}$$

where $f(v)$ is given by the relation (16.3.9); we obtain thus the velocity $v = v(t)$. Associating the relation $f = f(v)$ too, we determine the variation law of the particle mass, hence the optimal regime of work of the motor, in case of a rocket.

16.3.2.2 Motion by Simultaneous Capture and Emission

Let be a particle of variable mass which is moving in the same conditions as above in a uniform gravitational field, with the initial condition $v(0) = v_0$. In the case of a phenomenon of simultaneous capture and emission, the differential equation of motion is of the form

$$m\dot{v} = -mg\sin\theta - m_0 g\varphi(v) + \dot{m}^-(u_- - v) + \dot{m}^+(u_+ - v)\,, \tag{16.3.14}$$

where $\dot{m}^- < 0$ and $\dot{m}^+ > 0$ correspond to the variations of the emission and of the capture of mass, while u_- and u_+ are the absolute velocities, respectively. Denoting by $w_- = u_- - w$ the relative velocity – considered to be constant – of the emitted masses, with respect to the non-inertial frame of reference, assuming that $u_+ = 0$ (case considered by Levi-Civita) and that $\dot{m}^+ = -\dot{m}^- / \gamma$, we also can write

$$m\dot{v} = -mg[\sin\theta + \varphi(v)] + \dot{m}^-\left(w_- + \frac{v}{\gamma}\right). \tag{16.3.14'}$$

Because $w_- = \text{const}$ at the end of the active segment ($t = t_1$), when $u_- = 0$, we have $w_- = v_1$, where v_1 is the velocity at the respective moment.

If we denote $m^-(t) = m_0 f(t)$, with $f(0) = 1$, $m^-(0) = m_0$, then it results $\dot{m} = \dot{m}^- + \dot{m}^+ = \dot{m}^-(1 - 1/\gamma) = m_0\dot{f}(1 - 1/\gamma)$, because $m(t) = m_0 + m^-(t) + m^+(t)$; we get thus $m^+(t) = -m_0[bf(t) + a]$ and $m(t) = m_0[af(t) + b]$, $a = 1 - 1/\gamma$, $b = 1/\gamma$, $a + b = 1$. Replacing in the equation (16.3.14'), we obtain

$$(af + b)\dot{v} = -g[(af + b)\sin\theta + \varphi] - \dot{f}(v_1 - bv)\,. \tag{16.3.14''}$$

Choosing v as independent variable, we can write

$$gx = \int_{v_1}^{v_0} F(v; f, f_v')\mathrm{d}v = \int_{v_1}^{v_0} \frac{[(v_1 - bv)f_v'(v) + af(v) + b]v}{[af(v) + b]\sin\theta + \varphi(v)}\mathrm{d}v\,. \tag{16.3.15}$$

We impose a variation of mass given by $f(v)$, so that the displacement x have a maximum. Writing the corresponding Euler-Lagrange equation, we get

$$f(v) = \frac{v(v_1 - bv)\varphi_v'(v) + [(a + 2b)v - v_1]\varphi(v) - b(v_1 - 2bv)\sin\theta}{a(v_1 - 2bv)\sin\theta}\,. \tag{16.3.16}$$

As well, starting from the same relation (16.3.14''), we can write

$$gt = \int_{v_1}^{v} \frac{(v_1 - bv)f_v'(v) + af(v) + b}{[af(v) + b]\sin\theta + \varphi(v)}\mathrm{d}v\,. \tag{16.3.15'}$$

One obtains $v = v(t)$ and then $f = f(t)$, the consumption of mass being thus determined.

16.3.2.3 Considerations Concerning the Motion of the Rocket

In the case of a rocket launched along the ascendent vertical, Meshcherskiĭ's equation (see Chap. 10, Sect. 3.1.2 too) reads

$$m\dot{v} + mg + Q(v,x) + \dot{m}w = 0\,, \tag{16.3.17}$$

where $\mathbf{w} = \mathbf{u} - \mathbf{v}$ is the relative velocity of the evacuated masses, while $v = \dot{x}$. It is assumed that: (i) $g = \mathrm{const}$ for relatively small heights (100...200 km) from the Earth's surface; (ii) $w = \mathrm{const}$, hence a constant emission of mass; (iii) the influence of the mass variation in the resistance of the air, characterized by the function φ, is neglected; (iv) the variation of the momentum in the interior of the rocket, modelled as a particle, is neglected; (v) the influence of the motion of rotation of the Earth is not taken into consideration. The conditions $x = 0$, $v = v_0$, $m = m_0$ at the initial moment $t = 0$ and $x = x_1$, $v = v_1$, $m = m_1$ at a certain moment $t = t_1$ being given, G. Hamel, in 1936, put the problem to determine the minimal initial mass m_0, so that the rocket reach a given altitude x_1 (at the end of the active segment ($t = t_1$) for which we have the velocity v_1 and the mass m_1).

Noting that $m(t) = m_0 f(t)$, $f(0) = 1$, and applying the formula (10.3.13") for an equation of the form (10.3.13), (10.3.13'), we get

$$m_1 = \mathrm{e}^{-\int_0^{t_1}\{[g+\dot{v}(t)]/w\}\mathrm{d}t}\left\{m_0 - \int_0^{t_1}\frac{Q[v(t)x(t)]}{w}\mathrm{e}^{-\int_0^{t}\{[g+\dot{v}(\tau)]/w\}\mathrm{d}\tau}\mathrm{d}t\right\},$$

where we took into consideration also the initial condition $m(0) = m_0$; after calculations, we can write

$$\begin{aligned} m_0\mathrm{e}^{v_0/w} &= \frac{1}{w}\int_0^{t_1} Q[v(t)x(t)]\mathrm{e}^{[gt+v(t)]/w}\mathrm{d}t + m_1\mathrm{e}^{(gt_1+v_1)/w} \\ &= \int_0^{t_1} F(v,x;t)\mathrm{d}t + F_1(v_1,t_1)\,, \end{aligned} \tag{16.3.18}$$

obtaining a relation between the mechanical quantities at the end of the active segment.

After the consumption of the combustible, due to the existent kinetic energy, the rocket continues to ascend, in conformity to the equation of motion

$$m_1\dot{v} + m_1 g + Q(v,x) = 0\,, \tag{16.3.19}$$

which is obtained from (16.3.17), making $\dot{m} = 0$ and taking $m = m_1$; this equation is of the form $F(v,x;t) + \dot{F}_1(v;t) = 0$ and can also be written in the form

$$vv'_x = \overline{Q}(v,x), \quad \overline{Q}(v,x) = -\left[\frac{1}{m_1}Q(v,x) + g\right]. \tag{16.3.19'}$$

Assuming that $Q(v,x) \ll m_1 g$ and integrating, we get

$$v = \psi(x) = \sqrt{2g(h-x)}, \quad x \in [x_1, h], \tag{16.3.20}$$

hence a formula of Torricelli type, where $\psi(h) = 0$. In this case, $v_1 = \psi(x_1)$ and $F_1(v_1, t_1) = F_1(\psi(x_1), t_1)$.

In the variational problem which is put (the minimizing of the functional (16.3.18)) one takes, usually, x_1 and t_1 fixed, to avoid the difficulties of calculation, writing – consequently – the Euler-Lagrange equation for the function $F(v,x;t)$; if x_1 and t_1 are variable, then the problem becomes much more complicated.

16.3.3 The Motion of the Aircraft Fitted Out with Jet Propulsion Motors

In the case of modern aircraft fitted out with systems of jet propulsion of great traction, a great consumption of combustible is emphasized; thus, during the work of the system of propulsion, in the body of the aircraft circulate great masses of liquid, air and gases. To determine the equations of motion which allow the study of the flying qualities, it is necessary to take into account the displacement of the mass centre of the aircraft, the variation of the moments of inertia and of the non-inertial frame of reference of the principal axes of inertia with respect to a frame rigidly linked to the outer covering of the aircraft. In 1975, M.M. Niță and Gh. Drăgănoiu dealt with the motion of the mass centre of the aircraft, while M.M. Niță, considered, in 1979, its motion about the respective centre; in this case, it is necessary that the aircraft be modelled as a mechanical system of variable mass, by simultaneous emission and capture of mass (gas and air, respectively). In what follows, we make some general considerations concerning the general theorems corresponding to the mentioned case, applying then these results to the motion of the aircraft.

16.3.3.1 Theorems of Momentum and of Moment of Momentum

We assume that the aircraft (modelled as a mechanical system $\mathscr{S}$ of variable mass) the motion of which we are studying has, at the moment t, a mass M (contained in the interior of a surface Σ, corresponding to the outer covering of the aircraft) of the form

$$M = M_s + M_c + M_g, \tag{16.3.21}$$

where M_s is *the solid mass*, M_c *is the combustible mass* and M_g is *the mass of the particles of gas* (air or combustion products); the position vector of the mass centre C of the mechanical system $\mathscr{S}$, with respect to a non-inertial frame of reference $\mathscr{R}$ of axes $Ox_1x_2x_3$, rigidly connected to the outer covering of the aircraft (O is a point of the outer covering), is given by

$$\boldsymbol{\rho} = \frac{1}{M}\left(M_s \boldsymbol{\rho}_s + M_c \boldsymbol{\rho}_c + M_g \boldsymbol{\rho}_g \right), \tag{16.3.21'}$$

where $\boldsymbol{\rho}_s, \boldsymbol{\rho}_c, \boldsymbol{\rho}_g$ are the position vectors of the mass centres corresponding to the solid part, to the liquid combustible and to the particles of gas, respectively. The velocity of the centre C, relative to the frame of reference $\mathscr{R}$, is given by

$$\mathbf{w}_C = \frac{\partial \boldsymbol{\rho}}{\partial t} = \frac{1}{M}\left[\frac{\partial}{\partial t}\left(M_c \boldsymbol{\rho}_c \right) + \frac{\partial}{\partial t}\left(M_g \boldsymbol{\rho}_g \right) - \dot{M}\boldsymbol{\rho} \right], \tag{16.3.21''}$$

the derivative of the mass with respect to time being independent on the frame and having obviously, $\dot{M}_s = 0$ and $\partial \boldsymbol{\rho}_s / \partial t = \mathbf{0}$.

We can take

$$M_c = \sum_{j=1}^{p} \iiint_{V_l^j} \mu_l \,\mathrm{d}V + \sum_{i=1}^{q} \iiint_{V_l^i} \mu_l \,\mathrm{d}V, \tag{16.3.22}$$

where μ_l is the density of the liquid combustible, while V_l^j and V_l^i represent the volume of a room (bunker, tubing etc.) occupied by the liquid at a moment t, the indices j and i specifying the p commuted and the q non-commuted rooms respectively, in the consumption circuit. These integrals are defined on variable domains, so that, by the displacements of the free surfaces S_l^j and S_l^i, corresponding to the above mentioned volumes, respectively, it results

$$\dot{M}_c = \sum_{j=1}^{p} \iint_{S_l^j} \mu_l \left(\mathbf{w}_l \cdot \mathbf{n} \right) \mathrm{d}S + \sum_{i=1}^{q} \iint_{S_l^i} \mu_l \left(\mathbf{w}_l \cdot \mathbf{n} \right) \mathrm{d}S; \tag{16.3.22'}$$

it has been assumed that the liquid is incompressible and that $\mathbf{w}_l$ is the relative velocity with respect to the frame of reference $\mathscr{R}$ of a particle on the free surface, $\mathbf{n}$ being the unit vector of the external normal to this surface; obviously, the integrals corresponding to the q components non-commuted to the consumption circuit vanish. Analogously,

$$M_g = \iiint_{V_a} \mu_a \,\mathrm{d}V + \iiint_{V_g} \mu_g \,\mathrm{d}V, \tag{16.3.23}$$

where by μ_a and μ_g have been denoted the densities of the air and of the gas, respectively, V_a and V_g, being the corresponding volumes; noting that $V_a = \mathrm{const}$ and $V_g = \mathrm{const}$, it results

$$\dot{M}_g = \iiint_{V_a} \dot{\mu}_a \mathrm{d}V + \iiint_{V_g} \dot{\mu}_g \mathrm{d}V . \tag{16.3.23'}$$

Returning to the relation (16.3.22'), we notice that the first integral represents, in absolute value, the mass of the liquid contained in the volume delivered by the displacement of the free surface S_l^j in a unit of time and which, on the basis of the continuity condition, is equal to the rate of flow of the liquid combustible Q_l^j in the room V_l^j ; hence,

$$\iint_{S_l^j} \mu_l (\mathbf{w}_l \cdot \mathbf{n}) \mathrm{d}S = -Q_l^j , \quad \dot{M}_c = -\sum_{j=1}^{p} Q_l^j = -Q_l , \tag{16.3.24}$$

where Q_l is the total rate of flow of the liquid combustible. Noting that

$$M_c \boldsymbol{\rho}_c = \sum_{j=1}^{p} \iiint_{V_l^j} \mathbf{r}_l \mu_l \mathrm{d}V + \sum_{i=1}^{q} \iiint_{V_l^i} \mathbf{r}_l \mu_l \mathrm{d}V ,$$

it results

$$\frac{\partial}{\partial t}(M_c \boldsymbol{\rho}_c) = -Q_l \boldsymbol{\rho}_L + \frac{\partial}{\partial t} \sum_{i=1}^{q} \iiint_{V_l^i} \mathbf{r}_l \mu_l \mathrm{d}V , \tag{16.3.24'}$$

where

$$\boldsymbol{\rho}_L = \frac{1}{Q_l} \sum_{j=1}^{p} Q_l^j \boldsymbol{\rho}_{L_j} , \quad \boldsymbol{\rho}_{L_j} = \frac{\iint_{S_l^j} \mathbf{r}_l \mu_l (\mathbf{w}_l \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_l^j} \mu_l (\mathbf{w}_l \cdot \mathbf{n}) \mathrm{d}S} , \tag{16.3.24''}$$

$\mathbf{r}_l$ being the position vector of a particle of liquid of the free surface, with respect to the frame of reference $\mathscr{R}$. The vector $\boldsymbol{\rho}_{L_j}$ specifies the position of the mass centre of the mass of combustible contained in the volume delivered by the displacement of the free surface in a unit time; if the product $\mu_l (\mathbf{w}_l \cdot \mathbf{n})$ is the same at all the points of the surface S_l^j , then the vector $\boldsymbol{\rho}_{L_j}$ defines the centre of mass of this surface, while the vector $\boldsymbol{\rho}_L$ corresponds to the mass centre of all free surfaces in the rooms commuted in the consumption circuit.

If $\iiint_V \mathbf{r}_l \mu_l \mathrm{d}V$ is the static moment of the system of particles of liquid combustible contained in the volume V relative to the rooms commuted in the consumption circuit and if we take into account the formula (A.2.80'), then we get the momentum with respect to the non-inertial frame of reference $\mathscr{R}$ in the form $\iiint_V \mathbf{r}_l \dot{\mu}_l \mathrm{d}V + \iint_S \mathbf{r}_l \mu_l (\mathbf{w}_l \cdot \mathbf{n}) \mathrm{d}S$, where S is the surface which bounds the domain of volume V. In connection with these rooms, we consider the mixing surfaces S_m^k, $k = 1,2,...,r$, through which passes the liquid combustible, leaving the considered system (one assumes that, at the points of these surfaces, takes place the mixing of the particles of liquid with the particles of air or gas – at the systems with post-combustion– and the instantaneous transformation of this mixture in gas). Noting that $\dot{\mu}_l = 0$, the momentum of the mentioned liquid combustible is given by

$$\sum_{j=1}^{p} \iint_{S_l^j} \mathbf{r}_l \mu_l (\mathbf{w}_l \cdot \mathbf{n}) \mathrm{d}S + \sum_{k=1}^{r} \iint_{S_m^k} \mathbf{r}_l \mu_l (\mathbf{w}_m \cdot \mathbf{n}) \mathrm{d}S = Q_c (\boldsymbol{\rho}_M - \boldsymbol{\rho}_L),$$

where

$$\boldsymbol{\rho}_M = \frac{1}{Q_c} \sum_{k=1}^{r} Q_c^k \boldsymbol{\rho}_{M_k}, \quad \boldsymbol{\rho}_{M_k} = \frac{\iint_{S_m^k} \mathbf{r}_l \mu_l (\mathbf{w}_m \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_m^k} \mu_l (\mathbf{w}_m \cdot \mathbf{n}) \mathrm{d}S}, \tag{16.3.25}$$

with an analogous significance concerning the surfaces of mixing S_m^k, $\mathbf{w}_m$ being the relative velocity of a particle of liquid on this surface, in the frame of reference $\mathscr{R}$; obviously, we have

$$\sum_{k=1}^{r} Q_c^k = Q_c = Q_l, \quad Q_c^k = \iint_{S_m^k} \mathbf{r}_l \mu_l (\mathbf{w}_m \cdot \mathbf{n}) \mathrm{d}S, \tag{16.3.25'}$$

where Q_c^k is the rate of flow of the combustible which passes through the surface S_m^k, being transformed in gas. The relative momentum of the system of particles of liquid combustible contained in the rooms non-commuted to the consumption circuit is given by the second term in (16.3.24'); in this case, the momentum of the system of particles, corresponding to all the rooms, will be expressed in the form

$$\mathbf{H}_l = Q_c (\boldsymbol{\rho}_M - \boldsymbol{\rho}_L) + \frac{\partial}{\partial t} \sum_{i=1}^{q} \iiint_{V_l^i} \boldsymbol{\rho}_l \mu_l \mathrm{d}V, \tag{16.3.25''}$$

while the formula (16.3.24') becomes

$$\frac{\partial}{\partial t} (M_c \boldsymbol{\rho}_c) = \mathbf{H}_l - Q_c \boldsymbol{\rho}_M, \tag{16.3.26}$$

where we took into account (16.3.25'').

In what concerns the mass of the gas particles (air or combustion products) we can make analogous considerations. Thus, the condition of continuity of the air flow is given by

$$\iiint_{V_m} \dot{\mu}_m \mathrm{d}V = Q_e - Q_m, \tag{16.3.27}$$

where Q_e is the rate of flow of the air at the entrance sections S_e^h, $h = 1,2,\ldots\ldots,n_e$, while Q_m is the rate of flow of the air at the passing through the mixing surfaces; we have

$$\begin{aligned} Q_e &= \sum_{h=1}^{n_e} Q_e^h = -\sum_{h=1}^{n_e} \iint_{S_e^h} \mu_m (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S\,, \\ Q_m &= \sum_{k=1}^{r} Q_m^k = \sum_{k=1}^{r} \iint_{S_m^k} \mu_m (\mathbf{w}_{a_m} \cdot \mathbf{n}) \mathrm{d}S\,, \end{aligned} \tag{16.3.27'}$$

where $\mathbf{w}_{a_e}$ and $\mathbf{w}_{a_m}$ represent the velocities of a particle of air at the section of entrance and at the section of mixing, respectively. The condition of continuity of the flow of gases is of the form

$$\iiint_{V_g} \dot{\mu}_g \mathrm{d}V + \sum_{k=1}^{r} \iint_{S_m^k} \mu_g (\mathbf{w}_{g_m} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S = 0\,,$$

where $\mathbf{w}_{g_m}$ and $\mathbf{w}_{g_i}$ are the velocities of a particle of gas at the surface of mixing and at the surface of exit (issue) S_i^l, $l = 1,2,\ldots,n_i$, respectively. Assuming that, at the passing through the surfaces of mixing, the liquid is transformed instantaneously in gas, this condition becomes

$$\iint_{V_g} \dot{\mu}_g \mathrm{d}V = Q_c + Q_m - Q_i\,, \tag{16.3.28}$$

with

$$\sum_{k=1}^{r} \iint_{S_m^k} \mu_g (\mathbf{w}_{g_m} \cdot \mathbf{n}) \mathrm{d}S = -(Q_i + Q_m)\,, \quad Q_i = \sum_{l=1}^{n_i} Q_i^l = \sum_{l=1}^{n_i} \iint_{S_i^l} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S\,, \tag{16.3.28'}$$

Q_i being the rate of flow of gases at the sections of exit.

From (16.3.23'), (16.3.27) and (16.3.28) , it results

$$\dot{M}_g = Q_c + Q_e - Q_i\,. \tag{16.3.29}$$

Noting that

$$\dot{M}_c = -Q_c\,, \tag{16.3.29'}$$

from (16.3.24) and (16.3.25'), the relation (16.3.21) leads to

$$\dot{M} = Q_e - Q_i\,, \tag{16.3.30}$$

because $\dot{M}_s = 0$. If $\dot{\mu}_m = \dot{\mu}_g = 0$ and if we take into account the relation (16.3.27) (which leads to $Q_e = Q_m$) and the relation (16.3.28) (wherefrom $Q_i = Q_m + Q_c$), then it results

$$\dot{M} = -Q_c\,. \tag{16.3.30'}$$

Using the relations (16.3.27'), (16.3.28') and (16.3.30), the formula (13.2.12) allows to express *the theorem of momentum*, in the case of the considered problems, in the form

$$\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} + \sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{u}_{a_e}\,\mu_a\,(\mathbf{w}_{a_e} \cdot \mathbf{n})\mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{u}_{g_i}\,\mu_g\,(\mathbf{w}_{g_i} \cdot \mathbf{n})\mathrm{d}S = \mathbf{R}\,, \tag{16.3.31}$$

where $\mathbf{H}'$ is the momentum of the mechanical system $\mathscr{S}$ with respect to an inertial frame of reference $\mathscr{R}$, $\mathbf{u}_{a_e}$ and $\mathbf{u}_{g_i}$ are the velocities of the particles of air at the surface of entrance and of the particles of gas at the surface of exit, respectively, with respect to the same frame, while $\mathbf{R}$ is the resultant of the given external forces. As well, in conformity to the formula (13.2.15), extended to the generalized equation of Meshcherskiĭ, the theorem of moment of momentum for the above problem is given by

$$\frac{\mathrm{d}\mathbf{K}'_{O'}}{\mathrm{d}t} + \sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{r}'_a \times \mathbf{u}_{a_e}\,\mu_a\,(\mathbf{w}_{a_e} \cdot \mathbf{n})\mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{r}'_g \times \mathbf{u}_{g_i}\,\mu_g\,(\mathbf{w}_{g_i} \cdot \mathbf{n})\mathrm{d}S = \mathbf{M}_{O'}\,, \tag{16.3.32}$$

where $\mathbf{K}'_{O'}$ is the moment of momentum of the system $\mathscr{S}$ with respect to the inertial frame of reference, $\mathbf{r}'_a$ and $\mathbf{r}'_g$ are the position vectors of the particles of air at the surface of entrance and of the particles of gas at the section of exit, respectively, with respect to the same frame, while $\mathbf{M}_{O'}$ is the resultant moment of the given external forces with respect to the pole O' of the respective frame.

Obviously, starting from the results given in Sect. 13.2.1.2, we can give also other remarkable forms to these theorems.

16.3.3.2 Motion of the Centre of Mass

The mass centre of the particles of gas is given by the relation of static moments

$$M_g \boldsymbol{\rho}_g = \iiint_{V_a} \mathbf{r}_a \mu_a \mathrm{d}V + \iiint_{V_g} \mathbf{r}_g \mu_g \mathrm{d}V\,, \tag{16.3.33}$$

where $\mathbf{r}_a$ and $\mathbf{r}_g$ are the position vectors of the particles of air and of gas, respectively, with respect to the frame of reference $\mathscr{R}$; thus, it results

$$\frac{\partial}{\partial t}(M_g \boldsymbol{\rho}_g) = \iiint_{V_a} \mathbf{r}_a \dot{\mu}_a \mathrm{d}V + \iiint_{V_g} \mathbf{r}_g \dot{\mu}_g \mathrm{d}V\,.$$

As in the case of particles of liquid, we can write the momentum of the mass of the particles of air and of the particles of gas, respectively, with respect to the frame $\mathscr{R}$, in the form

$$\begin{aligned}\mathbf{H}_a &= \iiint_{V_a} \mathbf{r}_a \dot{\mu}_a \mathrm{d}V + \sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{r}_a \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S + \sum_{k=1}^{r} \iint_{S_m^k} \mathbf{r}_a \mu_a (\mathbf{w}_{a_m} \cdot \mathbf{n}) \mathrm{d}S ,\\ \mathbf{H}_g &= \iiint_{V_g} \mathbf{r}_g \dot{\mu}_g \mathrm{d}V + \sum_{h=1}^{r} \iint_{S_m^k} \mathbf{r}_g \mu_g (\mathbf{w}_{g_m} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{r}_g \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S .\end{aligned} \tag{16.3.34}$$

If the densities μ_a, μ_g and μ_l, as well as the scalar products $\mathbf{w}_{a_m} \cdot \mathbf{n}$, $\mathbf{w}_{g_m} \cdot \mathbf{n}$ and $\mathbf{w}_a \cdot \mathbf{n}$ are uniformly distributed on the mixing surface S_m^k, then we can write

$$\iint_{S_m^k} \mathbf{r}_a \mu_a (\mathbf{w}_{a_m} \cdot \mathbf{n}) \mathrm{d}S + \iint_{S_m^k} \mathbf{r}_g \mu_g (\mathbf{w}_{g_m} \cdot \mathbf{n}) \mathrm{d}S = -Q_c^k \boldsymbol{\rho}_{M_k} .$$

Introducing the notations

$$\begin{gathered}\boldsymbol{\rho}_E = \frac{1}{Q_e} \sum_{h=1}^{n_e} Q_e^h \boldsymbol{\rho}_{E_h} , \quad \boldsymbol{\rho}_I = \frac{1}{Q_i} \sum_{l=1}^{n_i} Q_i^l \boldsymbol{\rho}_{I_l} ,\\ \boldsymbol{\rho}_{E_h} = \frac{\iint_{S_e^h} \mathbf{r}_a \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_e^h} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S} , \quad \boldsymbol{\rho}_{I_l} = \frac{\iint_{S_i^l} \mathbf{r}_g \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_i^l} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S} ,\end{gathered} \tag{16.3.35}$$

where E and I represent the centre of the sections of entrance and the centre of the sections of exit (issue), respectively, the relations (16.3.27'), (16.3.28'), (16.3.33)-(16.3.35) lead to

$$\frac{\partial}{\partial t}(M_g \mathbf{r}_g) = \mathbf{H}_a + \mathbf{H}_g + Q_c \boldsymbol{\rho}_M + Q_e \boldsymbol{\rho}_E - Q_i \boldsymbol{\rho}_I . \tag{16.3.36}$$

If the products $\mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n})$ and $\mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n})$ are the same for all the points of the sections S_e^h and S_i^l, respectively, then the vectors $\boldsymbol{\rho}_{E_h}$ and $\boldsymbol{\rho}_{I_l}$ define the mass centres of these sections, while the points E and I represent *the mass centres of the sections of entrance and exit,* respectively. Taking into account the relations (16.3.26), (16.3.30), (16.3.36), the relation (16.3.21) allows to express the velocity of the mass centre C of the mechanical system $\mathscr{S}$ with respect to the frame of reference $\mathscr{R}$ in the form

$$\mathbf{w}_C = \frac{\partial \boldsymbol{\rho}}{\partial t} = \frac{1}{M} [\mathbf{H} + Q_e (\boldsymbol{\rho}_E - \boldsymbol{\rho}) - Q_i (\boldsymbol{\rho}_I - \boldsymbol{\rho})] , \tag{16.3.37}$$

where

$$\mathbf{H} = \mathbf{H}_l + \mathbf{H}_a + \mathbf{H}_g \tag{16.3.38}$$

is the momentum of this mechanical system with respect to the same frame.

In the case of the aircraft fitted with rocket motors we have $Q_e = 0$, so that

$$\mathbf{w}_C = \frac{1}{M}[\mathbf{H} - Q_i(\boldsymbol{\rho}_I - \boldsymbol{\rho})]. \tag{16.3.37'}$$

Let $\mathbf{v} = \mathbf{u} - \mathbf{w} = \mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r}$ be the velocity of a point of the outer covering of the aircraft, hence of the non-inertial frame of reference $\mathscr{R}$ with respect to the inertial frame $\mathscr{R}'$ (velocity of transportation), where $\mathbf{v}'_O$ is the velocity of the pole O with respect to the frame $\mathscr{R}'$, while $\boldsymbol{\omega}$ is the rotation angular velocity of the frame $\mathscr{R}$ with respect to the same frame $\mathscr{R}'$. The theorem of momentum (16.3.31) may be written also in the form

$$\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} - \sum_{h=1}^{n_e} Q_e^h \mathbf{w}_e^h - Q_e \mathbf{v}_E + \sum_{l=1}^{n_i} Q_i^l \mathbf{w}_i^l + Q_i \mathbf{v}_I = \mathbf{R}, \tag{16.3.39}$$

where we took into account the relations (16.3.27'), (16.3.28'), (16.3.35) and where $\mathbf{w}_e^h$, $\mathbf{w}_i^l$ are the velocities relative to the frame $\mathscr{R}$, considered to be constant, at the section of entrance S_e^h and at the section of exit S_i^l, respectively, $\mathbf{v}_E$ and $\mathbf{v}_I$ being the velocities of transportation of the centres E and I, respectively, given by the relations

$$\begin{aligned}
&\sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{v}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S = \mathbf{v}'_O \sum_{h=1}^{n_e} \iint_{S_e^h} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S \\
&+ \boldsymbol{\omega} \times \sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{r}_a \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S = -(\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}_E) Q_e = -Q_e \mathbf{v}_E, \\
&\sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{v}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S = \mathbf{v}'_O \sum_{l=1}^{n_i} \iint_{S_i^l} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S \\
&+ \boldsymbol{\omega} \times \sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{r}_g \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S = -(\mathbf{v}'_O + \boldsymbol{\omega} \times \boldsymbol{\rho}_I) Q_i = Q_i \mathbf{v}_I.
\end{aligned}$$

Taking into account (16.2.21), we can write $\mathbf{H}' = \mathbf{H}'_s + \mathbf{H}'_c + \mathbf{H}'_g$, so that

$$\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{H}'_s}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{H}'_c}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{H}'_g}{\mathrm{d}t}. \tag{16.3.40}$$

We have $\mathbf{H}'_s = M_s \mathbf{u}_{C_s} = M_s \mathbf{v}_{C_s}$, because the velocity relative to $\mathscr{R}$ of the mass centre of the solid part vanishes ($\mathbf{w}_{C_s} = \mathbf{0}$); hence,

$$\frac{\mathrm{d}\mathbf{H}'_s}{\mathrm{d}t} = M_s \frac{\mathrm{d}\mathbf{v}_{C_s}}{\mathrm{d}t} = \mathbf{a}'_{C_s}, \tag{16.3.41}$$

where $\mathbf{a}'_{C_s}$ is the acceleration of transportation of the mass centre of the solid part.

Calculating the momentum in the motion of transportation in the form

$$\sum_{j=1}^{p}\iiint_{V_l^j}\mathbf{v}\mu_l \mathrm{d}V + \sum_{i=1}^{q}\iiint_{V_l^i}\mathbf{v}\mu_l \mathrm{d}V = \mathbf{v}'_O\left(\sum_{j=1}^{p}\iiint_{V_l^j}\mu_l \mathrm{d}V + \sum_{i=1}^{q}\iiint_{V_l^i}\mu_l \mathrm{d}V\right)$$
$$+\boldsymbol{\omega}\times\left(\sum_{j=1}^{p}\iiint_{V_l^j}\mathbf{r}_l\mu_l \mathrm{d}V + \sum_{i=1}^{q}\iiint_{V_l^i}\mathbf{r}_l\mu_l \mathrm{d}V\right),$$

corresponding to the liquid combustible in the rooms occupied by it at the moment t, we can write ($\mathbf{H}_l$ is the momentum of the liquid part relative to the frame of reference $\mathscr{R}$)

$$\mathbf{H}'_c = \mathbf{H}_l + M_c(\mathbf{v}'_O + \boldsymbol{\omega}\times\boldsymbol{\rho}_c), \tag{16.3.42}$$

where we took into account (16.3.22) and the relation which gives the corresponding centre of mass. Taking into account (16.3.26), we get

$$\frac{\mathrm{d}\mathbf{H}'_c}{\mathrm{d}t} = \frac{\partial\mathbf{H}_l}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{H}_l - Q_c\mathbf{v}_M + M_c\mathbf{a}'_{C_c}, \tag{16.3.43}$$

where $\mathbf{v}_M$ is the velocity of transportation of *the centre M of the mixing surfaces*, $\mathbf{a}'_{C_c}$ is the acceleration of transportation of the centre of mass of the combustible, while the operator $\partial/\partial t$ corresponds to the derivative with respect to the frame $\mathscr{R}$.

Starting from (16.3.23), it results, analogously,

$$\mathbf{H}'_g = \mathbf{H}_g + M_g(\mathbf{v}'_O + \boldsymbol{\omega}\times\boldsymbol{\rho}_g) \tag{16.3.44}$$

for the particles gas-air. We obtain thus ($\mathbf{a}'_{C_g}$ is the acceleration of transportation of the mass centre the gas-air part)

$$\frac{\mathrm{d}\mathbf{H}'_g}{\mathrm{d}t} = \frac{\partial\mathbf{H}_g}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{H}_g + Q_c\mathbf{v}_M + Q_e\mathbf{v}_E - Q_i\mathbf{v}_I + M_g\mathbf{a}'_{C_g}, \tag{16.3.45}$$

where we took into account (16.3.29), (16.3.36) and the manner of introduction of the velocities of transportation $\mathbf{v}_E$ and $\mathbf{v}_I$.

Noting that at a fixed moment we can write

$$M\mathbf{a}'_C = M_s\mathbf{a}'_{C_s} + M_c\mathbf{a}'_{C_c} + M_g\mathbf{a}'_{C_g}, \tag{16.3.46}$$

where $\mathbf{a}'_C$ is the acceleration of transportation of the mass centre of the mechanical system $\mathscr{S}$, and denoting by $\mathbf{H} = \mathbf{H}_l + \mathbf{H}_g$ the momentum of this system with respect to the frame of reference $\mathscr{R}$ (the relative momentum of the solid part vanishes), the theorem of momentum (16.3.39), with (16.3.40), leads to

$$M\mathbf{a}'_C = \mathbf{R} + \sum_{h=1}^{n_e} Q_e^h\mathbf{w}_e^h - \sum_{l=1}^{n_i} Q_i^l\mathbf{w}_i^l - \frac{\partial\mathbf{H}}{\partial t} - 2\boldsymbol{\omega}\times\mathbf{H}. \tag{16.3.47}$$

This result can be used, e.g., in the study of the immobile flight of the aircraft with vertical take-off (we make $\mathbf{v}_C = \mathbf{a}_C = \mathbf{0}$ and $\boldsymbol{\omega} = \mathbf{0}$).

Introducing the relative acceleration $\partial^2 \boldsymbol{\rho} / \partial t^2$ of the mass centre C of the mechanical system, as well as the Coriolis acceleration $2\boldsymbol{\omega} \times \mathbf{w}_C$, we obtain the equation of motion of this centre with respect to the inertial frame of reference $\mathscr{R}'$ in the form

$$M \frac{\mathrm{d}\mathbf{u}_C}{\mathrm{d}t} = \mathbf{R} + \sum_{h=1}^{n_e} Q_e^h \mathbf{w}_e^h - \sum_{l=1}^{n_i} Q_i^l \mathbf{w}_i^l - \frac{\partial \mathbf{H}}{\partial t} - 2\boldsymbol{\omega} \times \mathbf{H} + M \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} + 2M\boldsymbol{\omega} \times \mathbf{w}_C , \tag{16.3.48}$$

where $\mathbf{u}_C$ is the absolute velocity of the centre C.

We decompose the resultant of the given forces in the form $\mathbf{R} = \mathbf{R}_p + \mathbf{R}_0$, where $\mathbf{R}_p$ is *the resultant of the forces of static pressure*, while $\mathbf{R}_0$ is the resultant of the other given external forces. If p_0 is the static pressure on the surface S of the outer-covering of the aircraft, then we will have

$$\iint_S \mathbf{p} \mathrm{d}S = p_0 \left(\sum_{h=1}^{n_e} A_e^h \mathbf{n}_e^h + \sum_{l=1}^{n_i} A_i^l \mathbf{n}_i^l \right),$$

where A_e^h and A_i^l are the areas of the sections of entrance and exit, respectively, supposed to be plane, while $\mathbf{n}_e^h$ and $\mathbf{n}_i^l$ are the unit vectors of the corresponding external normals. Denoting by p_e^h and p_i^l the mean pressure on the same surfaces of entrance and exit, respectively, it results

$$\iint_{S_e^h} \mathbf{p}_e \mathrm{d}S = -p_e^h A_e^h \mathbf{n}_e^h , \quad \iint_{S_i^l} \mathbf{p}_i \mathrm{d}S = -p_i^l A_i^l \mathbf{n}_i^l .$$

Finally, we can write

$$\mathbf{R}_p = \sum_{h=1}^{n_e} \left(p_0 - p_e^h \right) A_e^h \mathbf{n}_e^h + \sum_{l=1}^{n_i} \left(p_0 - p_i^l \right) A_i^l \mathbf{n}_i^l . \tag{16.3.49}$$

The equation of motion of the mass centre (16.3.48) becomes thus

$$M \frac{\mathrm{d}\mathbf{u}_C}{\mathrm{d}t} = \mathbf{R}_0 + \mathbf{T} - \frac{\partial \mathbf{H}}{\partial t} + M \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2} - 2\boldsymbol{\omega} \times \left(\mathbf{H} - M\mathbf{w}_C \right), \tag{16.3.50}$$

where *the force of traction* developed in the aeroreactive system of propulsion has been introduced in the form

$$\mathbf{T} = \sum_{h=1}^{n_e} \left[Q_e^h \mathbf{w}_e^h + \left(p_0 - p_e^h \right) A_e^h \mathbf{n}_e^h \right] - \sum_{l=1}^{n_i} \left[Q_i^l \mathbf{w}_i^l + \left(p_0 - p_i^l \right) A_i^l \mathbf{n}_i^1 \right]. \tag{16.3.50'}$$

We mention that the supplementary terms (excepting the forces $\mathbf{R}_0$ and $\mathbf{T}$) which appear in the equation (16.3.50) can present a particular interest, e.g., in the study of the motion and of the stability of the vehicles with a vertical take-off-landing. If, in particular, the vehicle is fitted out with rocket motors, then one takes $Q_e^h = 0$, $p_e^h = 0$, $h = 1, 2, \ldots, n_e$, in the expression of the force of traction.

Applying the theorem of momentum for the current of fluid which enters through the section S_e^h, in case of a motion of rectilinear and uniform translation, the flow is stationary and we can write $Q_e^h (\mathbf{w}_e^h + \mathbf{v}_c) + (p_0 - p_e^h) A_e^h \mathbf{n}_e^h = \mathbf{0}$, so that the function of traction becomes

$$\mathbf{T} = -Q_e \mathbf{v}_c - \sum_{l=1}^{n_i} \left[Q_i^l \mathbf{w}_i^l - (p_0 - p_i^l) A_i^l \mathbf{n}_i^l \right]. \tag{16.3.50''}$$

16.3.3.3 The Motion About the Centre of Mass

We notice that the position of a particle of solid, liquid, air or gas is specified by $\mathbf{r}' = \mathbf{r}'_O + \mathbf{r} = \boldsymbol{\rho}' + \bar{\mathbf{r}}$, where $\bar{\mathbf{r}}$ is the position vector in a Koenig frame of reference, its absolute velocity being written in the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where the velocity of transportation is given by $\mathbf{v} = \mathbf{v}'_O + \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}'_O + \boldsymbol{\omega} \times (\boldsymbol{\rho} + \bar{\mathbf{r}})$ $= \mathbf{v}_C + \boldsymbol{\omega} \times \bar{\mathbf{r}}$. The moment of momentum with respect to the pole O', in the frame $\mathscr{R}'$, reads

$$\mathbf{K}'_{O'} = \boldsymbol{\rho}' \times \mathbf{H}' + \mathbf{K}_C + \bar{\mathbf{K}}_C, \quad \mathbf{K}_C = \iiint_V \bar{\mathbf{r}} \times \mathbf{w} \mu \, \mathrm{d}V, \quad \bar{\mathbf{K}}_C = \iiint_V \bar{\mathbf{r}} \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}) \mu \, \mathrm{d}V, \tag{16.3.51}$$

where $\mathbf{K}_C$ and $\bar{\mathbf{K}}_C$ are the momenta of momentum with respect to the mass centre C, in the frame $\mathscr{R}$ and in the frame $\bar{\mathbf{R}}^{(C)}$ of Koenig, respectively, and where we have taken into account that the static moment of the mechanical system $\mathscr{S}$ with respect to the mass centre vanishes. Differentiating with respect to time, in the fixed frame of reference, and taking into account that $\mathbf{u}_C = \mathbf{v}_C + \mathbf{w}_C$, $\mathbf{w}_C = \partial \boldsymbol{\rho} / \partial t$ and $\mathbf{H}' = \mathbf{H} + M \mathbf{v}_C$ (we use the expressions of the velocities $\mathbf{u}$ and $\mathbf{v}$, as well as the same property of the static moments), we obtain

$$\frac{\mathrm{d}\mathbf{K}'_{O'}}{\mathrm{d}t} = \mathbf{w}_C \times \mathbf{H} + M \mathbf{w}_C \times \mathbf{v}_C + \mathbf{v}_C \times \mathbf{H} + \boldsymbol{\rho}' \times \frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{K}_C}{\mathrm{d}t} + \frac{\mathrm{d}\bar{\mathbf{K}}_C}{\mathrm{d}t}, \tag{16.3.51'}$$

We can define the centres E and I by the formulae

$$\boldsymbol{\rho}_E = \frac{1}{Q_e} \sum_{h=1}^{n_e} Q_e^h \bar{\boldsymbol{\rho}}_{E_h}, \quad \boldsymbol{\rho}_I = \frac{1}{Q_i} \sum_{l=1}^{n_i} Q_i^l \bar{\boldsymbol{\rho}}_{I_l},$$

$$\bar{\boldsymbol{\rho}}_{E_h} = \frac{\iint_{S_e^h} \bar{\mathbf{r}}_a \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_e^h} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S}, \quad \bar{\boldsymbol{\rho}}_{I_l} = \frac{\iint_{S_i^l} \bar{\mathbf{r}}_g \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S}{\iint_{S_i^l} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S}, \tag{16.3.52}$$

analogous to the formulae (16.3.35). Noting that $\bar{\boldsymbol{\rho}}_E = \boldsymbol{\rho}_E - \boldsymbol{\rho}$, $\bar{\boldsymbol{\rho}}_I = \boldsymbol{\rho}_I - \boldsymbol{\rho}$ and taking into account (16.3.37), we get

$$\sum_{h=1}^{n_e} \iint_{S_e^h} \bar{\mathbf{r}}_a \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \bar{\mathbf{r}}_g \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S = \mathbf{H} - M\mathbf{w}_C .$$

In this case, the sums of the integrals in the formula (16.3.32) become

$$\begin{aligned}
&\sum_{h=1}^{n_e} \iint_{S_e^h} \mathbf{r}_a' \times \mathbf{u}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \mathbf{r}_g' \times \mathbf{u}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S \\
&= \boldsymbol{\rho}' \times \left(\mathbf{R} - \frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t} \right) - \mathbf{v}_C \times (\mathbf{H} - M\mathbf{w}_C) \\
&+ \sum_{h=1}^{n_e} \iint_{S_e^h} \bar{\mathbf{r}}_a \times \mathbf{w}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \bar{\mathbf{r}}_g \times \mathbf{w}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S \\
&+ \sum_{h=1}^{n_e} \iint_{S_e^h} \bar{\mathbf{r}}_a \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}_a) \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S + \sum_{l=1}^{n_i} \iint_{S_i^l} \bar{\mathbf{r}}_g \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}_g) \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S ,
\end{aligned}$$

where we took into account the above result, the formula (16.3.31) and the expressions of the position vectors $\mathbf{r}_a'$ and $\mathbf{r}_g'$, as well the decomposition of the velocities $\mathbf{u}_{a_e}$ and $\mathbf{u}_{g_i}$. Replacing in (16.3.32) and taking into account the previous results and the relation (16.3.51'), we get, finally,

$$\begin{aligned}
\frac{\mathrm{d}\bar{\mathbf{K}}_C}{\mathrm{d}t} &= \mathbf{M}_C - \mathbf{w}_C \times \mathbf{H} - \frac{\mathrm{d}\mathbf{K}_C}{\mathrm{d}t} \\
&- \sum_{h=1}^{n_e} \iint_{S_e^h} \bar{\mathbf{r}}_a \times \mathbf{w}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S - \sum_{l=1}^{n_i} \iint_{S_i^l} \bar{\mathbf{r}}_g \times \mathbf{w}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S \\
&- \sum_{h=1}^{n_e} \iint_{S_e^h} \bar{\mathbf{r}}_a \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}_a) \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S - \sum_{l=1}^{n_i} \iint_{S_i^l} \bar{\mathbf{r}}_g \times (\boldsymbol{\omega} \times \bar{\mathbf{r}}_g) \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S ,
\end{aligned} \tag{16.3.53}$$

where we have noticed that $\mathbf{M}_{O'} = \mathbf{M}_C + \boldsymbol{\rho}' \times \mathbf{R}$.

To give to this equation of motion of rotation about the mass centre a form as useful as possible from the point of view of the practical calculation, we mention that

$$\frac{\mathrm{d}\bar{\mathbf{K}}_C}{\mathrm{d}t} = \frac{\partial \bar{\mathbf{K}}_C}{\partial t} + \boldsymbol{\omega} \times \bar{\mathbf{K}}_C , \quad \frac{\mathrm{d}\mathbf{K}_C}{\mathrm{d}t} = \frac{\partial \mathbf{K}_C}{\partial t} + \boldsymbol{\omega} \times \mathbf{K}_C , \tag{16.3.54}$$

where we have used the derivatives with respect to the non-inertial frame of reference $\mathscr{R}$. Noting that $\bar{\mathbf{K}}_C = \mathbf{I}_C \boldsymbol{\omega}$, we have

$$\frac{\partial \overline{\mathbf{K}}_C}{\partial t} = \left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\mathbf{I}_C = \mathrm{const}}} + \left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\boldsymbol{\omega} = \mathrm{const}}},$$
$$\left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\mathbf{I}_C = \mathrm{const}}} = \mathbf{I}_C \dot{\boldsymbol{\omega}}, \quad \left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\boldsymbol{\omega} = \mathrm{const}}} = \frac{\partial \mathbf{I}_C}{\partial t} \boldsymbol{\omega}, \tag{16.3.55}$$

because the mechanical system $\mathscr{S}$ is of variable mass.

We can decompose the resultant moment of the given forces in the form $\mathbf{M}_C = \mathbf{M}_{C_p} + \mathbf{M}_{C_0}$, where $\mathbf{M}_{C_p}$ is *the resultant moment of the forces of pressure* while $\mathbf{M}_{C_0}$ is the resultant moment of the other external forces.

Noting that $\mathbf{w}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S$ and $\mathbf{w}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S$ can be considered elementary forces due to the capture of the particles of air and to the detachment of particles of gas, respectively, through an elementary surface of area $\mathrm{d}S$, we introduce *the moment of traction* in the form

$$\mathbf{M}_T = \mathbf{M}_{C_p} - \sum_{h=1}^{n_e} \iint_{S_e^h} \overline{\mathbf{r}}_a \times \mathbf{w}_{a_e} \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S - \sum_{l=1}^{n_i} \iint_{S_i^l} \overline{\mathbf{r}}_g \times \mathbf{w}_{g_i} \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S . \tag{16.3.56}$$

As well,

$$M_a = -\left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\boldsymbol{\omega} = \mathrm{const}}} - \sum_{h=1}^{n_e} \iint_{S_e^h} \overline{\mathbf{r}}_a \times (\boldsymbol{\omega} \times \overline{\mathbf{r}}_a) \mu_a (\mathbf{w}_{a_e} \cdot \mathbf{n}) \mathrm{d}S$$
$$- \sum_{l=1}^{n_i} \iint_{S_i^l} \overline{\mathbf{r}}_g \times (\boldsymbol{\omega} \times \overline{\mathbf{r}}_g) \mu_g (\mathbf{w}_{g_i} \cdot \mathbf{n}) \mathrm{d}S \tag{16.3.57}$$

is *the moment of gas-dynamical damping.* The equation of motion (16.3.3) takes thus the form

$$\left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\mathbf{I}_C = \mathrm{const}}} + \boldsymbol{\omega} \times \overline{\mathbf{K}}_C = \mathbf{M}_{C_0} + \mathbf{M}_T - \frac{\partial \mathbf{K}_C}{\partial t} - \boldsymbol{\omega} \times \mathbf{K}_C + \mathbf{M}_a - \mathbf{w}_C \times \mathbf{H}, \tag{16.3.58}$$

generalizing Euler's equations.

Considering that the aircraft is a system of constant mass and assuming that $\partial \mathbf{K}_C / \partial t = \mathbf{0}$, we can write

$$\left(\frac{\partial \overline{\mathbf{K}}_C}{\partial t} \right)_{\overrightarrow{\mathbf{I}_C = \mathrm{const}}} + \boldsymbol{\omega} \times \overline{\mathbf{K}}_C = \mathbf{M}_{C_0} + \mathbf{M}_T + \mathbf{K}_C \times \boldsymbol{\omega}, \tag{16.3.58'}$$

where the last term of gyroscopic nature corresponds to solid masses which give raise to the spin. Comparing the equations (16.3.58) (where we make $\partial \mathbf{K}_C / \partial t = \mathbf{0}$) and

(16.3.58'), we notice that, considering that the aircraft is a system of variable mass, appears the supplementary moment

$$\mathbf{M}_s = \mathbf{M}_a + \mathbf{H} \times \mathbf{w}_C \,. \tag{16.3.59}$$

Because the vectors $\mathbf{H}$ and $\mathbf{w}_C$ are, often, nearly parallel, it results $\mathbf{M}_S \cong \mathbf{M}_a$. Hence, $\mathbf{M}_s = \mathbf{0}$ if $\boldsymbol{\omega} = \mathbf{0}$; the condition in which takes place a motion of translation are thus formally the same, immaterial if the mechanical system $\mathscr{S}$ is of constant or of variable mass.

If we suppress the capture of particles of air, then the equation (16.3.58) becomes the equation of motion of the rocket about the centre of mass.

Chapter 17

Dynamics of Systems of Rigid Solids

A mechanical system $\mathscr{S}$ can be formed of a finite number (let be n) of rigid solids and of a finite number (let be p) of particles, its position with respect to an inertial frame of reference $\mathscr{R}'$ being specified by means of $6n + 3p = 3(2n + p)$ parameters; hence, such a system has $3(2n + p)$ degrees of freedom. In the case in which the system $\mathscr{S}$ is subjected to m constraints, the number of degrees of freedom is reduced with m units, equating $3(2n + p) - m$. We study, in this chapter, the motion of the mechanical system $\mathscr{S}$, free or with constraints, including the case in which appear discontinuities (the collision problem). The results thus obtained will be applied to some problems of dynamics of machines.

17.1 Motion of Systems of Rigid Solids

After some results with a general character, one considers the contact problem of two rigid solids; in this order of ideas, one studies various particular cases too.

17.1.1 General Results

In what follows, we make some general considerations concerning the motion of systems of rigid solids; a special attention is given to some particular cases (the double pendulum and the sympathetic pendulum).

17.1.1.1 General Considerations

Let be, in general, a free mechanical system $\mathscr{S}$, formed of the rigid solids $\mathscr{S}_k$, $k = 1,2,...,n$, and of the particles P_i, of position vectors $\mathbf{r}_i'$, $i = 1,2,...,p$, with respect to an inertial frame of reference $\mathscr{R}'$, of pole O'; the elements of this system are acted upon by given external forces, originated in other systems, and by given internal forces, corresponding to their reciprocal actions. The rigid solid $\mathscr{S}_l$ acts upon the rigid solid $\mathscr{S}_k$ by a set of forces (analogously modelled as sliding vectors) of torsor $\{\mathbf{R}_{kl}, \mathbf{M}_{kl}\}$, while the rigid solid $\mathscr{S}_k$ acts upon the rigid solid $\mathscr{S}_l$ by a set of forces (analogously modelled) of torsor $\{\mathbf{R}_{lk}, \mathbf{M}_{lk}\}$; corresponding to the theorem of action and reaction (the Theorem 12.1.7 stated for a continuous mechanical system, can be applied in the case of a system of rigid solids too), we can write

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$\{\mathbf{R}_{kl}, \mathbf{M}_{kl}\} + \{\mathbf{R}_{lk}, \mathbf{M}_{lk}\} = \mathbf{0}$, hence $\mathbf{R}_{kl} + \mathbf{R}_{lk} = \mathbf{0}$, $\mathbf{M}_{kl} + \mathbf{M}_{lk} = \mathbf{0}$, $k \neq l$, $k, l = 1, 2, \ldots, n$. The particles P_i and P_j are interacting with the forces $\mathbf{F}_{ij}$ (applied upon the particle P_i) and $\mathbf{F}_{ji}$ (applied upon the particle P_j), respectively, having as support the straight line $P_i P_j$ and verifying the relation $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$, $i \neq j$, $i, j = 1, 2, \ldots, p$; these forces are modelled as bound vectors. Analogously, if the rigid solid $\mathcal{S}_k$ acts upon the particle P_i with a force $\mathbf{\Phi}_{ik}$ (modelled as a bound vector), then the particle P_i reacts upon the rigid solid $\mathcal{S}_k$ with a force $\mathbf{\Phi}_{ki}$ (modelled as a sliding vector), these forces having the same support and verifying the relation $\mathbf{\Phi}_{ik} + \mathbf{\Phi}_{ki} = \mathbf{0}$, $i \neq k$, $i = 1, 2, \ldots, p$, $k = 1, 2, \ldots, n$, as free vectors. As in the case of a discrete system of particles (see Sect. 11.1.1.4 too), applying the principle of action of forces (the theorem of torsor in case of a rigid solid; see Sect. 14.1.1.7 too), we can write the equations of motion of the free mechanical system $\mathcal{S}$ in the form (for the sake of simplicity, it is convenient to denote by "prime" the sums of terms with two indices, if the case of equal indices is excluded)

$$m\ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{p}{}' \mathbf{F}_{ij} + \sum_{k=1}^{n}{}' \mathbf{\Phi}_{ik}, \quad i = 1, 2, \ldots, p, \tag{17.1.1}$$

$$\begin{aligned} \dot{\mathbf{H}}'_k &= \mathbf{R}_k + \sum_{i=1}^{p}{}' \mathbf{\Phi}_{ki} + \sum_{l=1}^{n}{}' \mathbf{R}_{kl}, \quad k = 1, 2, \ldots, n, \\ \dot{\mathbf{K}}'_{O'k} &= \mathbf{M}_{O'k} + \sum_{i=1}^{p}{}' \mathbf{r}'_i \times \mathbf{\Phi}_{ki} + \sum_{l=1}^{n}{}' \mathbf{M}_{kl}, \quad k = 1, 2, \ldots, n, \end{aligned} \tag{17.1.1'}$$

where $\mathbf{F}_i$ is the resultant of the given external forces which act upon the particle $\mathbf{P}_i$, $\{\mathbf{R}_k, \mathbf{M}_{O'k}\}$ is the torsor of the given external forces which act upon the rigid solid $\mathcal{S}_k$, while $\mathbf{H}'_k$ and $\mathbf{K}'_{O'k}$ represent the momentum and the moment of momentum, respectively, of the rigid solid $\mathcal{S}_k$, with respect to the pole O', in the frame of reference $\mathcal{R}'$. In general, the given forces can depend on the position vectors $\mathbf{r}'_i$, $i = 1, 2, \ldots p$, on the position vectors $\mathbf{r}'_{Ok}$ and on Euler's angles ψ_k, θ_k, φ_k, which specify the position of a point of the rigid solid $\mathcal{S}_k$, $k = 1, 2, \ldots, n$, as well as the motion of rotation about this one.

The mechanical system $\mathcal{S}$ is considered to be free, so that the $2n + p$ vector equations (17.1.1), (17.1.1') (or, in components, the $3(2n + p)$ scalar corresponding equations) can determine the $3(2n + p)$ parameters (co-ordinates and Euler's angles), which specify its position. These equations are completed by the initial conditions (at the moment $t = t_0$), i.e.: the positions and the velocities of the particles, as well as the positions and the velocities (the parameters which specify them) of the rigid solids which form the mechanical system $\mathcal{S}$. In this case too, starting from the Theorem 11.1.1, corresponding to a system of particles, and taking into account the Theorem 14.1.12, corresponding to a rigid solid, we can state a *theorem of existence and* *uniqueness of Cauchy-Lipschitz type* for the free discrete mechanical system considered above.

In the case of a mechanical system $\mathcal{S}$ subjected to constraints (external or internal, because of the contact between the rigid solids or because of other causes), one must introduce also the constraint forces (external or internal). If a particle $P_i \in \mathcal{S}$, e.g., is constrained to move on a surface $S \in \mathcal{S}$, then the constraint is internal, while if $S \notin \mathcal{S}$, then the constraint is external. Analogously, if a rigid solid $S_k \in \mathcal{S}$ remains in contact with another rigid solid appears a torsor of internal or external constraint forces, as this solid belongs or not to the mechanical system $\mathcal{S}$. The holonomic constraint relations are of the form

$$f_\alpha\left(x_1'^{(i)}, x_2'^{(i)}, x_3'^{(i)}, x_{O1}'^{(k)}, x_{O2}'^{(k)}, x_{O3}'^{(k)}, \psi_k, \theta_k, \varphi_k; t\right) = 0, \quad \alpha = 1,2,\ldots,q, \tag{17.1.2}$$

the non-holonomic constraints being expressed by the relations

$$\begin{aligned} g_\beta\Big(&x_1'^{(i)}, x_2'^{(i)}, x_3'^{(i)}, \dot{x}_1'^{(i)}, \dot{x}_2'^{(i)}, \dot{x}_3'^{(i)}, x_{O1}'^{(k)}, x_{O2}'^{(k)}, x_{O3}'^{(k)}, \\ &\dot{x}_{O1}'^{(k)}, \dot{x}_{O2}'^{(k)}, \dot{x}_{O3}'^{(k)}, \psi_k, \theta_k, \varphi_k, \dot{\psi}_k, \dot{\theta}_k, \dot{\varphi}_k; t\Big) = 0, \quad \beta = 1,2,\ldots,r, \end{aligned} \tag{17.1.2'}$$

linear in the linear and angular velocities. Obviously, the number of the unknowns increases by $m = q + r$, corresponding to the constraint forces; but also the number of relations which link the unknown functions increases analogously (by the relations (17.1.2), (17.1.2')). We can enounce a *first basic problem* (*the direct problem*), the solution of which is unique in certain conditions, sufficiently large (e.g. the Cauchy-Lipschitz theorem). A *second basic problem* (*the inverse problem*), as well as *the mixed basic problem* have not, in general, a unique solution; to have uniqueness, it is necessary to impose some supplementary conditions (see Sect. 11.1.1.4 too).

17.1.1.2 The Double Pendulum

The double pendulum is a mechanical system $\mathcal{S}$ formed of two rigid solids $\mathcal{S}_1$ and $\mathcal{S}_2$, subjected to the action of the own weights $\mathbf{G}_1 = M_1\mathbf{g}$ and $\mathbf{G}_2 = M_2\mathbf{g}$ at the centres of mass C_1 and C_2, respectively; the rigid solid $\mathcal{S}_1$ oscillates about a fixed horizontal axis O_1x_3', while the rigid solid $\mathcal{S}_2$ oscillates about an axis $O_2\overline{x}_3$, parallel to the first axis and rigidly connected to the solid $\mathcal{S}_1$. We assume, for the sake of simplicity, that the vertical plane normal to these axes, which passes through the poles O_1 and O_2, is a plane of geometric and mechanical symmetry for the mechanical system $\mathcal{S}$, hence containing also the mass centres C_1 and C_2; we assume that $C_1 \in O_1O_2$ too. We choose the O_1x_1'-axis along the descendent vertical, the O_1x_2'-axis being horizontal; the axes $O_2\overline{x}_1$ and $O_2\overline{x}_2$ are chosen analogously. The position of the system $\mathcal{S}$ (the motion is plane-parallel, the point O_1 is fixed (two constraint relations), while the point O_2 is moving on a fixed circle (one constraint relation), hence the mechanical system has $2 \cdot 3 - (2 + 1) = 3$ degrees of freedom) is specified by the distance $\overline{O_1 O_2} = l$ between the two poles (constant for a given double pendulum) and by the angles θ_1 and θ_2 made by $O_1 C_1$ and $O_2 C_2$, respectively, with the descendent vertical; we notice $\overline{O_1 C_1} = l_1$, $\overline{O_2 C_2} = l_2$ too (Fig. 17.1,a). Upon the rigid solid $\mathscr{S}_1$ act the own weight $\mathbf{G}_1 = M_1 \mathbf{g}$, the constraint force $\mathbf{R}_1$ at O_1 and the constraint force $\mathbf{R}_2$, exerted by the rigid solid $\mathscr{S}_2$ upon the rigid solid $\mathscr{S}_1$ at the cylindrical hinge O_2, while upon the rigid solid $\mathscr{S}_2$ act the own weight $\mathbf{G}_2 = M_2 \mathbf{g}$ and the constraint force $\mathbf{R}_2'$ exerted by the rigid solid $\mathscr{S}_1$ upon the rigid solid $\mathscr{S}_2$ at O_2; obviously, we have $\mathbf{R}_2 + \mathbf{R}_2' = \mathbf{0}$. We neglect the friction couples which appear on the axes $O_1 x_3'$ and $O_2 \overline{x}_3$. The motion of both centres of mass is given by the equation (the differentiation takes place with respect to the fixed frame of reference)

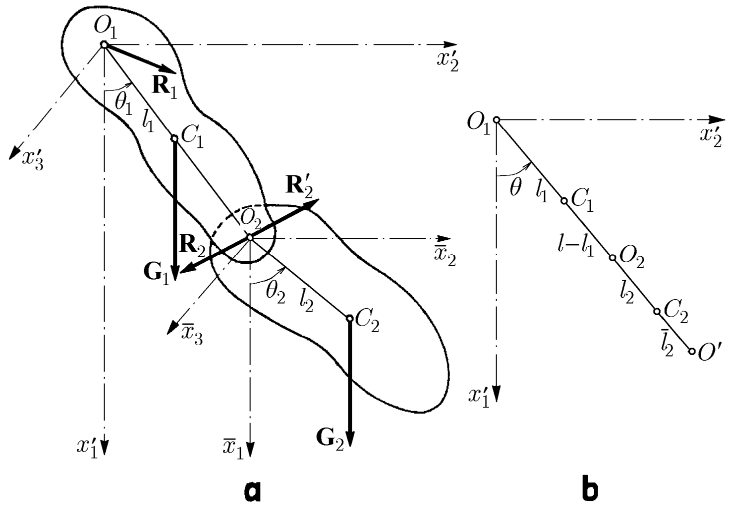

Fig. 17.1 Double pendulum

$$M_1 \ddot{\boldsymbol{\rho}}_1 = M_1 \mathbf{g} + \mathbf{R}_1 + \mathbf{R}_2, \quad M_2 \ddot{\boldsymbol{\rho}}_2 = M_2 \mathbf{g} + \mathbf{R}_2',$$

where

$$\boldsymbol{\rho}_1 = l_1 (\cos\theta_1 \mathbf{i}_1' + \sin\theta_1 \mathbf{i}_2'),$$
$$\boldsymbol{\rho}_2 = (l\cos\theta_1 + l_2 \cos\theta_2)\mathbf{i}_1' + (l\sin\theta_1 + l_2 \sin\theta_2)\mathbf{i}_2'$$

are the corresponding position vectors; projecting on the axes $O_1 x_1'$, $O_1 x_2'$ and calculating the components of the accelerations, one obtain the components of the constraint forces in the form

$$\begin{aligned} R_{11} &= -(M_1 l_1 + M_2 l)\left(\cos\theta_1 \dot{\theta}_1^2 + \sin\theta_1 \ddot{\theta}_1\right) \\ &\quad - M_2 l_2 \left(\cos\theta_2 \dot{\theta}_2^2 + \sin\theta_2 \ddot{\theta}_2\right) - (M_1 + M_2) g, \\ R_{12} &= (M_1 l_1 + M_2 l)\left(\cos\theta_1 \ddot{\theta}_1 - \sin\theta_1 \dot{\theta}_1^2\right) + M_2 l_2 \left(\cos\theta_2 \ddot{\theta}_2 - \sin\theta_2 \dot{\theta}_2^2\right), \end{aligned} \tag{17.1.3}$$

$$\begin{aligned} R_{21} &= -R'_{21} = M_2\left[l\left(\cos\theta_1\dot{\theta}_1^2 + \sin\theta_1\ddot{\theta}_1\right) + l_2\left(\cos\theta_2\dot{\theta}_2^2 + \sin\theta_2\ddot{\theta}_2\right) + g\right], \\ R_{22} &= -R'_{22} = -M_2\left[l\left(\cos\theta_1\ddot{\theta} - \sin\theta_1\dot{\theta}_1^2\right) + l_2\left(\cos\theta_2\ddot{\theta}_2 - \sin\theta_2\dot{\theta}_2^2\right)\right]. \end{aligned} \tag{17.1.3'}$$

Taking into account the Huygens-Steiner theorem (formulae (3.1.113), (3.1.113'), we can write the moment of inertia of the rigid solid $\mathscr{S}_1$ with respect to the $O_1x'_3$-axis in the form $M_1\left(l_1^2 + i_1^2\right)$, where i_1 is the gyration radius corresponding to the mass centre C_1, taken with respect to an axis parallel to $O_1x'_3$; the moment of inertia of the rigid solid $\mathscr{S}_2$ with respect to an axis parallel to the $O_2\overline{x}_3$, which passes through the mass centre C_2, is given by $M_2 i_2^2$, where i_2 has an analogous significance. The theorem of moment of momentum for the rigid solid $\mathscr{S}_1$, in the frame $\mathscr{R}'$, with respect to the $O_1x'_3$-axis, reads

$$M_1\left(l_1^2 + i_1^2\right)\ddot{\theta}_1 = -M_1 g l_1 \sin\theta_1 + l\left(R_{22}\cos\theta_1 - R_{21}\sin\theta_1\right).$$

As well, writing the theorem of moment of momentum for the rigid solid $\mathscr{S}_2$, in a frame of reference of Koenig, with respect to an axis parallel to the $O_1x'_3$-axis, we get

$$M_2 i_2^2\ddot{\theta}_2 = l_2\left(R'_{21}\sin\theta_2 - R'_{22}\cos\theta_2\right).$$

Replacing the constraint forces given by (17.1.3'), it results the system of differential equations of second order

$$\begin{aligned} &\alpha_1\ddot{\theta}_1 + \cos(\theta_2 - \theta_1)\ddot{\theta}_2 - \sin(\theta_2 - \theta_1)\dot{\theta}_2^2 + \frac{g}{l}\beta\sin\theta_1 = 0, \\ &\alpha_2\ddot{\theta}_2 + \cos(\theta_2 - \theta_1)\ddot{\theta}_1 + \sin(\theta_2 - \theta_1)\dot{\theta}_1^2 + \frac{g}{l}\sin\theta_2 = 0, \end{aligned} \tag{17.1.4}$$

where

$$\alpha_1 = \frac{\mu\left(l_1^2 + i_1^2\right) + l^2}{l_2 l} > 0, \quad \alpha_2 = \frac{l_2^2 + i_2^2}{l_2 l} > 0, \quad \beta = \frac{\mu l_1 + l}{l_2} > 0, \quad \mu = \frac{M_1}{M_2}.$$

To determine the motion, one must add also the conditions $\theta_1(t_0) = \theta_1^0$, $\theta_2(t_0) = \theta_2^0$, $\dot{\theta}_1(t_0) = \dot{\theta}_1^0$, $\dot{\theta}_2(t_0) = \dot{\theta}_2^0$, where θ_1^0, θ_2^0 and $\dot{\theta}_1^0$, $\dot{\theta}_2^0$ specify the position and the velocity, respectively, of the double pendulum at the initial moment.

Being impossible to give an exact solution to this non-linear problem, we will consider the case of small motions for which $\cos(\theta_2 - \theta_1) \cong 1$, $\sin\theta_1 \cong \theta_1$, $\sin\theta_2 \cong \theta_2$ and $\sin(\theta_2 - \theta_1) \cong \theta_2 - \theta_1$; we find thus

$$\begin{aligned} &\alpha_1\ddot{\theta}_1 + \ddot{\theta}_2 - (\theta_2 - \theta_1)\dot{\theta}_2^2 + \frac{g}{l}\beta\theta_1 = 0, \\ &\alpha_2\ddot{\theta}_2 + \ddot{\theta}_1 + (\theta_2 - \theta_1)\dot{\theta}_1^2 + \frac{g}{l}\theta_2 = 0. \end{aligned} \tag{17.1.4'}$$

For linearization, we assume – further – that $\theta_1 \cong \theta_2$ so that $(\theta_2 - \theta_1)\dot{\theta}_1^2 \ll \ddot{\theta}_1$ and $(\theta_2 - \theta_1)\dot{\theta}_2^2 \ll \ddot{\theta}_2$; we may write

$$\begin{aligned} \alpha_1 \ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l}\beta\theta_1 &= 0, \\ \alpha_2 \ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{l}\theta_2 &= 0. \end{aligned} \tag{17.1.4''}$$

We search solutions of the form $\theta_1(t) = \lambda_1 e^{\sigma t}$, $\theta_2(t) = \lambda_2 e^{\sigma t}$ and are led to

$$(\alpha_1\lambda_1 + \lambda_2)\sigma^2 + \frac{g}{l}\beta\lambda_1 = 0, \quad (\alpha_2\lambda_2 + \lambda_1)\sigma^2 + \frac{g}{l}\lambda_2 = 0; \tag{17.1.5}$$

one must have

$$(1 - \alpha_1\alpha_2)\sigma^4 - \frac{g}{l}(\alpha_1 + \beta\alpha_2)\sigma^2 - \left(\frac{g}{l}\right)^2 \beta = 0,$$

wherefrom

$$\sigma^2 = -\frac{g}{2l(\alpha_1\alpha_2 - 1)}\left[\alpha_1 + \beta\alpha_2 \pm \sqrt{(\alpha_1 - \beta\alpha_2)^2 + 4\beta}\,\right], \tag{17.1.5'}$$

so that the linear algebraic system in λ_1 and λ_2 be compatible. Noting that the discriminant of the biquadratic equation is always positive, that the magnitude between the square brackets is also positive and that $\lambda_1\lambda_2 > 1$ we can state that $\sigma^2 < 0$. Denoting by $\pm\mu_1 i$, $\pm\mu_2 i$ the corresponding roots and determining λ_1, λ_2 from the system (17.1.5), one obtains the general solution of the linearized system (17.1.4'') in the form

$$\begin{aligned} \theta_1(t) &= C_1\left(\frac{g}{l} - \alpha_2\mu_1^2\right)\cos(\mu_1 t - \varphi_1) + C_2\left(\frac{g}{l} - \alpha_2\mu_2^2\right)\cos(\mu_2 t - \varphi_2), \\ \theta_2(t) &= C_1\mu_1^2 \cos(\mu_1 t - \varphi_1) + C_2\mu_2^2 \cos(\mu_2 t - \varphi_2), \end{aligned} \tag{17.1.6}$$

where C_1, C_2, φ_1, φ_2 are arbitrary constants, which are determined by the initial conditions. The system $\mathscr{S}$ oscillates so that the amplitudes of any oscillation are small (hence, the constants C_1 and C_2 too), the pulsations μ_1 and μ_2 being, in general, distinct.

We can make an analogous study of the double pendulum choosing as unknown functions the co-ordinates $x_2'(t) = l_1 \sin\theta_1$ and $\overline{x}_2(t) = l_2 \sin\theta_2$ of the mass centres C_1 and C_2, respectively.

We may put the problem to find the conditions in which the mechanical system $\mathscr{S}$ oscillates as a unitary rigid solid; in this case, we must have $\dot{\theta}_1(t) = \dot{\theta}_2(t)$. Assuming that $\theta_1(t) = \theta_2(t) = \theta(t)$ too, the equations (17.1.4) become

$$(1+\alpha_1)\ddot{\theta}+\frac{g}{l}\beta\sin\theta=0\,,\quad (1+\alpha_2)\ddot{\theta}+\frac{g}{l}\sin\theta=0\,; \tag{17.1.7}$$

we are led to the same solution if the relation $1+\alpha_1=\beta(1+\alpha_2)$ takes place, hence if

$$M_1\left[l_2\left(i_1^2+l_1^2\right)-l_1\left(i_2^2+l_2^2\right)-ll_1l_2\right]=M_2 l i_2^2\,, \tag{17.1.7'}$$

even in case of finite amplitudes. It is sufficient to have $\theta_1\cong\theta_2$ in the case of small motions. The two equations (17.1.7) correspond thus to a synchronous mathematical pendulum of length $l'=l(1+\alpha_2)$ hence of length

$$l'=l+l_2+\overline{l}_2\,, \tag{17.1.7''}$$

where $\overline{l}_2=i_2^2/l_2>0$; we notice thus that $l'=\overline{O_1O'}$, the point O' being on the other part of the mass centre C_2, with respect to the point O_2 (Fig. 17.1,b).

A bell together with its tongue forms a double pendulum, subjected to small oscillations. If, by construction, the condition (17.1.7') takes place, then the bell does not ring (no sound is heard, because the tongue cannot strike the bell). Such a phenomenon took place in reality, remaining famous, at the inauguration, in 1876, of the bell of the cathedral in Cologne; this bell has been put in form from the bronze of the guns conquered by the Prussian army at Sedan, in 1870, in the campaign against France.

Returning to the non-linear system of differential equations of second order (17.1.4), we introduce the notation $\theta_1=z_1$, $\theta_2=z_2$, $\dot{\theta}_1=z_3$, $\dot{\theta}_2=z_4$, so that the system becomes now of first order and reads

$$\begin{gathered}\dot{z}_1=z_3\,,\quad \dot{z}_2=z_4\,,\\ \dot{z}_3+\alpha\left[\dot{z}_4\cos(z_2-z_1)-z_4^2\sin(z_2-z_1)\right]+\beta\sin z_1=0\,,\\ \dot{z}_4+\gamma\left[\dot{z}_3\cos(z_2-z_1)+z_3^2\sin(z_2-z_1)\right]+\sin z_2=0\,,\end{gathered} \tag{17.1.8}$$

where

$$\alpha=\frac{rs}{r^2+(4/3)m}\,,\quad \beta=\frac{4}{3}s\frac{r+m}{r^2+(4/3)m}\,,\quad \gamma=\frac{3}{4}\frac{r}{s} \tag{17.1.8'}$$

are non-dimensional coefficients with

$$m=\frac{M_1}{M_2}\,,\quad r=\frac{l}{l_1}\,,\quad s=\frac{l_2}{l_1}\,. \tag{17.1.8''}$$

Using *the theory of Lie transform* applied by L. Morino, F. Mastroddi and M. Cutroni in 1992, Anca Zlătescu studied, in 1998, in her doctor thesis, the system (17.1.8), assuming that

$$\frac{1}{1 - \alpha\gamma\cos^2(z_2 - z_1)} \cong 1 + \alpha\gamma\cos^2(z_2 - z_1)\,. \tag{17.1.8'''}$$

She put in evidence also the effect of this approximation, as well as the influence of the generalized forces of vibrating type applied at the points C_2 or O', which introduce perturbing terms in the second members of the last two equations (17.1.8). In this order of ideas, she dealt with the conditions of stability of the motion; the passing to chaos has been taken into consideration too.

Analogously, one can study the problem of the triple pendulum.

17.1.1.3 Sympathetic Pendulums

Let be a mechanical system $\mathscr{S}$ formed of two identical (*in tune*) or distinct (*out of time*) physical pendulums, linked between them (*stronger* or *weaker*) by an elastic spring; these pendulums which oscillate simultaneously are called *sympathetic* (*coupled*) *pendulums*. We mention the analogy between this system and a device formed of two electrical circuits (one primary and one secondary) inductively connected. If l_1 and l_2 are the lengths of the mathematical pendulums synchronous with the considered sympathetic pendulums, then the pulsations are given by $\omega_1^2 = g / l_1$, $\omega_2^2 = g / l_2$; the elastic constants involved are $k_1 = \sigma / M_1$, $k_2 = \sigma / M_2$, where M_1 and M_2 are the masses of the pendulums, while σ is the stress in the spring for a unit linear strain (Fig. 17.2).

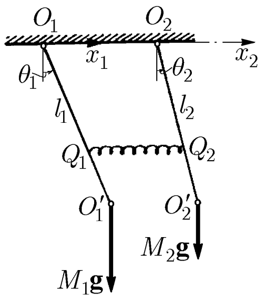

Fig. 17.2 Sympathetic Pendulums

If, in particular, the sympathetic pendulums are perfectly in tune, then we can write the equations of motion in the form ($l_1 = l_2 = l$, $\omega_1 = \omega_2 = \omega_0$, $M_1 = M_2 = M$, $k_1 = k_2 = k$)

$$\ddot{x}_1 + \omega_0^2 x_1 = -k(x_1 - x_2), \quad \ddot{x}_2 + \omega_0^2 x_2 = -k(x_2 - x_1), \tag{17.1.9}$$

where x_1 and x_2 are the elongations of the corresponding centres of oscillation O_1' and O_2'. The change of variable $\xi_1 = x_1 - x_2$, $\xi_2 = x_1 + x_2$ leads to the equations

$$\ddot{\xi}_1 + (\omega_0^2 + 2k)\xi_1 = 0, \quad \ddot{\xi}_2 + \omega_0^2 \xi_2 = 0, \tag{17.1.9'}$$

with the pulsations $\omega = \sqrt{\omega_0^2 + 2k} \cong \omega_0 + k/\omega_0$, $\omega' = \omega_0$. Assuming that at the initial moment $t = 0$ we have $x_1^0 = a$, $\dot{x}_1^0 = 0$, $x_2^0 = 0$, $\dot{x}_2^0 = 0$, hence $\xi_1^0 = \xi_2^0 = a$, $\dot{\xi}_1^0 = \dot{\xi}_2^0 = 0$, we get $\xi_1(t) a\cos\omega t$, $\xi_2(t) a\cos\omega' t$ and then $(x_1 = (\xi_1 + \xi_2)/2$, $x_2 = (\xi_2 - \xi_1)/2)$

$$\begin{aligned} x_1(t) &= \frac{a}{2}(\cos\omega' t + \cos\omega t) = a\cos\frac{(\omega - \omega')t}{2}\cos\frac{(\omega + \omega')t}{2}, \\ x_2(t) &= \frac{a}{2}(\cos\omega' t - \cos\omega t) = a\sin\frac{(\omega - \omega')t}{2}\sin\frac{(\omega + \omega')t}{2}. \end{aligned} \qquad (17.1.9'')$$

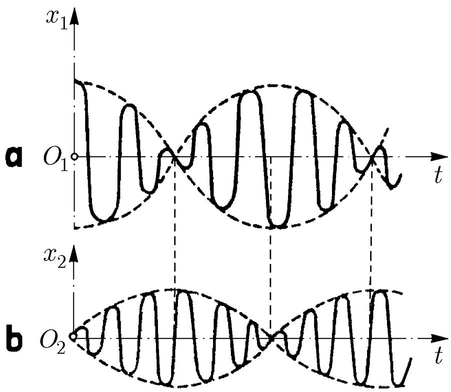

Fig. 17.3 Sympathetic pendulums: Graphics of the elongations of the points O_1' (**a**) and O_2' (**b**)

The graphics of the elongations of the points O_1' and O_2' are drawn in Fig. 17.3a and b, where the variation of the amplitude (vibrations with modulation in amplitude) is represented by broken lines; if $(\omega - \omega')/2 \cong k/2\omega_0 \ll \omega_0$, then the connection of the pendulums is a *weak connection*, the amplitude varying slowly in time (we can say that a *fluctuation* of the amplitude takes place). One can see that the vibration of each physical pendulum can be obtained by the superposition of two fundamental vibrations; to a constructive interference (maximal amplitude) of one pendulum corresponds a destructive interference (which is an *extinction*) for the second pendulum. Each of the two pendulums leads to a phenomenon of *beats* (see Chap. 8, Sect. 2.2.4 too). The mechanical energy passes from a pendulum to another one.

There are two cases in which this energy transfer does not take place: the symmetric case (Fig. 17.4a) in which $x_1^0 = x_2^0 = a/2$, $\dot{x}_1^0 = \dot{x}_2^0 = 0$, resulting $x_1(t) = x_2(t) = (a/2)\cos\omega' t$ and the antisymmetric case (Fig. 17.4b) in which $x_1^0 = -x_2^0 = a/2$, $\dot{x}_1^0 = \dot{x}_2^0 = 0$, obtaining $x_1(t) = -x_2(t) = (a/2)\cos\omega t$. These vibrations represent the two *fundamental vibrations* (*principal vibrations* or *proper vibrations*) of the sympathetic pendulums, which take place without transfer of energy, corresponding to the number of degrees of freedom of the mechanical system. One observes that the equations (17.1.9') are just the differential equations of these two vibrations. The vibrations (17.1.9") are obtained by superposing the effects of the two fundamental vibrations.

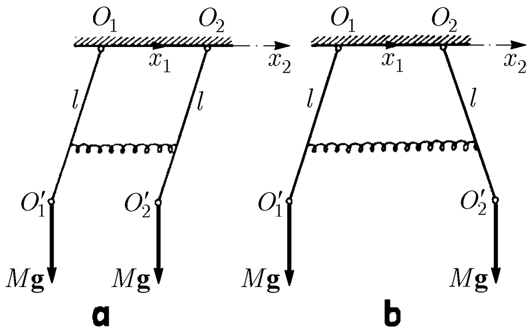

Fig. 17.4 Sympathetic pendulums: a) symmetric case; b) antisymmetric case

If the two pendulums are *out of tune* (the general case) then the equations of motion read

$$\ddot{x}_1 + \omega_1^2 x_1 = -k_1(x_1 - x_2), \quad \ddot{x}_2 + \omega_2^2 x_2 = -k_2(x_2 - x_1). \tag{17.1.10}$$

Putting $x_1 = A_1 e^{\pm i\nu t}, x_2 = A_2 e^{\pm i\nu t}$, we get the homogeneous algebraic system

$$\begin{aligned}(\omega_1^2 - \nu^2 + k_1)A_1 - k_1 A_2 &= 0, \\ (\omega_2^2 - \nu^2 + k_2)A_2 - k_2 A_1 &= 0;\end{aligned}$$

the condition of compatibility leads to the biquadratic equation

$$[\nu^2 - (\omega_1^2 + k_1)][\nu^2 - (\omega_2^2 + k_2)] = k_1 k_2,$$

wherefrom

$$\nu^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 + k_1 + k_2 \pm \sqrt{(\omega_1^2 - \omega_2^2 + k_1 - k_2)^2 + 4k_1 k_2}\right],$$

both roots being positive. Expanding the radical after Newton's binomial, we get

$$\nu^2 \cong \frac{1}{2}\left\{\omega_1^2 + \omega_2^2 + k_1 + k_2 \pm (\omega_1^2 - \omega_2^2 + k_1 - k_2)\left[1 + \frac{2k_1 k_2}{(\omega_1^2 - \omega_2^2 + k_1 - k_2)^2}\right]\right\}$$

for k_1 and k_2 small (hence for a weak connection); the pulsations ω and ω' will be given by

$$\omega^2 = \omega_1^2 + k_1 + \frac{k_1 k_2}{\omega_1^2 + k_1 - (\omega_2^2 + k_2)}, \quad \omega'^2 = \omega_2^2 + k_2 - \frac{k_1 k_2}{\omega_1^2 + k_1 - (\omega_2^2 + k_2)}. \tag{17.1.10'}$$

One obtains thus elongations of the form

$$\begin{aligned} x_1(t) &= \gamma(A\cos\omega t + B\sin\omega t) + \gamma'(A'\cos\omega' t + B'\sin\omega' t), \\ x_2(t) &= A\cos\omega t + B\sin\omega t + A'\cos\omega' t + B'\sin\omega' t, \end{aligned} \tag{17.1.10''}$$

where γ and γ' correspond to the non-determinate solutions of the homogeneous linear algebraic system. With the initial conditions (for $t = 0$) $x_1^0 = a$, $\dot{x}_1^0 = 0$, $x_2^0 = 0$, $\dot{x}_2^0 = 0$, it results

$$\begin{aligned} x_1(t) &= \frac{a}{\gamma' - \gamma}(\gamma'\cos\omega' t - \gamma\cos\omega t), \\ x_2(t) &= \frac{a}{\gamma' - \gamma}(\cos\omega' t - \cos\omega t) \\ &= \frac{2a}{\gamma' - \gamma}\sin\frac{(\omega - \omega')t}{2}\sin\frac{(\omega + \omega')t}{2}. \end{aligned} \tag{17.1.10'''}$$

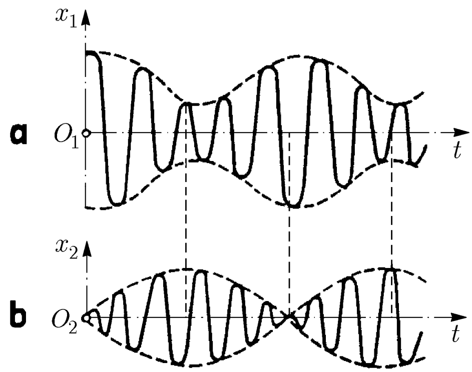

Fig. 17.5 Sympathetic pendulums out of tune. Graphics of the elongations of the points O_1' (**a**) and O_2' (**b**)

The graphics of these elongations are drawn in Fig. 17.5a and b. One observes that, in this case, the destructive interference of the first pendulum does not lead to extinction after intervals of time equal to $2\pi/(\omega - \omega')$.

An analogous study can be made choosing as unknown functions the angles $\theta_1(t) = \arcsin(x_1/l_1)$ and $\theta_2(t) = \arcsin(x_2/l_2)$ made by O_1O_1' and O_2O_2' with the descendent vertical line, respectively.

We denote $\overline{O_1O_1} = l$ and $\overline{O_1Q_1} = \overline{O_2Q_2} = a$, where Q_1 and Q_2 are the ends of the elastic spring, characterized by the elastic constant k; the elastic force in the spring is thus given by $k\left|\overline{O_1O_2} - \overline{Q_1Q_2}\right|$. We notice that

$$\begin{aligned} \overline{Q_1Q_2}^2 &= [l + a(\sin\theta_2 - \sin\theta_1)]^2 + a^2(\cos\theta_2 - \cos\theta_1)^2 \\ &= l^2 + 2al(\sin\theta_2 - \sin\theta_1) + 2a^2[1 - \cos(\theta_2 - \theta_1)]. \end{aligned} \tag{17.1.11}$$

The non-linear system of equations of motion is obtained in the form

$$\begin{aligned} I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 + \frac{k}{2}\frac{\partial}{\partial\theta_1}\left(l - \overline{Q_1Q_2}\right)^2 &= 0\,, \\ I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 + \frac{k}{2}\frac{\partial}{\partial\theta_2}\left(l - \overline{Q_1Q_2}\right)^2 &= 0\,. \end{aligned} \tag{17.1.12}$$

Introducing the expression of $\overline{Q_1Q_2}$, it results

$$\begin{aligned} I_1\ddot{\theta}_1 + M_1 g l_1 \sin\theta_1 + kaH\left[l\cos\theta_1 + a\sin(\theta_2 - \theta_1)\right] &= 0\,, \\ I_2\ddot{\theta}_2 + M_2 g l_2 \sin\theta_2 - kaH\left[l\cos\theta_2 + a\sin(\theta_2 - \theta_1)\right] &= 0\,, \end{aligned} \tag{17.1.12'}$$

where

$$H(\theta_1,\theta_2) = \frac{l - \Psi(\theta_1,\theta_2)}{\Psi(\theta_1,\theta_2)}\,, \tag{17.1.11'}$$

with

$$\Psi(\theta_1,\theta_2) = \overline{Q_1Q_2} = \left\{l^2 + 2al(\sin\theta_2 - \sin\theta_1) + 2a^2\left[1 - \cos(\theta_2 - \theta_1)\right]\right\}^2. \tag{17.1.11''}$$

Using the notations

$$\begin{aligned} w = \frac{M_1^2 g l_1}{I_1 k}\,, \quad \beta w = \frac{M_1 M_2 g l_2}{I_2 k}\,, \quad \Phi = \frac{\Psi}{l}\,, \\ \xi = \frac{a}{l}\,, \quad \alpha = \frac{alM_1}{I_1}\,, \quad \tilde{\alpha} = \frac{alM_1}{I_2}\,, \end{aligned} \tag{17.1.13}$$

as well as

$$\gamma(\theta_1,\theta_2) = \Phi^{-1/2} - 1\,, \tag{17.1.13'}$$

with

$$\Phi(\theta_1,\theta_2) = 1 + 2\xi(\sin\theta_2 - \sin\theta_1) + 2\xi^2\left[1 - \cos(\theta_2 - \theta_1)\right]. \tag{17.1.13''}$$

Introducing the unknown functions $z_1 = \theta_1$, $z_2 = \theta_2$, $z_3 = \dot{\theta}_1$, $z_4 = \dot{\theta}_2$, the system of differential equations of second order takes the form of a system of first order

$$\begin{aligned} \dot{z}_1 &= z_3\,, \quad \dot{z}_2 = z_4\,, \\ \dot{z}_3 &= -w\sin z_1 - \gamma(z_1,z_2)\,\alpha\left[\cos z_1 + \xi\sin(z_2 - z_1)\right], \\ \dot{z}_4 &= -\beta w\sin z_2 + \gamma(z_1,z_2)\,\tilde{\alpha}\left[\cos z_2 + \xi\sin(z_2 - z_1)\right]. \end{aligned} \tag{17.1.12''}$$

These equations have been introduced in 2001 by Ştefania Donescu in her doctor thesis. She studied them using *the linear equivalence method* (LEM) established by Ileana Toma in her doctor thesis, in 1980 (see Chap. 24, Sect. 1.2), assuming that

$$\left|2\xi(\sin\theta_2 - \sin\theta_1) + 2\xi^2[1 - \cos(\theta_2 - \theta_1)]\right| < 1, \tag{17.1.13'''}$$

wherefrom

$$\gamma(\theta_1, \theta_2) \cong -\xi\sin(\theta_2 - \theta_1) - \xi^2[1 - \cos(\theta_2 - \theta_1)]; \tag{17.1.13$^{\text{iv}}$}$$

this approximation has been thoroughly discussed. Şt. Donescu dealt also with cnoidal solutions, obtaining numerical results plotted into diagrams, as well as interesting results concerning the stability of the mechanical system, both for the approximate case and for the exact one.

17.1.2 Contact of Two Rigid Solids

In what follows, we consider firstly the general case of the rigid solid in contact with a fixed surface, assuming that one can have constraints with friction too; the results thus obtained will be then used to the study of some particular problems (the motion of the gyroscope in contact with a fixed plane, the motion of a heavy circular disc or of a heavy sphere on a fixed plane etc.).

The first study in this direction has been made by S.-D. Poisson (the motion of a heavy rigid solid in contact with a fixed plane); Cournot took again the problem considering the friction too. The particular case of the motion of a billiard ball has been considered by Coriolis in 1835. Puiseux studied the motion of a heavy rigid solid of rotation on a perfectly smooth horizontal plane; Slesser tackled in 1861 the same case, assuming that the rigid solid can roll and pivot without sliding, Neumann taking again the problem in 1886. Other results are due to Scouten, Ferrers, Carvallo, Korteweg and Appell.

17.1.2.1 General Considerations

Let be two rigid solids $\mathscr{S}$ and $\mathscr{S}'$, bounded by the surfaces S and S', respectively, having – at every moment – the ordinary common point $P \equiv P'$, $P \in S$, $P' \in S'$ (obviously, the points P and P' of the two rigid solids are always other ones, the mentioned situation being instantaneous), at which they have the same tangent plane Π; the distribution of the velocities in the relative motion of a rigid solid with respect to another one has been considered in Chap. 5, Sect. 3.3.1. In this study, we neglect any interaction which can intervene between the two solids, excepting the actions of contact. The condition of impenetrability of the rigid solids (which can be separated or in contact) is expressed, in general by an inequality; e.g., if the rigid solids are two spheres (O, R) and (O', R'), the unilateral constraint relation is expressed in the form $\left|\overrightarrow{OO'}\right| \geq R + R'$. We assume, in what follows, that the constraints are bilateral, having contact relations at one point $P \equiv P'$, expressed by equalities.

Let $\mathbf{R}$ be the constraint force which represents the action of the rigid solid $\mathscr{S}'$ upon the rigid solid $\mathscr{S}$, at the point P, and let $\mathbf{R}'$ be the constraint force corresponding to the action of a rigid solid $\mathscr{S}$ upon the rigid solid $\mathscr{S}'$, at the point P'; in conformity to the theorem of action and reaction, these forces verify the relation $\mathbf{R}' + \mathbf{R} = \mathbf{0}$. We mention that the equations of motion of the rigid solid (the theorems of momentum and of moment of momentum), the theorem of action and reaction and the relation of impenetrability (relation of bilateral constraint) are not sufficient to determine the unknowns which intervene (as a matter of fact, the constraint forces $\mathbf{R}$ and $\mathbf{R}'$ are unknowns); to solve the problem, one must add some supplementary conditions.

In case of a contact without friction (the surfaces S and S' are smooth), the constraint forces $\mathbf{R}$ and $\mathbf{R}'$ must be normal to the tangent plane Π at the point of contact, being denoted by $\mathbf{N}$ and $\mathbf{N}'$ respectively (we have $\mathbf{N} + \mathbf{N}' = \mathbf{0}$; eventual tangential components $\mathbf{T}$ and $\mathbf{T}'$ would correspond to an infinite relative displacement in the plane Π. We notice that these constraint forces are pressures, the sense of which is towards the interior of the rigid solid upon which they act and which vanish when the contact ceases (see Chap. 3, Sect. 2.2.9 too). In some cases, a contact takes place along a curve or on a certain surface and one can make analogous considerations.

If the contact is with friction, remaining punctual, then the friction is a sliding friction. We can decompose the constraint force in the form $\mathbf{R} = \mathbf{N} + \mathbf{T}$ (the same for the constraint force $\mathbf{R}'$) where $\mathbf{N}$ is the ideal constraint force (normal, without friction), while $\mathbf{T}$ is the sliding constraint force (see Chap. 3, Sect.. 2.2.12 too). We denote by f the Coulombian coefficient of friction ($f \geq 0$), introduced in Chap. 3, Sect.. 2.2.11. Further, we assume that the rigid solid $\mathscr{S}'$ is at rest with respect to an inertial frame of reference, studying the motion of the rigid solid $\mathscr{S}$ on the surface S'. If the sliding velocity of the rigid solid $\mathscr{S}$ on the surface S' at the contact point (the velocity of a point Q which coincides with $P \equiv P'$ at any moment t, hence the velocity of transportation with respect to the surface S', contained in the plane Π and given by (5.3.21)) vanishes ($\mathbf{v}_Q = \mathbf{0}$), then we have $|\mathbf{T}| \leq f|\mathbf{N}|$; to $\mathbf{v}_Q \neq \mathbf{0}$ corresponds $|\mathbf{T}| = f|\mathbf{N}|$, with $\mathbf{T} = -\lambda \mathbf{v}_Q$, $\lambda = f|\mathbf{N}|/|\mathbf{v}_Q| > 0$ (in this limit case, the vector $\mathbf{R}$ has its support along a generatrix of the cone of friction).

In fact, in the dynamic case one uses *the Coulomb-Morin relation* in the form

$$\mathbf{T} = -f_d N \operatorname{vers} \mathbf{v}, \tag{17.1.14}$$

where f_d is a *coefficient of a dynamic friction*; we notice that $f_d \leq f_s$, $f_s = f$, where f_s is *the coefficient of static friction*. In general, we can have $f_d = f_d(v)$.

The equation of motion of the theoretical contact point P is of the form

$$M\ddot{\mathbf{r}} = \mathbf{F} - f_d N \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}, \tag{17.1.15}$$

where $\mathbf{F}$ is the resultant of the given forces, while $\mathbf{M}$ is the mass of the rigid solid $\mathscr{S}$, mathematically modelled as a particle (considered to be reduced to the point P); by integration, we get

$$\frac{1}{2}Mv^2 = h + \int_0^s \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_0^s f_d N \mathrm{d}\overline{s} , \tag{17.1.16}$$

where $\overline{s} \in [0, s]$ is a curvilinear co-ordinate along the trajectory of the point P. In case of a conservative force, we can write

$$T + V = h - \int_0^s f_d N \mathrm{d}\overline{s} , \tag{17.1.16'}$$

where h is the energy constant. Because the above integral is always positive, it results a decrease of the mechanical energy of the rigid solid $\mathscr{S}$ which slides on the surface of another fixed rigid solid $\mathscr{S}'$. This energy is lost, being transformed in a degraded energy (e.g., heat).

But, practically, the contact between rigid solids is not punctual taking place on a small part Σ of the surface, where a process of deformation takes place (the model of rigid is only an approximation); the actions $\mathbf{R}_\Sigma$ of a solid upon the other one are distributed on Σ after a unknown law (as a mater fact, even the surface Σ is, in general, difficult to specify). Therefore, assuming some simplifying hypotheses, the torsor of these actions at a point $P \equiv P' \in \Sigma$ is determined.

We assume, further, the rigid model for the two solids, the contact taking place – theoretically – at the point $P \equiv P'$; the action of a solid upon the other will be modelled by a torsor (a force and a moment (couple)), applied at the theoretical point of contact, the theorem of action and reaction being used. If we suppose, further, that the rigid solid $\mathscr{S}'$ is at rest with respect to an inertial frame of reference $\mathscr{R}'$, at the point P of the solid $\mathscr{S}$ will act, in general, a constraint torsor $\tau_P\{\mathbf{R}_\Sigma\}$ of resultant $\mathbf{R}$ and resultant moment $\mathbf{M}_P$ (see Chap. 3, Sect.. 2.2.12 too). We effect a decomposition, so that one component be along the normal to the surface S at P, the other component being contained in the plane Π tangent to this surface, at the same point; one obtains thus $\mathbf{R} = \mathbf{N} + \mathbf{T}$ and $\mathbf{M}_P = \mathbf{M}_p + \mathbf{M}_r$ where $\mathbf{M}_p$ is the pivoting friction moment (along the normal), while $\mathbf{M}_r$ is the rolling friction moment (in the tangent plane). As well, we notice that the motion of the rigid solid $\mathscr{S}$ with respect to the surface S' is characterized by a translation of velocity $\mathbf{v}_Q(t)$ and by a rotation of angular velocity $\boldsymbol{\omega}(t)$ (see Chap. 5, Sect.. 3.3.1 too); analogously, we decompose the angular velocity in the form $\boldsymbol{\omega} = \boldsymbol{\omega}_n + \boldsymbol{\omega}_t$, where $\boldsymbol{\omega}_n(t)$ is *the pivoting angular velocity*, while $\boldsymbol{\omega}_t(t)$ is *the rolling angular velocity*. If the rigid solid $\mathscr{S}$ is at rest with respect to the frame of reference $\mathscr{R}'$ ($\mathbf{v}_Q = \mathbf{0}$, $\boldsymbol{\omega} = \mathbf{0}$), then the inequalities

$$T \leq fN , \quad M_r \leq sN , \quad M_p \leq aN , \quad f, s, a \geq 0 , \tag{17.1.17}$$

take place, where f is the (non-dimensional) coefficient of sliding friction, s is the rolling friction coefficient, while a is the pivoting coefficient of friction (s and a have the dimension of a length), as it has been shown in detail in Chap. 5, Sect.. 3.3.1. In the case in which a sliding, a rolling and a pivoting of the rigid solid $\mathscr{S}$ on the surface S' take place ($\mathbf{v}_Q \neq \mathbf{0}$, $\boldsymbol{\omega}_t \neq \mathbf{0}$, $\boldsymbol{\omega}_n \neq \mathbf{0}$), the inequalities (17.1.17) are replaced by equalities, in the form

$$\begin{gathered} \mathbf{T} = -\lambda \mathbf{v}_Q, \quad \mathbf{M}_r = -\lambda_r \boldsymbol{\omega}_t, \quad \mathbf{M}_p = -\lambda_p \boldsymbol{\omega}_n, \\ \lambda = f\frac{N}{v_Q} > 0, \quad \lambda_r = s\frac{N}{\omega_t} > 0, \quad \lambda_p = a\frac{N}{\omega_n} > 0. \end{gathered} \tag{17.1.17'}$$

If one or two of the types of motion mentioned above do not take place, then the corresponding relations of equality (17.1.17') are replaced by the inequalities (17.1.17), which put in evidence the respective friction phenomena. E.g., if $\mathbf{v}_Q \neq \mathbf{0}$, $\boldsymbol{\omega}_t = \boldsymbol{\omega}_n = \mathbf{0}$, then we have a (pure) sliding without rolling and pivoting (the surfaces S and S' are *perfectly smooth*), while if $\mathbf{v}_Q = \mathbf{0}$, $\boldsymbol{\omega}_t \neq \mathbf{0}$, $\boldsymbol{\omega}_n = \mathbf{0}$, then we have a (pure) rolling without sliding and pivoting (the surfaces S and S' are *perfectly rough*).

We have assumed, in the above considerations, that the friction coefficients *f*, s and a are the same both in the static (relations (17.1.17)) and in the dynamic case (relations (17.1.17')).

If one has a contact at several points between the surfaces S and S', then at one of these points (let be the point P) is concentrated the contribution due to the constraint forces et each point. As well, if between these surfaces takes place a contact along an arc of curve C, then one considers the constraint corresponding to an element $\mathrm{d}s$ of the curve, the constraint forces being in direct proportion to this element; the contribution of all these constraints is obtained calculating the corresponding torsor by integration along the arc of curve C. In general, if the surfaces S and S' have in common a part Σ of the surface, so that $P \in \Sigma$, the constraint forces are in direct proportion to the element of area $\mathrm{d}\Sigma$, while $\mathbf{R}\mathrm{d}\Sigma = \mathbf{N}\mathrm{d}\Sigma + \mathbf{T}\mathrm{d}\Sigma$, $\mathbf{M}_P\mathrm{d}\Sigma = \mathbf{M}_r\mathrm{d}\Sigma + \mathbf{M}_p\mathrm{d}\Sigma$; the torsor of these forces is $\boldsymbol{\tau}_P\{\mathbf{R}_\Sigma\} = \left\{\iint_\Sigma \mathbf{R}\mathrm{d}\Sigma, \iint_\Sigma \mathbf{M}_P\mathrm{d}\Sigma\right\}$, where $\mathbf{R}$ and $\mathbf{M}_P$ are vector functions.

We assume now that also the rigid solid $\mathscr{S}'$ (hence, the surface S' too) is in motion with respect to the inertial frame of reference $\mathscr{R}'$. In this case, the velocity of the point Q, which coincides with $P \equiv P'$ at any moment t, with respect to this frame, is given by $\mathbf{v}_Q = \mathbf{v}_P + \mathbf{w}_P = \mathbf{v}_{P'} + \mathbf{w}_{P'}$ where $\mathbf{w}_P$ and $\mathbf{w}_{P'}$, are the velocities of these points with respect to the rigid solids $\mathscr{S}$ and $\mathscr{S}'$, respectively, and one can write $\mathbf{v}_P - \mathbf{v}_{P'} = -(\mathbf{w}_P - \mathbf{w}_{P'})$; because the second difference is contained in the common tangent plane Π, the difference $\mathbf{v}_P - \mathbf{v}_{P'}$ has the same property. As well, if $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are the instantaneous rotations of the rigid solid $\mathscr{S}$ and $\mathscr{S}'$, respectively, with respect to the frame of reference $\mathscr{R}'$, then $\boldsymbol{\omega} - \boldsymbol{\omega}'$ is the instantaneous rotation of the rigid solid $\mathscr{S}$ relative to the rigid solid $\mathscr{S}'$. The forces of friction exerted by $\mathscr{S}'$ upon $\mathscr{S}$ are expressed also by the torsor $\boldsymbol{\tau}_P\{\mathbf{R}_\Sigma\}$, which is decomposed analogously, being led to the same inequalities (17.1.17). If a relative sliding, rolling and pivoting takes place, then the relations (17.1.17') read

$$\mathbf{T} = -\lambda(\mathbf{v}_P - \mathbf{v}'_P), \ \mathbf{M}_r = -\lambda_r(\boldsymbol{\omega}_t - \boldsymbol{\omega}'_t), \ \mathbf{M}_p = -\lambda_p(\boldsymbol{\omega}_n - \boldsymbol{\omega}'_n), \ \lambda, \lambda_r, \lambda_p > 0, \tag{17.1.18}$$

where we have put in evidence the tangential and the normal components of the angular velocities. Corresponding to the theorem of action and reaction, the rigid solid $\mathscr{S}$ acts upon the rigid solid $\mathscr{S}'$ by forces the torsor of which is $\{\mathbf{R}', \mathbf{M}'_P\}$, so that $\mathbf{R} + \mathbf{R}' = \mathbf{0}$, $\mathbf{M}_P + \mathbf{M}'_P = \mathbf{0}$; as well, the decompositions $\mathbf{R}' = \mathbf{N}' + \mathbf{T}'$, $\mathbf{M}'_P = \mathbf{M}'_p + \mathbf{M}'_r$, $\mathbf{N} + \mathbf{N}' = \mathbf{0}$, $\mathbf{T} + \mathbf{T}' = \mathbf{0}$, $\mathbf{M}_p + \mathbf{M}'_p = \mathbf{0}$, $\mathbf{M}_r + \mathbf{M}'_r = \mathbf{0}$ take place too.

The formula (14.1.37) allows to express the elementary work of the forces which rise on the contact surface Σ of the rigid solids $\mathscr{S}$ and $\mathscr{S}'$ in the form

$$\begin{aligned} \mathrm{d}W' &= (\mathbf{R} \cdot \mathbf{v}_P + \mathbf{M}_P \cdot \boldsymbol{\omega})\mathrm{d}t + (\mathbf{R}' \cdot \mathbf{v}_{P'} + \mathbf{M}_{P'} \cdot \boldsymbol{\omega}')\mathrm{d}t \\ &= [\mathbf{R} \cdot (\mathbf{v}_P - \mathbf{v}_{P'}) + \mathbf{M}_P \cdot (\boldsymbol{\omega} - \boldsymbol{\omega}')]\mathrm{d}t; \end{aligned} \tag{17.1.19}$$

one can notice that this work corresponds to the motion of the rigid solid $\mathscr{S}$ with respect to the rigid solid $\mathscr{S}'$. Decomposing $\mathbf{R}$, $\mathbf{M}_P$ and $\boldsymbol{\omega}$ so as to obtain the normal and the tangential components and observing that $\mathbf{N} \cdot (\mathbf{v}_P - \mathbf{v}_{P'}) = 0$ (the difference $\mathbf{v}_P - \mathbf{v}_{P'}$ is contained in the plane Π), $\mathbf{M}_P \cdot (\boldsymbol{\omega}_t - \boldsymbol{\omega}'_t) = 0$, $\mathbf{M}_r \cdot (\boldsymbol{\omega}_n - \boldsymbol{\omega}'_n) = 0$, it results

$$\begin{aligned} \mathrm{d}W' &= [\mathbf{T} \cdot (\mathbf{v}_P - \mathbf{v}_{P'}) + \mathbf{M}_r \cdot (\boldsymbol{\omega}_t - \boldsymbol{\omega}'_t) + \mathbf{M}_p \cdot (\boldsymbol{\omega}_n - \boldsymbol{\omega}'_n)]\mathrm{d}t \\ &= -\left[\lambda(\mathbf{v}_P - \mathbf{v}_{P'})^2 + \lambda_r(\boldsymbol{\omega}_t - \boldsymbol{\omega}'_t)^2 + \lambda_p(\boldsymbol{\omega}_n - \boldsymbol{\omega}'_n)^2\right]\mathrm{d}t < 0, \end{aligned} \tag{17.1.19'}$$

where we took into account the relations (17.1.18). Hence, the elementary work of the friction forces in case of the contact of two rigid solids is negative. Taking into account the formula (14.1.37), we get the power of these forces in the form

$$P' = \mathbf{R} \cdot (\mathbf{v}_P - \mathbf{v}_{P'}) + \mathbf{M}_P \cdot (\boldsymbol{\omega} - \boldsymbol{\omega}'), \tag{17.1.19''}$$

quantity which – obviously – is negative too.

We mention that, in a first approximation, the effect of the couples of rolling and pivoting can be neglected.

17.1.2.2 Painlevé's Paradox

Painlevé called attention to some contradictions in the study of the motion of a non-homogeneous heavy circular disc, which moves in a vertical plane, laying on a fixed rough horizontal straight line; thus, the motion is a plane-parallel one (see Sect. 14.2.2.7 too). Let be thus a non-homogeneous disc of centre O, radius R and weight $M\mathbf{g}$, the centre of gravity of which is at C, of position vector $\boldsymbol{\rho}$ with respect to a movable frame of reference $\mathscr{R}$ of axes Ox_1x_2 parallel to those of a fixed frame $\mathscr{R}'$ of axes $O'x_1'x_2'$, the horizontal axis $O'x_1'$ being tangent to the disc (Fig. 17.6). At the point of contact P act the normal constraint force $\mathbf{N}$ and the sliding friction force $\mathbf{T}$. The velocity of the point C with respect to the frame $\mathscr{R}'$ is given by $\mathbf{v}_C' = \mathbf{v}_O' + \boldsymbol{\omega} \times \boldsymbol{\rho}$, where the angular velocity $\boldsymbol{\omega}$ corresponds to a positive rotation.

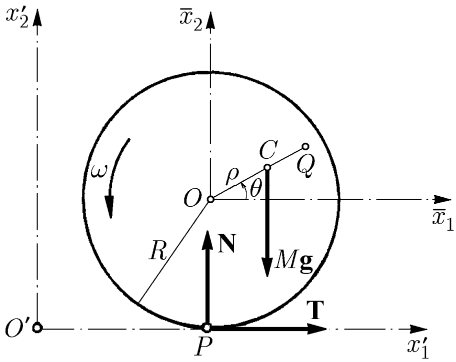

Fig. 17.6 Painlevé's paradox

The theorem of motion of the mass centre gives the equation of motion

$$M\frac{\mathrm{d}}{\mathrm{d}t}(v_O' - \omega\rho_2) = T, \quad M\frac{\mathrm{d}}{\mathrm{d}t}(\omega\rho_1) = N - Mg. \tag{17.1.20}$$

As well, the theorem of moment of momentum with respect to a frame of Koenig leads to

$$I_C\dot{\omega} = (R + \rho_2)T - \rho_1 N, \tag{17.1.20'}$$

where $I_C = Mi_C^2$ is the central moment of inertia with respect to an axis parallel to the $O'x_3'$-axis, while i_C is the corresponding radius of the gyration.

Eliminating $\dot{\omega}$ between the second equation (17.1.20) and (17.1.20'), we get (we mention that $\dot{\rho}_1 = \mathrm{d}(\rho\cos\theta)/\mathrm{d}t = -\rho\sin\theta\dot{\theta} = -\omega\rho_2$)

$$-\rho_2\omega^2 + \frac{\rho_1}{I_C}[(R + \rho_2)T - \rho_1 N] - \frac{N}{M} + g = 0,$$

a relation which must take place at the initial moment too. We assume that at this moment $\rho_1 = \rho$, $\rho_2 = 0$, the centre C being on the Ox_1-axis; as well, we take $\omega_0 = 0$ and we consider that $v_O' < 0$, so that $T = fN$ (we have $N > 0$). Hence, at the initial moment, the relation

$$\frac{N}{M}\left[1 - \frac{\rho}{i_C^2}(fR - \rho)\right] = g \tag{17.1.21}$$

must take place.

Assuming that the non-homogeneity is obtained by attaching to a homogeneous disc of radius R and mass M_0 a second homogeneous disc of radius r and mass M_Q at the point Q, situated at the distance a from the centre O, the relation of static moments gives $M_Q(a-\rho) = M_O\rho$; with the aid of the non-dimensional ratio $\varepsilon = M_Q / M_0$ it results $\rho = [\varepsilon/(1+\varepsilon)]a$. According to the Huygens – Steiner theorem (formula (3.1.113)), we may write $Mi_C^2 = M_0\left(R^2/2+\rho^2\right) + M_Q\left[r^2/2+(a-\rho)^2\right]$, wherefrom

$$i_C^2 = \frac{1}{1+\varepsilon}\left\{\frac{1}{2}\left(R^2+r^2\right)+\frac{\varepsilon}{1+\varepsilon}a^2\right\}.$$

We obtain thus the ratio

$$\eta = \frac{\rho}{i_C^2}(fR-\rho) = 2\varepsilon\frac{(1+\varepsilon)f(a/R)-\varepsilon(a/R)^2}{(1+\varepsilon)\left[1+(r/R)^2\right]+2\varepsilon(a/R)^2}. \qquad (17.1.21')$$

We observe that one can find a technical solution so that the subunitary non-dimensional ratio r/R be sufficiently small, the subunitary non-dimensional ratio a/R be close to $1/2$, the coefficient of sliding friction be sufficiently great, while the non-dimensional ratio ε be superunitary, sufficiently great, so that $\eta > 1$. In this case, it results from (17.1.21) that one must have $N < 0$, contradicting thus the Coulombian model of sliding friction.

F. Klein showed later that the motion considered by Painlevé does not satisfy the continuity hypotheses assumed in the Newtonian modelling of mechanics relative to the initial conditions. As a matter of fact, at the initial moment intervenes an impulse which modifies the static laws assumed for the friction, the deterministic aspect of mechanics being thus preserved.

17.1.2.3 Motion of a Rigid Solid Which Slides Frictionless on a Fixed Plane

Let us consider a rigid solid $\mathcal{S}$, bounded by the surface S, which slides without friction on a fixed plane $\mathcal{P}$, remaining during the motion in contact to this one. We assume that the fixed frame of reference $\mathcal{R}'$ has the axes $O'x_1'$ and $O'x_2'$ contained in this plane, the $O'x_3'$-axis being normal to the plane and directed towards the part in which is the rigid solid $\mathcal{S}$; as well, we introduce the movable frame of reference $\mathcal{R}$, having the pole at the mass centre C of the solid and being rigidly linked to it, and the movable frame $\overline{\mathcal{R}}$, with the pole at the same point and the axes parallel to the axes of the frame $\mathcal{R}'$. To specify the position of the rigid solid $\mathcal{S}$, we determine the position of the mass C (three parameters $\rho_1', \rho_2', \rho_3'$) and the three angles ψ, θ and φ of Euler (three parameters), which give the position of the frame $\mathcal{R}$ with respect to the frame $\overline{\mathcal{R}}$. But these six parameters are not independent.

Indeed, let $f(x_1, x_2, x_3) = 0$ be the equation of the surface S with respect to the frame of reference $\mathscr{R}$, while $\mathbf{r}_P$ and $\mathbf{r}$ are the position vectors corresponding to the contact point P and to an arbitrary point Q of the plane $\mathscr{P}$ (Fig. 17.7), respectively; the equation of this plane is written in the form

$$(\mathbf{r}_P - \mathbf{r}) \cdot \operatorname{grad} f = 0. \tag{17.1.22}$$

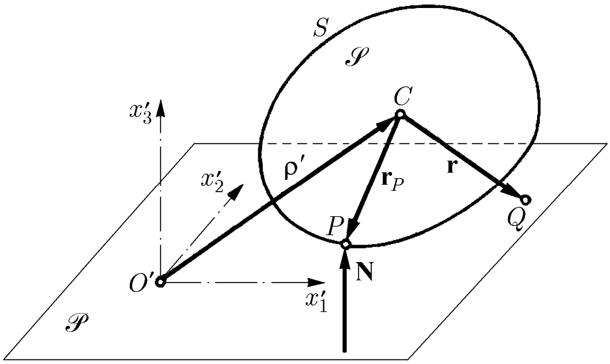

Fig. 17.7 Motion of a rigid solid which slides frictionless on a fixed plane

The rigid solid $\mathscr{S}$ being tangent to the plane $\mathscr{P}$, there must exist a point $(x_1, x_2, x_3) \in S$ at which the tangent plane be normal to $\mathbf{i}_3'(\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)$, and the relations

$$\frac{f_{,1}}{\sin\theta\sin\varphi} = \frac{f_{,2}}{\sin\theta\cos\varphi} = \frac{f_{,3}}{\cos\theta} = \Phi(x_1, x_2, x_3; \psi, \theta, \varphi) \tag{17.1.23}$$

take place. As well, this tangent plane must pass through the pole O', so that we must have $\mathbf{r} \cdot \mathbf{i}_3' = x_1 \sin\theta\sin\varphi + x_2 \sin\theta\cos\varphi + x_3 \cos\theta = -\rho_3'$. The relations (17.1.23) read

$$\frac{\mathbf{r} \cdot \operatorname{grad} f}{\mathbf{r} \cdot \mathbf{i}_3'} = \frac{\mathbf{r}_P \cdot \operatorname{grad} f}{-\rho_3'} = \Phi(x_1, x_2, x_3; \psi, \theta, \varphi). \tag{17.1.23'}$$

Eliminating x_1, x_2, x_3 between the equations (17.1.23), (17.1.23') and taking into account the equation of the surface S, we find a relation of the form

$$\rho_3' = \rho_3'(\psi, \theta, \varphi), \tag{17.1.24}$$

which is – in fact – the constraint relation of the rigid solid $\mathscr{S}$; hence, only five independent parameters are necessary to specify the motion, the rigid solid $\mathscr{S}$ having only five degrees of freedom.

The theorem of motion of the mass centre, in projection on the $O'x'_j$-axes, $j = 1,2,3$, reads

$$M\frac{\mathrm{d}^2\rho'_\alpha}{\mathrm{d}t^2} = \mathbf{R}\cdot\mathbf{i}'_\alpha\,,\quad \alpha = 1,2\,,\quad M\frac{\mathrm{d}^2\rho'_3}{\mathrm{d}t^2} = \mathbf{R}\cdot\mathbf{i}'_3 + N\,, \tag{17.1.25}$$

where $\mathbf{R}$ is the resultant of the given forces which act upon the rigid solid $\mathscr{S}$; the constraint force $\mathbf{N}$ at the contact point P is normal to the plane $\mathscr{P}$, having thus only one component N along the $O'x'_3$-axis, which is given by the third equation (17.1.25). The theorem of moment of momentum written in the frame $\mathscr{R}$ with respect to the movable axis Cx_3 leads to Euler's equation

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = M_{C3}\,, \tag{17.1.26}$$

where M_{C3} is the projection on Cx_3 of the moment with respect to C of the given forces and where we notice that the moment of the constraint force $\mathbf{R}$ has not a non-zero component along this axis; as well, projecting the equation given by the theorem of moment of momentum on the $C\overline{x}_3$-axis (of fixed direction) and noting that $[\mathrm{d}(\mathbf{I}_C\boldsymbol{\omega})/\mathrm{d}t]\cdot\mathbf{i}'_3 = \mathrm{d}[(\mathbf{I}_C\boldsymbol{\omega})\cdot\mathbf{i}'_3]/\mathrm{d}t$, we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_1\omega_1\sin\theta\sin\varphi + I_2\omega_2\sin\theta\cos\varphi + I_3\omega_3\cos\theta) = M_{C3'}\,, \tag{17.1.26'}$$

where we have introduced the moment of the given forces with respect to this axis. Taking into account the results in Sects. 14.1.1.6 and 14.1.1.7, we can write the theorem of kinetic energy in the form

$$\mathrm{d}\left[\frac{1}{2}M\left(\frac{\mathrm{d}\rho'}{\mathrm{d}t}\right)^2 + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)\right] = \mathbf{R}\cdot\mathrm{d}\boldsymbol{\rho}'\,. \tag{17.1.27}$$

We obtain thus five scalar equations, which do not contain the unknown constraint force N, to determine the five independent parameters characterizing the motion of the rigid solid $\mathscr{S}$.

17.1.2.4 Motion of a Heavy Homogeneous Rigid Solid of Rotation Which Slides Frictionless on a Fixed Horizontal Plane

We consider, in particular, the case of a rigid solid $\mathscr{S}$ for which the surface S is of rotation with respect to the Cx_3-axis, this axis being – at the same time – an axis of symmetry for the corresponding central ellipsoid of inertia. These conditions are fulfilled, e.g., by a heavy homogeneous rigid solid of rotation. We represent in Fig. 17.8 a meridian curve $\mathscr{C}$ of a rigid solid $\mathscr{S}$, the point Q being the projection of the mass centre C on the fixed plane $\mathscr{P}$.

If we denote by θ the angle formed by the straight line CQ with the Cx_3-axis, then we can write a relation of the form

$$\rho_3' = f(\theta), \tag{17.1.28}$$

which is determined by the meridian curve $\mathscr{C}$. Choosing the Ox_1-axis in the considered meridian plane, the equation of the tangent PQ is given by

$$x_1 \sin\theta - x_3 \cos\theta = f(\theta). \tag{17.1.29}$$

Because the meridian curve $\mathscr{C}$ is the envelope of this tangent, the co-ordinates of the contact point P are obtained associating the equation

$$x_1 \cos\theta + x_3 \sin\theta = f'(\theta), \tag{17.1.29'}$$

obtained by differentiation with respect to θ; this is just the equation of the straight line which passes through P and is parallel to the $O'x_3'$-axis. The distance $\delta = \overline{PQ}$ is, in this case, given by

$$\delta = |f'(\theta)|. \tag{17.1.29''}$$

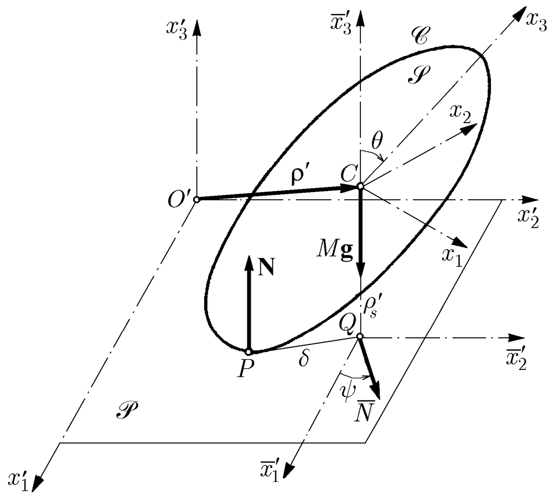

Fig. 17.8 Motion of a heavy homogeneous rigid solid of rotation which slides frictionless on a fixed horizontal plane

Taking into account that $\mathbf{R} = M\mathbf{g}$, corresponding to the own weight of the rigid solid $\mathscr{S}$, the equations of motion (17.1.25) of the mass centre read

$$\frac{\mathrm{d}^2\rho_1'}{\mathrm{d}t^2} = 0\,, \quad \frac{\mathrm{d}^2\rho_2'}{\mathrm{d}t^2} = 0\,, \quad M\frac{\mathrm{d}^2\rho_3'}{\mathrm{d}t^2} = -Mg + N\,. \tag{17.1.30}$$

Hence, the point Q has a rectilinear and uniform motion in the plane $\mathscr{P}$.

Firstly, we assume that – at the initial moment – the velocity of the mass centre is directed towards the vertical line (along the $O'x_3'$-axis) or vanishes; because the horizontal component of this velocity is constant, it results that it will be – further – equal to zero. Hence, the point Q remains fixed, while the centre C oscillates along the vertical of this point.

Applying the theorem of moment of momentum, we notice that $\mathbf{M}_C = \mathbf{0}$. Because $I_1 = I_2 = J$, the equation (17.1.26) leads to $\dot{\omega}_3 = 0$, hence $\omega_3 = \omega_3^0 = \text{const}$. As well, the equation (17.1.26') leads to the first integral

$$J \sin\theta\,(\omega_1 \sin\varphi + \omega_2 \cos\varphi) + I_3 \omega_3^0 \cos\theta = K'_{C3'}, \tag{17.1.31}$$

where $K'_{C3'}$ is the constant component of the moment of momentum along the $C\overline{x}_3$-axis, in the frame of reference $\mathscr{R}'$, with respect to the centre C. From (17.1.27) we get a conservation theorem of mechanical energy in the form (in the considered hypothesis we have $\mathrm{d}\rho_1'/\mathrm{d}t = \mathrm{d}\rho_2'/\mathrm{d}t = 0$)

$$M\left(\frac{\mathrm{d}\rho_3'}{\mathrm{d}t}\right)^2 + J\left(\omega_1^2 + \omega_2^2\right) + I_3\left(\omega_3^0\right)^2 = -2Mg\rho_3' + 2h, \tag{17.1.32}$$

where h is the energy integration constant.

Noting that the relation (17.1.28) allows to write $\mathrm{d}\rho_3'/\mathrm{d}t = f'(\theta)\dot{\theta}$, $f'(\theta) = \mathrm{d}f/\mathrm{d}\theta$, and, using the relations (5.2.35), the first integrals (17.1.31), (17.1.32) take the form (we associate the third relation (5.2.35))

$$\begin{gathered}\dot{\psi}\sin^2\theta = \alpha - a\omega_3^0\cos\theta, \\ \dot{\psi}^2\sin^2\theta + \left[1 + cf'^2(\theta)\right]\dot{\theta}^2 = \beta - bf(\theta), \\ \dot{\psi}\cos\theta + \dot{\varphi} = \omega_3^0,\end{gathered} \tag{17.1.33}$$

where we have introduced the notations

$$\alpha = \frac{K_{C3'}}{J},\ \beta = \frac{2h - I_3\left(\omega_3^0\right)^2}{J},\ a = \frac{I_3}{J} > 0,\ b = \frac{2Mg}{J} = 2gc > 0,\ c = \frac{M}{J} > 0;$$

we notice that α and β are constants which depend on the initial conditions, while a, b and c depend only on the geometry and the mechanical properties of the rigid solid. We obtain thus a system of differential equations which determine Euler's angles $\psi = \psi(t)$, $\theta = \theta(t)$ and $\varphi = \varphi(t)$. Eliminating $\dot{\psi}$ between the first two equations, we get the equation

$$\left[1 + cf'^2(\theta)\right]\sin^2\theta\dot{\theta}^2 = \left[\beta - bf(\theta)\right]\sin^2\theta - \left(\alpha - a\omega_3^0\cos\theta\right)^2, \tag{17.1.34}$$

which determines t as a function of θ by a quadrature; if $f(\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$ and if we take as a new variable $\tan(\theta/2)$, then we obtain a hyperelliptic integral. Taking into account the geometric significance of the function $f(\theta)$, specified by the relation (17.1.28), it results that the integral can take only finite values. We notice that, for $\theta = 0$ and $\theta = \pi$, the function in the second member of the equation (17.1.34) takes negative values; for $\theta = \theta_0$, at the initial moment, the function can take only a positive value, corresponding to a real value of $\dot{\theta}$. It results $\theta_1 < \theta_0 < \theta_2$, where θ_1 and θ_2 are two real zeros of the mentioned function. Thus, we can make for the equation (17.1.34) a study analogous to that in Sects. 15.2.1.1 and 15.2.1.2. In this context, Puiseux showed that one can choose an initial angular velocity ω_3^0 sufficiently great so that $\theta(t)$ remain close to θ_0 at any moment t, the motion of the rigid solid being thus stable. But Thomson showed that supplementary constraints, instead to increase the stability of the rigid solid, could lead to a loss of the stability of its motion (the rigid solid overturns).

We notice that the straight line $Q\bar{N}$, normal to the meridian plane, is parallel to the line of nodes; in this case, if we use the $O\bar{x}_1'$-axis, parallel to the fixed axis $O'x_1'$, the point Q being chosen as pole, then the contact point P will be specified by the polar co-ordinates δ and $\chi = \psi + 3\pi/2$. The first equation (17.1.33) leads thus to the equation

$$\dot{\chi} = \frac{1}{\sin^2\theta}\left(\alpha - a\omega_3^0\cos\theta\right). \tag{17.1.35}$$

Eliminating $\mathrm{d}t$ between the equations (17.1.34) and (17.1.35), we find an equation with separate variables which gives the angle χ by a quadrature as a function of the angle θ; taking into account (17.1.29"), it results a relation which links χ to δ. We obtain thus the curve described by the contact point P in the fixed plane $\mathscr{P}$.

If the mass centre is not projected at a fixed point Q on the plane $\mathscr{P}$, then we observe that this point has a rectilinear and uniform motion; we report the relative motion to a frame of reference $Q\bar{x}_1'\bar{x}_2'\bar{x}_3'$, observing that this frame is inertial too. Hence, the relative motion is governed by the same differential equations as the absolute motion; in the movable frame, the point Q is fixed, so that the problem is reduced to that studied above.

As an application, one can study Gervat's gyroscope (the "equilibrist foot") previously considered in Sect. 16.2.1.1.

17.1.2.5 Frictionless Motion of a Heavy Gyroscope on a Fixed Horizontal Plane

The gyroscope is a rigid solid for which the ellipsoid of inertia relative to a fixed point of it is of rotation. In our case, the fixed point is the contact point P (at a given moment), assuming that the gyroscope is bounded at the vicinity of this point by a smooth surface, e.g., spherical (even if the radius of the respective sphere is very small); if the gyroscope is a solid of rotation, then the PC-axis is a principal axis of inertia, being just the symmetry axis of it. If $\overline{CP} = l$, then the function $f(\theta)$ is given by (Fig. 17.9)

$$f(\theta) = l\cos\theta\,. \tag{17.1.36}$$

As well, we have

$$\delta = l\sin\theta\,. \tag{17.1.36'}$$

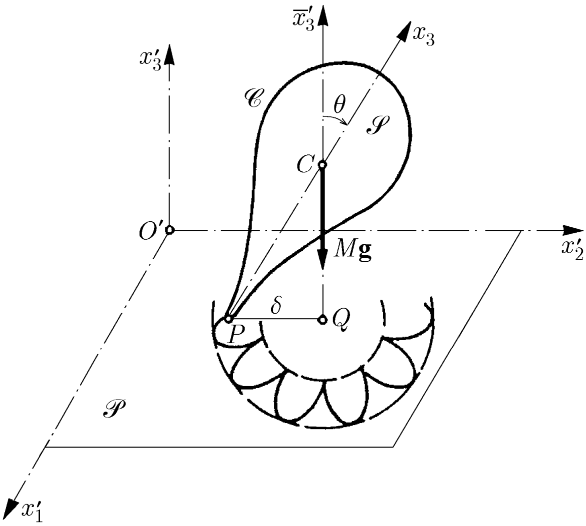

Fig. 17.9 Frictionless motion of a heavy gyroscope on a fixed horizontal plane

The initial conditions (at the moment $t = 0$) read

$$\omega_1(0) = \omega_2(0) = 0\,,$$
$$\omega_3(0) = \omega_3^0 = \omega_0\,,$$
$$\theta(0) = \theta_0\,,\quad \psi(0) = \psi_0\,;$$

we associate to them $\dot\theta(0) = 0$ (we use the second relation (14.1.15)) and the constants of integration are obtained in the form

$$K'_{C3'} = I_3\omega_0\cos\theta_0\,,\quad 2h = I_3(\omega_0)^2 + 2Mgl\cos\theta_0\,,$$

so that

$$\begin{gathered}\dot\psi\sin^2\theta = \frac{I_3}{J}\omega_0(\cos\theta_0 - \cos\theta)\,,\\ \dot\psi^2\sin^2\theta + \left(1 + \frac{M}{J}l^2\sin^2\theta\right)\dot\theta^2 = \frac{2Mgl}{J}(\cos\theta_0 - \cos\theta)\,.\end{gathered} \tag{17.1.37}$$

Finally, the equation (17.1.34) takes the form

$$\dot{\theta}^2 = 2Mgl\frac{\cos\theta_0 - \cos\theta}{J + Ml^2\sin^2\theta}\left(1 - \frac{I_3^2\omega_0^2}{2MglJ}\frac{\cos\theta_0 - \cos\theta}{\sin^2\theta}\right). \tag{17.1.38}$$

From the first equation (17.1.37) it results that $\cos\theta_0 > \cos\theta$, while the relation (17.1.38) shows that one must have $I_3^2\omega_0^2(\cos\theta_0 - \cos\theta)/2MglJ\sin^2\theta < 1$. If, at the initial moment, we impart to the gyroscope a very great rotation ω_0, then we must have $\cos\theta_0 - \cos\theta << 1$; taking $\theta = \theta_0 + \overline{\theta}$, $\overline{\theta} << 1$, it results $\cos\theta_0 - \cos\theta = \cos\theta_0(1 - \cos\overline{\theta}) + \sin\theta_0\sin\overline{\theta} \cong \overline{\theta}\sin\theta_0$ and $\sin\theta \cong \sin\theta_0$. The equation (17.1.38) becomes

$$\dot{\overline{\theta}}^2 = \lambda\overline{\theta}(1 - \mu\overline{\theta}), \tag{17.1.38'}$$

where

$$\lambda = \frac{2Mgl\sin\theta_0}{J + Ml^2\sin^2\theta_0}, \quad \mu = \frac{I_3^2\omega_0^2}{2MglJ\sin\theta_0}, \tag{17.1.38''}$$

with the initial condition $\overline{\theta}(0) = 0$. By integration, using a substitution of the form $\mu\overline{\theta} = \sin^2 z$, we get

$$\overline{\theta} = \frac{1}{\mu}\sin^2\left(\frac{1}{2}\sqrt{\lambda\mu}t\right), \tag{17.1.38'''}$$

whence $\overline{\theta}_{\max} = 1/\mu = 2MglJ\sin\theta_0/I_3^2\omega_0^2$ has a small value. As in Chap. 15, Sect..2.1.2, to can appreciate easier the mode in which the motion of the gyroscope is obtained and to determine easier its position, we consider the motion on the unit sphere of centre C of the piercing point of the movable axis Cx_3 with the very same sphere; one obtains thus, in the motion of nutation, a curve contained in the interior of the spherical zone bounded by the circles $\theta = \theta_0$ and $\theta = \theta_0 + \overline{\theta}_{\max}$, the latter one being the inferior circle. The period of motion is given by $2\pi\cdot 2/\sqrt{\lambda\mu} = 4\pi\sqrt{J(J + Ml^2\sin^2\theta_0)}/I_3\omega_0$, being as much smaller as the initial angular velocity ω_0 is greater. We notice that, in fact, the period of motion is in inverse proportion to ω_0, while the amplitude of the motion (see the formula (17.1.38''')) is in inverse proportion to ω_0^2. From the first equation (17.1.37) we get, analogously,

$$\dot{\psi} = \frac{I_3\omega_0}{J\sin\theta_0}\overline{\theta} = \frac{2Mgl}{I_3\omega_0}\sin^2\left(\frac{1}{2}\sqrt{\lambda\mu}t\right), \tag{17.1.39}$$

wherefrom, by integration,

$$\psi(t) = \frac{Mgl}{I_3\omega_0}\left[t - \frac{1}{\sqrt{\lambda\mu}} \sin\left(\sqrt{\lambda\mu} t\right) \right] + \psi_0 . \tag{17.1.39'}$$

Observing that $\omega_0 > 0$, the meridian plane being rotated in the same sense, it results that the rotation axis of the gyroscope performs a motion of precession with an angular velocity, which varies periodically between the value zero and a maximal value $\dot{\psi}_{\max} = 2Mgl / I_3\omega_0$. The curve described in the motion of nutation on the unit sphere between the circles $\theta = \theta_0$ and $\theta = \theta_0 + \overline{\theta}_{\max}$ is of the nature of the curve in Fig. 15.21c, from the corresponding Lagrange-Poisson motion, having cuspidal points for $\theta = \theta_0$ and being tangent to the circle $\theta = \theta_0 + \overline{\theta}_{\max}$. Indeed, if V is the angle made by this curve with a meridian circle, then we may write (see Sect. 15.2.1.2 too)

$$\tan V = \sin\theta \frac{d\psi}{d\theta} \cong \sin\theta_0 \frac{\dot{\psi}}{\dot{\overline{\theta}}} = \frac{1}{\sin\theta_0}\sqrt{1 + \frac{Ml^2}{J}\sin^2\theta_0}\, \tan\left(\frac{1}{2}\sqrt{\lambda\mu} t\right). \tag{17.1.40}$$

For $\theta = \theta_0$ we have $\overline{\theta} = 0$, while – if we take into account (17.1.38''') – it results $\tan V = 0$, hence cuspidal points; for $\theta \to \theta_0 + \overline{\theta}_{\max}$ one obtains $\tan V \to \infty$, the curve being normal to the meridian circle.

The point P describes an analogous curve in the plane $\mathscr{P}$ (Fig. 17.9).

17.1.2.6 Frictionless Motion of a Homogeneous Rigid Solid of Cylindrical Form on a Fixed Horizontal Plane

We consider, analogously, a homogeneous rigid solid $\mathscr{S}$ of cylindrical form, which lays on a fixed horizontal plane $\mathscr{P}$ along a generatrix $P'P'$. The fixed frame of reference $\mathscr{R}'$ is linked to the plane $\mathscr{P}$, the $O'x_3'$-axis being normal to it, in the part in which is the solid $\mathscr{S}$. The movable frame $\mathscr{R}$ is chosen so that the axes Cx_2, Cx_3 be in the median transverse section, which attains the plane $\mathscr{P}$ at the point P; the Cx_1-axis is taken parallel to the generatrices of the cylinder (Fig. 17.10). The applicate $\overline{CQ} = \rho_3'$ of the centre C will be given by a relation of the form (17.1.28) too. We also notice that the Ox_1-axis is contained in the plane Cx_1x_2 (we use also a frame of reference $\overline{\mathscr{R}}$, the axes of which are parallel to the axes of the frame $\mathscr{R}'$), so that Euler's angle is $\varphi = 0$; the relations (5.2.35) become

$$\omega_1 = \dot{\theta}, \quad \omega_2 = \dot{\psi}\sin\theta, \quad \omega_3 = \dot{\psi}\cos\theta . \tag{17.1.41}$$

The own weight $M\mathbf{g}$ and the constraint forces along the generatrix $P'P'$ are along the direction of the $O'x_3'$-axis. From the theorem of motion of the mass centre, as in the preceding cases, it results that the point Q has a rectilinear and uniform motion in the plane $\mathscr{P}$; analogously, we can reduce the general case to the case in which the point Q is fixed. Observing that the moment of the given and constraint external forces with respect to the $O'x_3'$-axis vanishes, we can write a scalar conservation theorem of moment of momentum in the form $\mathbf{K}'_{O'} \cdot \mathbf{i}'_3 = K'_{O'3'} = \text{const}$; the formulae (14.1.23'), (14.1.24'), with $\boldsymbol{\rho} = \mathbf{0}$ and $\mathbf{v}'_C \parallel \mathbf{i}'_3$, lead to $\mathbf{K}'_{O'} = \mathbf{I}_C \boldsymbol{\omega}$, so that

$$\sin\theta\,(I_{12}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3) + \cos\theta\,(I_{31}\omega_1 + I_{23}\omega_2 + I_{33}\omega_3) = K'_{O'3'},$$

where we took into account the relations (5.2.36) with $\varphi = 0$ and so that the axes chosen for the frame $\mathscr{R}$ are not, in general, principal axes. Introducing the relations (17.1.41) too, we get

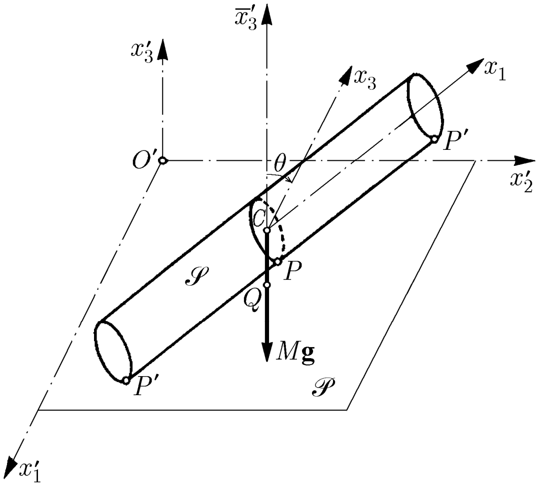

Fig. 17.10 Frictionless motion of a homogeneous rigid solid of cylindrical form on a fixed horizontal plane

$$(I_{12}\sin\theta + I_{31}\cos\theta)\dot{\theta} + \left(I_{22}\sin^2\theta + I_{33}\cos^2\theta + 2I_{23}\sin\theta\cos\theta\right)\dot{\psi} = K'_{O'3'}. \tag{17.1.42}$$

We express the kinetic energy of the rigid solid $\mathscr{S}$ with respect to the inertial frame of reference $\mathscr{R}'$ by means of the formulae (14.1.29'), (14.1.30'), with $\boldsymbol{\rho} = \mathbf{0}$ in the form $2T' = \boldsymbol{\omega} \cdot (\mathbf{I}_C \boldsymbol{\omega}) + M v_C'^2$; if we take into account also the relations (17.1.41), the conservation theorem of mechanical energy leads to the relation

$$\begin{aligned} &\left[I_{11} + Mf'^2(\theta)\right]\dot{\theta}^2 + \left(I_{22}\sin^2\theta + I_{33}\cos^2\theta + 2I_{23}\sin\theta\cos\theta\right)\dot{\psi}^2 \\ &\qquad + 2(I_{12}\sin\theta + I_{31}\cos\theta)\dot{\theta}\dot{\psi} = -2Mgf(\theta) + 2h. \end{aligned} \tag{17.1.43}$$

The differential equations (17.1.42), (17.1.43) allow to obtain Euler's angles $\psi = \psi(t)$ and $\theta = \theta(t)$. Eliminating $\dot{\psi}$ between these equations, we get a differential equation with separate variables, which gives the time t as a function of θ, by a quadrature; returning to the equation (17.1.42), we find ψ as a function of θ by a quadrature too. As in the preceding cases, one can make a qualitative study of the solution. An interesting particular case is that of the heavy homogeneous circular cylinder, case in which the points P and Q coincide, while $f(\theta) = \text{const}$.

17.1.2.7 Slidingless Motion of a Sphere on a Fixed Plane

Let be now a rigid solid sphere for which the repartition of masses is such that the mass centre C is just the centre of the sphere, the corresponding ellipsoid of inertia being a sphere ($I_{11} = I_{22} = I_{33} = J$); for the sake of simplicity, we can assume that the sphere is homogeneous. We suppose that the sphere of radius l moves without sliding on a fixed plane $\mathscr{P}$, taken as $O'x_1'x_2'$-plane, the sphere being on that part of the plane for which $x_3' > 0$; we assume, as well, that the sphere is acted upon by a force **F**, applied at the centre C (which, eventually, includes its own weight too), and by a constraint force **R**, applied at the contact point P (Fig. 17.11). We are in the hypothesis in which, at the point P, there does not exist sliding and the moments of rolling and pivoting friction can be neglected. The mass centre C moves in a fixed plane, parallel to the plane $\mathscr{P}$ and situated at a distance l of it; hence, $\mathbf{v}_C' \cdot \mathbf{i}_3' = 0$. From the imposed conditions, it results $\mathbf{v}_P' = \mathbf{v}_C' + \boldsymbol{\omega} \times \mathbf{r}_P = \mathbf{0}$ ($\mathbf{v}_P'$ is the velocity of the point of the sphere which is in contact with the plane $\mathscr{P}$ at the moment t), where we use a frame of Koenig. From the decomposition

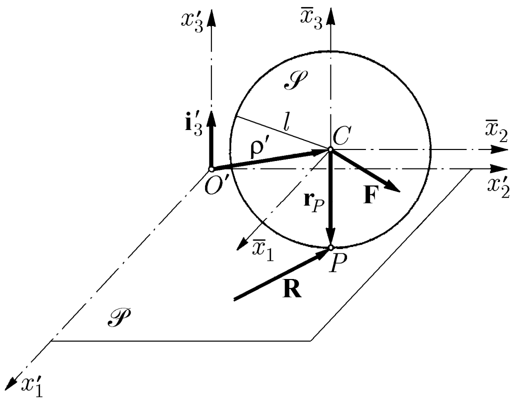

Fig. 17.11 Slidingless motion of a sphere on a fixed plane

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n + \boldsymbol{\omega}_t\,, \quad \boldsymbol{\omega}_t \cdot \mathbf{i}_3' = 0\,, \quad \boldsymbol{\omega}_n \times \mathbf{i}_3' = \mathbf{0}\,,$$

we obtain $\mathbf{v}_C' = l\boldsymbol{\omega}_t \times \mathbf{i}_3'$, whence

$$\boldsymbol{\omega}_t = \frac{1}{l}\mathbf{i}_3' \times \mathbf{v}_C' . \tag{17.1.44}$$

The constraint forces which arise between two rigid solids in contact are of the nature of pressures, so that one must have

$$\mathbf{R} \cdot \mathbf{i}_3' \geq 0 . \tag{17.1.45}$$

The theorem of motion of the mass centre gives

$$M\ddot{\boldsymbol{\rho}}' = \mathbf{F} + \mathbf{R} , \tag{17.1.46}$$

where M is the mass of the sphere. In Koenig's frame $\mathscr{R}$, the moment of momentum with respect to the centre C is $\overline{\mathbf{K}}_C = J\boldsymbol{\omega}$, so that the corresponding theorem of moment of momentum reads

$$J\dot{\boldsymbol{\omega}} = \mathbf{r}_P \times \mathbf{R} . \tag{17.1.47}$$

Using the decomposition of the vector $\boldsymbol{\omega}$ and taking into account (17.1.46), we also can write

$$J\dot{\boldsymbol{\omega}}_n = l\mathbf{i}_3' \times (\mathbf{F} - M\ddot{\boldsymbol{\rho}}') - J\dot{\boldsymbol{\omega}}_t .$$

Because the second member of this relation is a vector parallel to the fixed plane $\mathscr{P}$, it results $\dot{\boldsymbol{\omega}}_n = \mathbf{0}$. Hence, the pivoting angular velocity $\boldsymbol{\omega}_n$, is constant, so that it can be specified by the initial conditions. The given force $\mathbf{F}$ can be decomposed, as well, in the form $\mathbf{F} = \mathbf{F}_t + \mathbf{F}_n$, with $\mathbf{F}_t \cdot \mathbf{i}_3' = 0$, $\mathbf{F}_n \times \mathbf{i}_3' = \mathbf{0}$, so that we get $J\dot{\boldsymbol{\omega}}_t + Ml\mathbf{i}_3' \times \ddot{\boldsymbol{\rho}}' = l\mathbf{i}_3' \times \mathbf{F}_t$, . Taking into account (17.1.44), we obtain, finally, the equation of motion of the mass centre in the form

$$\mathbf{i}_3' \times \left(\frac{J}{l} + Ml \right) \ddot{\boldsymbol{\rho}}' = l\mathbf{i}_3' \times \mathbf{F}_t .$$

Because $\rho_3' = \text{const}$, hence $\ddot{\rho}_3' = 0$, it results that the centre of mass C moves as a particle of reduced mass $M_0 = M + J/l^2$, acted upon only by the given force $\mathbf{F}_t$, hence corresponding to the equation

$$M_0\ddot{\boldsymbol{\rho}}' = \mathbf{F}_t . \tag{17.1.48}$$

The angular velocity vector is then given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n + \frac{1}{l}\mathbf{i}_3' \times \dot{\boldsymbol{\rho}}' , \quad \boldsymbol{\omega}_n = \overrightarrow{\text{const}} . \tag{17.1.49}$$

The constraint force

$$\mathbf{R} = -\frac{1}{M_0}\left(M\mathbf{F}_n + \frac{J}{l^2}\mathbf{F}\right) \tag{17.1.50}$$

is given by (17.1.46), (17.1.48). The condition (17.1.45) leads thus to the condition

$$\mathbf{F}_n \cdot \mathbf{i}_3' \leq 0\,, \tag{17.1.45'}$$

which must be fulfilled by the given force $\mathbf{F}$; the mechanical interpretation of this condition is obvious. The conditions which ensure that the motion is without sliding must be also fulfilled.

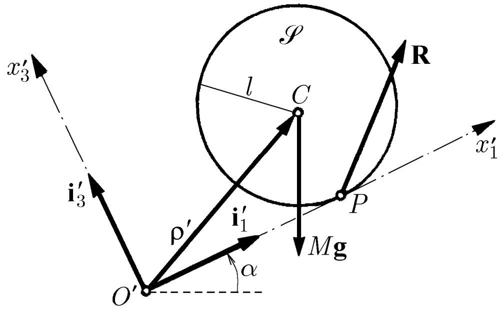

Fig. 17.12 Slidingless motion of a homogeneous sphere of weight $M\mathbf{g}$ on a plane inclined with the angle α

In particular, let us consider a homogeneous sphere of weight $M\mathbf{g}$, which moves slidingless on a plane inclined with the angle α with respect to the horizontal line. Observing that $J = (2/5)Ml^2$, $M_0 = (7/5)M$, $\mathbf{F}_t = -Mg\sin\alpha\mathbf{i}_1'$, where $\mathbf{i}_1'$ is the unit vector of the line of greatest inclination of the plane in an ascendant orientation (Fig. 17.12), we can write the equation (17.1.48) in the form

$$\ddot{\boldsymbol{\rho}}' = -\frac{5}{7}g\sin\alpha\mathbf{i}_1'\,. \tag{17.1.51}$$

Hence, the motion of the centre C is uniformly accelerated, the plane trajectory (in the fixed plane $x_3' = l$) being rectilinear or parabolical, corresponding to the initial conditions. The formula (17.1.50) gives the constraint force in the form

$$\mathbf{R} = Mg\left[\frac{2}{7}\mathbf{i}_1'\sin\alpha + \mathbf{i}_3'\cos\alpha\right]. \tag{17.1.52}$$

Because

$$N = Mg\cos\alpha\,,\quad T = \frac{2}{7}Mg\sin\alpha\,,$$

the motion takes place without sliding if the angle α verifies the condition $\tan\alpha \le (7/2) f = (7/2)\tan\varphi$, where f is the coefficient of sliding friction.

We notice that in the equation (17.1.51) does not intervene the radius l of the sphere $\mathscr{S}$. As small would be this radius, the homogeneous sphere $\mathscr{S}$, which rolls without sliding on an inclined plane, cannot be assimilated to a particle; indeed, in case of a particle the equation of motion is $\ddot{\mathbf{r}}' = -g\sin\alpha\mathbf{i}_1'$.

17.1.2.8 Motion of a Heavy Homogeneous Sphere on a Fixed Horizontal Plane

Let us consider a sphere $\mathscr{S}$, the mass centre C of which coincides with the centre of the sphere, of radius l and weight $M\mathbf{g}$, which moves with friction on the horizontal plan $\mathscr{P}$. As till now, we choose the axes $O'x_1'$ and $O'x_2'$ in the plane $\mathscr{P}$, the $O'x_3'$-axis being directed towards the part in which is the sphere $\mathscr{S}$ (Fig. 17.13). At the contact point P acts a constraint force having the component $\mathbf{N}$ along the normal to the plane $\mathscr{P}$ and the component $\mathbf{T}$ contained in this plane. We use a movable frame of reference $\mathscr{R}$ with the centre at C, rigidly connected to the solid, the axes of which are specified by Euler's angles with respect to a frame $\overline{\mathscr{R}}$ of Koenig. We neglect the moments of rolling and pivoting friction.

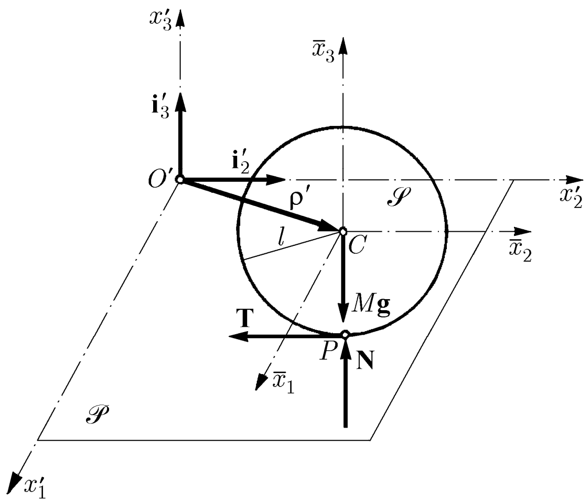

Fig. 17.13 Motion of a heavy homogeneous sphere on a fixed horizontal plane

The equation of motion of the mass centre C is

$$M\ddot{\boldsymbol{\rho}}' = M\mathbf{g} + \mathbf{N} + \mathbf{T}, \tag{17.1.53}$$

the motion of rotation about this centre being specified by

$$J\dot{\boldsymbol{\omega}} = \mathbf{r}_P \times \mathbf{T}. \tag{17.1.54}$$

Noting that $\rho_3' = l = \text{const}$, it results that $M\mathbf{g} + \mathbf{N} = \mathbf{0}$ and $\mathbf{N} = M\mathbf{g}$, wherefrom $T = fMg$ during the sliding; as a matter of fact, during this motion one must have a relation of the form $\mathbf{T} = -\lambda \mathbf{v}_P'$, where $\lambda = \lambda(t)$ is a positive scalar. The equations (17.1.53), (17.1.54) are reduced to

$$M\ddot{\boldsymbol{\rho}}' = \mathbf{T}, \tag{17.1.53'}$$

$$J\dot{\boldsymbol{\omega}}_t + Ml\mathbf{i}_3' \times \ddot{\boldsymbol{\rho}}' = \mathbf{0}, \quad \dot{\boldsymbol{\omega}}_n = \mathbf{0}. \tag{17.1.54'}$$

The velocity of the contact point P of the sphere is given by $\mathbf{v}_P' = \mathbf{v}_C' + \boldsymbol{\omega} \times (-l\mathbf{i}_3') = \mathbf{v}_C' - l\boldsymbol{\omega}_t \times \mathbf{i}_3'$, so that it results

$$\dot{\mathbf{v}}_P' = \dot{\mathbf{v}}_C' - l\dot{\boldsymbol{\omega}}_t \times \mathbf{i}_3' = \ddot{\boldsymbol{\rho}}' + \frac{Ml^2}{J}(\mathbf{i}_3' \times \ddot{\boldsymbol{\rho}}') \times \mathbf{i}_3' = \left(1 + \frac{Ml^2}{J}\right)\ddot{\boldsymbol{\rho}}'$$
$$= \left(\frac{1}{M} + \frac{l^2}{J}\right)\mathbf{T} = -\lambda\left(\frac{1}{M} + \frac{l^2}{J}\right)\mathbf{v}_P'.$$

Assuming that the sphere is homogeneous, one can write

$$\dot{\mathbf{v}}_P' = -\frac{7}{2M}\lambda(t)\mathbf{v}_P'. \tag{17.1.55}$$

Hence, the velocity $\mathbf{v}_P'$ has a fixed direction of unit vector $\mathbf{i}_2'$ (we choose the $O'x_2'$-axis so that to be parallel to this fixed direction, without any loss of generality). As well, the constraint force $\mathbf{T}$ will have a fixed direction; because $T = \text{const}$, it results $\mathbf{T} = -fMg\mathbf{i}_2' = \overrightarrow{\text{const}}$ too. Integrating the equation (17.1.55), we get

$$\mathbf{v}_P'(t) = \left[v_P'^0 - \frac{7}{2}fg(t - t_0)\right]\mathbf{i}_2'. \tag{17.1.55'}$$

We notice now, from the equation (17.1.53'), that the acceleration $\ddot{\boldsymbol{\rho}}'$ of the mass centre is constant in time, its trajectory being rectilinear or parabolical in the plane $\rho_3' = l$, corresponding to the initial conditions; indeed,

$$\boldsymbol{\rho}' = -\frac{fg}{2}(t - t_0)^2\mathbf{i}_2' + \mathbf{v}_C'^0(t - t_0) + \boldsymbol{\rho}'^0. \tag{17.1.56}$$

The angular velocity vector is given by the equations (17.1.54') in the form

$$\boldsymbol{\omega}_t(t) = -\frac{fMgl}{J}(t - t_0)\mathbf{i}_1' + \boldsymbol{\omega}_t^0, \quad \boldsymbol{\omega}_n = \boldsymbol{\omega}_n^0. \tag{17.1.57}$$

The magnitude v'_P of the velocity of the point P with respect to the frame of reference $\mathscr{R}'$ decreases in time till vanishing at the moment $t_1 = t + (2/7)v'^0_P / fg$, so that $\mathbf{v}'_P(t_1) = \mathbf{0}$ and $\mathbf{T}(t_1) = \mathbf{0}$. Hence, the motion of the sphere takes place with sliding, after the laws established above for $t \in [t_0, t_1]$; for $t \geq t_1$ the motion takes place with rolling and pivoting but without sliding, corresponding to the results in the preceding subsection. We notice that we must make $\mathbf{F}_t = \mathbf{0}$ in (17.1.48), while $\dot{\boldsymbol{\rho}}' = \mathbf{v}'^0_C = \overrightarrow{\mathrm{const}}$ in (17.1.49); it results that the motion of the mass centre C becomes rectilinear and uniform, the angular velocity vector $\boldsymbol{\omega}$ being constant in time (in space and with respect to the sphere) .

Let us choose the axes Cx_1, Cx_2, Cx_3 of the non-inertial frame of reference $\mathscr{R}$ so that the Cx_3-axis be after the fixed support of the vector $\boldsymbol{\omega} = \boldsymbol{\omega}_0$; from Euler's equations (5.2.35), it results

$$\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi = 0, \quad -\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi = 0, \quad \dot{\varphi} + \dot{\psi}\cos\theta = \omega_0 .$$

The determinant of the coefficients of the first two equations is $\sin\theta$; if $\theta \neq 0$, then it results $\dot{\theta} = \dot{\psi} = 0$, while the third equation gives $\dot{\varphi} = \omega_0$. We get $\theta = \theta_0$, $\psi = \psi_0$, $\varphi = \omega_0(t - t_0) + \varphi_0$ corresponding to the initial moment.

Coriolis observed that the equations (17.1.54') do not depend on the constraint force **T**, taking place at any moment t (immaterial on the phase of motion); by integration, we can write

$$\boldsymbol{\omega} = \frac{Ml}{J}\mathbf{v}'_C \times \mathbf{i}'_3 + \boldsymbol{\omega}^0, \quad \boldsymbol{\omega}^0 = \overrightarrow{\mathrm{const}}, \tag{17.1.58}$$

as well as

$$\mathbf{v}'_C = \frac{J}{Ml}\mathbf{i}'_3 \times (\boldsymbol{\omega} - \boldsymbol{\omega}^0). \tag{17.1.58'}$$

Choosing a point $Q(0,0,J/Ml)$ over the mass centre C, so that $\mathbf{r}_Q = (J/Ml)\mathbf{i}'_3$, it results $\mathbf{v}'_Q = \mathbf{v}'_C + \boldsymbol{\omega} \times \mathbf{r}_Q = (J/Ml)\boldsymbol{\omega}^0 \times \mathbf{i}'_3 = \overrightarrow{\mathrm{const}}$; the vector $\boldsymbol{\omega}^0$ depends on the initial conditions. In the second phase of the motion we have $\mathbf{v}'_P = \mathbf{0}$, hence

$$\mathbf{v}'_C = l\boldsymbol{\omega} \times \mathbf{i}'_3 . \tag{17.1.59}$$

Replacing in (17.1.58'), we obtain

$$\mathbf{v}'_C = \frac{J}{M_0 l}\boldsymbol{\omega}^0 \times \mathbf{i}'_3, \quad M_0 = M + \frac{J}{l^2}. \tag{17.1.59'}$$

From (17.1.58') and (17.1.59'), one sees that – by sliding – the sphere $\mathscr{S}$ moves away from the pole O', in the first phase of the motion, and then comes near to the very same pole, by rolling and pivoting without sliding.

In the first phase of the motion takes place the holonomic constraint relation

$$\rho_3' = l\,, \tag{17.1.60}$$

the sphere $\mathscr{S}$ having $6 - 1 = 5$ degrees of freedom. In the second phase of the motion one has the relation (17.1.59), hence $v_{C1}' = \mathrm{d}\rho_1' / \mathrm{d}t = l\omega_2'$, $v_{C2}' = \mathrm{d}\rho_2' / \mathrm{d}t = -l\omega_1'$ (after the axes of the frame of reference $\mathscr{R}'$); taking into account (5.2.35'), there results two constraint relations of the form

$$\begin{aligned} &\mathrm{d}\rho_1' - l(\sin\psi\mathrm{d}\theta - \sin\theta\cos\psi\mathrm{d}\varphi) = 0\,, \\ &\mathrm{d}\rho_2' + l(\cos\psi\mathrm{d}\theta + \sin\theta\sin\psi\mathrm{d}\varphi) = 0\,. \end{aligned} \tag{17.1.60'}$$

Assuming that one can write an integrable relation of the form $f(\rho_1', \rho_2', \psi, \theta, \varphi) = 0$, imposing the condition $\mathrm{d}f = 0$ and taking into account (17.1.60'), we get

$$\frac{\partial f}{\partial \theta} + l\left(\sin\psi\frac{\partial f}{\partial \rho_1'}\right) = 0\,,\quad \frac{\partial f}{\partial \varphi} - l\sin\theta\left(\cos\psi\frac{\partial f}{\partial \rho_1'} + \sin\psi\frac{\partial f}{\partial \rho_2'}\right) = 0\,,\quad \frac{\partial f}{\partial \psi} = 0\,.$$

A partial differentiation of the first two relations with respect to ψ, where we took into account the third relation, leads to

$$\cos\psi\frac{\partial f}{\partial \rho_1'} + \sin\psi\frac{\partial f}{\partial \rho_2'} = 0\,,\quad \sin\theta\left(\sin\psi\frac{\partial f}{\partial \rho_1'} - \cos\psi\frac{\partial f}{\partial \rho_2'}\right) = 0\,.$$

It results $\partial f / \partial\rho_1' = \partial f / \partial\rho_2' = 0$ one obtains then $\partial f / \partial\theta = \partial f / \partial\varphi = 0$ too. The function f depends on no one of the five variables; it results that one cannot determine any integrable form, starting from the constraint relations (17.1.60'). Hence, these relations are non-holonomic constraints. One can obtain this result also by using the Theorem 3.2.1 of Frobenius. Thus, the sphere $\mathscr{S}$ remains only with $6 - (1 + 2) = 3$ degrees of freedom. The above results are verified, e.g., in case of the billiard ball.

17.1.2.9 Slidingless Motion of a Heavy Circular Disc on a Fixed Horizontal Plane

Let be a heavy rigid solid $\mathscr{S}$, the centre of gravity C of which coincides with its centre, bounded by a contour $\mathscr{C}$ of radius l, which allows a motion without sliding on a fixed horizontal plane $\mathscr{P}$; we assume that the central ellipsoid of inertia is of rotation about an axis Cx_3 normal to the plane of the contour $\mathscr{C}$ ($I_1 = I_2 = J$). The fixed frame of reference $\mathscr{R}'$ has the axes $O'x_1'$ and $O'x_2'$ contained in the plane $\mathscr{P}$, while the $O'x_3'$-axis is directed along the ascendent vertical; the frame $C\overline{x}_1\overline{x}_2\overline{x}_3$ is a frame of Koenig.

We draw the horizontal line of nodes CN, normal to the plane formed by the axes $C\overline{x}_3$ and Cx_3; the CN'-axis, normal to the line of nodes, is situated along the line of greatest slope of the plane of the contour $\mathscr{C}$. The movable frame of reference $\mathscr{R}$, rigidly connected to the solid $\mathscr{S}$, is specified – with respect to the frame $\overline{\mathscr{R}}$ – by Euler's angles ψ, θ and φ. Upon the rigid solid $\mathscr{S}$ acts the own weight $M\mathbf{g}$, at the centre C, and the constraint force $\mathbf{R}$, at the contact point P; the moments of rolling and pivoting friction are neglected (Fig. 17.14).

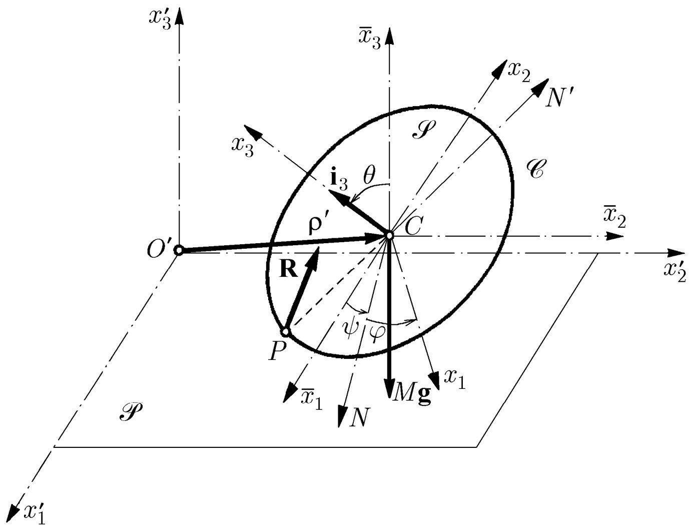

Fig. 17.14 Slidingless motion of a heavy circular disc on a fixed horizontal plane

We notice that the angular velocity vector will be given by $\boldsymbol{\omega} = \boldsymbol{\omega}' + \dot{\varphi}\mathbf{i}_3$, where $\boldsymbol{\omega}'$ corresponds to the instantaneous rotation of the movable frame $\widetilde{\mathscr{R}}$, of axes CN, CN' and Cx_3, with respect to the fixed frame $\mathscr{R}'$; the components of these vectors along the axes of the frame $\mathscr{R}$, will be

$$\tilde{\omega}_1 = \tilde{\omega}'_1 = \dot{\theta}\,, \quad \tilde{\omega}_2 = \tilde{\omega}'_2 = \dot{\psi}\sin\theta\,, \quad \tilde{\omega}'_3 = \dot{\psi}\cos\theta = \tilde{\omega}_2\cot\theta\,, \quad \tilde{\omega}_3 = \dot{\varphi} + \dot{\psi}\cos\theta\,.$$

The equation of motion of the mass centre

$$M(\dot{\mathbf{v}}'_C + \boldsymbol{\omega}' \times \mathbf{v}'_C) = M\mathbf{g} + \mathbf{R} \tag{17.1.61}$$

is written in the form

$$\begin{aligned} &M[\dot{\tilde{v}}'_{C1} + \tilde{\omega}_2(\tilde{v}'_{C3} - \tilde{v}'_{C2}\cot\theta)] = \tilde{R}_1\,, \\ &M(\dot{\tilde{v}}'_{C2} + \tilde{\omega}_2\tilde{v}'_{C1}\cot\theta - \tilde{\omega}_1\tilde{v}'_{C3}) = -Mg\sin\theta + \tilde{R}_2\,, \\ &M(\dot{\tilde{v}}'_{C3} + \tilde{\omega}_1\tilde{v}'_{C2} - \tilde{\omega}_2\tilde{v}'_{C1}) = -Mg\cos\theta + \tilde{R}_3\,, \end{aligned} \tag{17.1.61'}$$

with respect to the frame $\widetilde{\mathscr{R}}$. Because the rigid solid $\mathscr{S}$ has a central ellipsoid of inertia of rotation, the moment of momentum with respect to the centre C, in the frame $\widetilde{\mathscr{R}}$, will have the components $\tilde{K}_{C1} = J\tilde{\omega}_1$, $\tilde{K}_{C2} = J\tilde{\omega}_2$, $\tilde{K}_{C3} = I_3\tilde{\omega}_3$; Euler's equations

$$\mathbf{I}_C\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}' \times (\mathbf{I}_C\boldsymbol{\omega}) = \mathbf{M}_C + \overline{\mathbf{M}}_C \tag{17.1.62}$$

can be written (we have $\mathbf{M}_C = \mathbf{0}$ and $\overline{\mathbf{M}}_C = \mathbf{r}_P \times \mathbf{R}$, the components of the vector $\mathbf{r}_P$ in the frame $\widetilde{\mathscr{R}}$, being 0, –l, 0)

$$\begin{aligned} J\dot{\tilde{\omega}}_1 + (I_3\tilde{\omega}_3 - J\tilde{\omega}_2\cot\theta)\tilde{\omega}_2 &= -l\tilde{R}_3, \\ J\dot{\tilde{\omega}}_2 - (I_3\tilde{\omega}_3 - J\tilde{\omega}_2\cot\theta)\tilde{\omega}_1 &= 0, \\ I_3\dot{\tilde{\omega}}_3 &= l\tilde{R}_1, \end{aligned} \tag{17.1.62'}$$

in the frame $\widetilde{\mathscr{R}}$.

The rigid solid $\mathscr{S}$ is rolling and pivoting without sliding on the plane $\mathscr{P}$ only if the velocity $\mathbf{v}'_P = \mathbf{v}'_C + \boldsymbol{\omega} \times \mathbf{r}_P$ of its contact point P vanishes; in components on the axes of the frame $\widetilde{\mathscr{R}}$, it results

$$\tilde{v}'_{C1} + l\tilde{\omega}_3 = 0, \quad \tilde{v}'_{C2} = 0, \quad \tilde{v}'_{C3} - l\tilde{\omega}_1 = 0. \tag{17.1.63}$$

We get thus nine equation (17.1.61'), (17.1.62'), (17.1.63) for the unknowns $\tilde{v}'_{Cj}$, $\tilde{\omega}_j$ and $\tilde{R}_j$, $j = 1,2,3$.

Eliminating the velocity of the mass centre between the equations (17.1.61') and (17.1.63), we obtain

$$\begin{aligned} Ml(\dot{\tilde{\omega}}_3 - \tilde{\omega}_1\tilde{\omega}_2) &= -\tilde{R}_1, \\ Ml(\tilde{\omega}_1^2 + \tilde{\omega}_2\tilde{\omega}_3\cot\theta) &= Mg\sin\theta - \tilde{R}_2, \\ Ml(\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3) &= -Mg\cos\theta + \tilde{R}_3. \end{aligned} \tag{17.1.64}$$

We can eliminate $\tilde{R}_1$ between the last equation (17.1.62') and the first equation (17.1.64); it results $(I_3 + Ml^2)\dot{\tilde{\omega}}_3 - Ml^2\tilde{\omega}_1\tilde{\omega}_2 = 0$. Associating the second equation (17.1.62') and using the relation $\tilde{\omega}_1 = \dot{\theta}$, we get

$$\begin{aligned} (I_3 + Ml^2)\frac{\mathrm{d}\tilde{\omega}_3}{\mathrm{d}\theta} - Ml^2\tilde{\omega}_2 &= 0, \\ J\frac{\mathrm{d}\tilde{\omega}_2}{\mathrm{d}\theta} - (I_3\tilde{\omega}_3 - J\tilde{\omega}_2\cot\theta) &= 0. \end{aligned} \tag{17.1.65}$$

This system of differential equations determines the functions $\tilde{\omega}_2 = \tilde{\omega}_2(\theta)$, $\tilde{\omega}_3 = \tilde{\omega}_3(\theta)$. The first equation (17.1.65) gives

$$\tilde{\omega}_2 = \left(1 + \frac{I_3}{Ml^2}\right)\frac{\mathrm{d}\tilde{\omega}_3}{\mathrm{d}\theta}. \tag{17.1.66}$$

Replacing in the second equation (17.1.65), we find the resolvent equation of the problem in the form

$$\frac{\mathrm{d}^2\tilde{\omega}_3}{\mathrm{d}\theta^2} + \cot\theta\frac{\mathrm{d}\tilde{\omega}_3}{\mathrm{d}\theta} - \frac{Ml^2 I_3}{J(I_3 + Ml^2)}\tilde{\omega}_3 = 0. \tag{17.1.67}$$

The constraint at P being scleronomous, the elementary work of the constraint forces vanishes; the conservation theorem of the mechanical energy gives

$$M(\tilde{v}_{C1}'^2 + \tilde{v}_{C2}'^2 + \tilde{v}_{C3}'^2) + J(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + I_3\tilde{\omega}_3^2 = -2Mg\rho_3' + 2h. \tag{17.1.68}$$

Taking into account (17.1.63) and the relation $\rho_3' = l\sin\theta$, we can also write

$$(J + Ml^2)\tilde{\omega}_1^2 + J\tilde{\omega}_2^2 + (I_3 + Ml^2)\tilde{\omega}_3^2 = 2Mgl\sin\theta + 2h. \tag{17.1.68'}$$

After determination of the components $\tilde{\omega}_2 = \tilde{\omega}_2(\theta)$ and $\tilde{\omega}_3 = \tilde{\omega}_3(\theta)$, this equation allows the calculation of the component $\tilde{\omega}_1 = \dot{\theta}$. As a matter of fact, this equation is of the form $\dot{\theta}^2 = \Phi(\theta)$; its discussion can be made as in the previous cases. It results, finally, $\tilde{\omega}_j = \tilde{\omega}_j(t)$, $j = 1,2,3$; from (17.1.63), we obtain $\tilde{v}_{Cj}' = \tilde{v}_{Cj}'(t)$, while (17.1.61') gives the components $\tilde{R}_j$, $j = 1,2,3$, of the constraint force.

Making the change of variable $\cos^2\theta = s$, the problem is reduced to *the hypergeometric Euler-Gauss differential equation*

$$s(1-s)\frac{\mathrm{d}^2\tilde{\omega}_3(s)}{\mathrm{d}s^2} + [\gamma - (\alpha+\beta+1)s]\frac{\mathrm{d}\tilde{\omega}_3(s)}{\mathrm{d}s} - \alpha\beta\tilde{\omega}_3(s) = 0, \tag{17.1.69}$$

with

$$\alpha + \beta = \frac{1}{2}, \quad \alpha\beta = \frac{Ml^2 I_3}{J(I_3 + Ml^2)}, \quad \gamma = \frac{1}{2}. \tag{17.1.70}$$

The general solution of this equation depends on two constants of integration λ and μ and is expressed by means of *the hypergeometric function* in the form

$$\tilde{\omega}_3(s) = \lambda F(\alpha,\beta,\gamma;s) + \mu s^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;s). \tag{17.1.69'}$$

In the particular case in which the rigid solid is a homogeneous circular disc of mass M and radius l (e.g., a coin), we have $J = Ml^2/4$ and $I_3 = Ml^2/2$, so that $\alpha\beta = 1/3$, and if the rigid solid is a torus of mass M and of radius (of its axis) l, considered reduced to its axis, then we have $J = Ml^2/2$ and $I_3 = Ml^2$, wherefrom $\alpha\beta = 1/4$. Obviously, a and β have imaginary values and we can write

$$\tilde{\omega}_3(\theta) = \lambda F\left(\alpha, \beta, \frac{1}{2}; \cos^2\theta\right) + \mu \cos\theta F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \frac{3}{2}; \cos^2\theta\right). \tag{17.1.71}$$

By a change of variable of the form $\cos\theta = s$, Korteweg obtains an equation of the form (17.1.69) too, with

$$\alpha + \beta = 1, \quad \alpha\beta = \frac{Ml^2 I_3}{J(I_3 + Ml^2)}, \quad \gamma = 1. \tag{17.1.70'}$$

In the particular case of the torus, $\alpha\beta = 1/4$, so that $\alpha = \beta = 1/2$, obtaining the solution

$$\tilde{\omega}_3(\theta) = \lambda F\left(\frac{1}{2}, \frac{1}{2}, 1; \cos\theta\right), \tag{17.1.71'}$$

which depends only on one arbitrary parameter.

If we take into account

$$\begin{gathered}
\boldsymbol{\omega} = \omega_j' \mathbf{i}_j' = \left(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi\right)\mathbf{i}_1' \\
+ \left(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi\right)\mathbf{i}_2' + \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\mathbf{i}_3', \\
\mathbf{r}_P = l(\cos\theta\sin\psi\mathbf{i}_1' - \cos\theta\cos\psi\mathbf{i}_2' - \sin\theta\mathbf{i}_3'),
\end{gathered}$$

the condition $\mathbf{v}_C' + \boldsymbol{\omega} \times \mathbf{r}_P = \mathbf{0}$ leads to the constraint relations

$$\begin{gathered}
\dot{\rho}_1' + l\left(\dot{\psi}\cos\theta\cos\psi - \dot{\theta}\sin\theta\sin\psi + \dot{\varphi}\cos\psi\right) = 0, \\
\dot{\rho}_2' + l\left(\dot{\psi}\cos\theta\sin\psi + \dot{\theta}\sin\theta\cos\psi + \dot{\varphi}\sin\psi\right) = 0, \\
\dot{\rho}_3' - l\dot{\theta}\cos\theta = 0.
\end{gathered} \tag{17.1.72}$$

The last relations is holonomic, obtaining the obvious relation $\rho_3 = l\sin\theta$, by integration. The first two relations can be written, by simple linear combinations, in the equivalent Pfaff forms

$$\begin{gathered}
\cos\psi \mathrm{d}\rho_1' + \sin\psi \mathrm{d}\rho_2' + l(\cos\theta \mathrm{d}\psi + \mathrm{d}\varphi) = 0, \\
\sin\psi \mathrm{d}\rho_1' - \cos\psi \mathrm{d}\rho_2' - l\sin\theta \mathrm{d}\theta = 0
\end{gathered} \tag{17.1.72'}$$

too, which are not integrable, the corresponding constraints being non-holonomic. The rigid solid $\mathscr{S}$ remains thus with $6 - (1 + 2) = 3$ degrees of freedom.

17.2 Motion with Discontinuities of the Rigid Solids. Collisions

The problem of motion with discontinuities of the discrete mechanical systems has be considered in Chap. 13, §1. In what follows, we will complete these results in case of the rigid solids, studying the general phenomenon of collision of two arbitrary rigid solids as well as some interesting particular cases. We put in evidence also the motion of a rigid solid subjected to the action of a percussive force.

17.2.1 Percussion of Two Rigid Solids

After some general considerations concerning the phenomenon of collision, we present a basic particular case: the centrical or the oblique collision of two spheres. We consider then some technical applications which are of interest, as well as the general case of collision of two rigid solids.

17.2.1.1 General Considerations on the Phenomenon of Collision

The general considerations in Sect. 13.1.1.1 concerning the phenomenon of collision in case of a discrete mechanical system of particles remain valid in case of a discrete mechanical system of rigid solids. *The basic problem* which is put consists in the determination of the velocities of the points of the rigid solids after collision, assuming that the corresponding velocities before this mechanical phenomenon are known. As well, we mark out a *phase of compression* and a *phase of relaxation* (*restitution*), so that the model of rigid solid is no more sufficient. We use, further, the notion of *percussion*, as it has been defined in Chap. 10, Sect.. 1.2.3, starting from the notions of force and impulse of the generalized force.

Using the results in Sect. 13.1.1.3 and Sect. 14.1.2.1, we can put in evidence *the jump relations* corresponding to the discrete mechanical system $\mathscr{S}$ of rigid solids, reported to an inertial frame of reference $\mathscr{R}'$ and subjected to the action of given and constraint, external and internal, percussive and non-percussive forces. By a process of passing to limit in the sense of the theory of distributions, we express the theorems of momentum and of motion of the mass centre in the form

$$(\Delta\mathbf{H})_0 = M(\Delta\mathbf{v}_C)_0 = \mathbf{R} + \overline{\mathbf{R}}, \tag{17.2.1}$$

where $(\Delta\mathbf{H})_0$ and $(\Delta\mathbf{v}_C)_0$ represent the jumps of the momentum of the mechanical system $\mathscr{S}$ and of the velocity of the mass centre C of this system, respectively, at a moment of discontinuity, while by $\mathbf{R}$ and $\overline{\mathbf{R}}$ one has denoted *the resultants of the given* and *constraint external percussions*, respectively, which act upon this system at that moment.

The corresponding jump relation of the moment of momentum will be

$$(\Delta\mathbf{K}_O)_0 = \mathbf{M}_O + \overline{\mathbf{M}}_O, \tag{17.2.2}$$

where one has denoted by $(\Delta\mathbf{K}_O)_0$ the jump of the moment of momentum of the mechanical system $\mathscr{S}$, with respect to the fixed pole O, at a moment of discontinuity, while $\mathbf{M}_O$ and $\overline{\mathbf{M}}_O$ are *the resultant moments of the given resultant percussions* and *of the constraint external percussions*, respectively, which act upon this system, with respect to the very same pole, at that moment.

These results can be expressed synthetically in the form of a theorem of torsor, which maintains its form with respect to a movable pole Q too. As well, this theorem can be stated also with respect to a non-inertial frame of reference, in a continuous motion with respect to an inertial frame; the basic equations of the mathematical model of the collision phenomenon do not need a privileged frame. In the phenomenon of collision of the considered mechanical system $\mathcal{S}$, the jump relations are thus invariant to a change of frame or pole.

The theorems of Carnot and Kelvin in Sect. 13.1.1.5 can be applied also in the case of the mechanical system $\mathcal{S}$ considered above.

If $\mathbf{R} + \overline{\mathbf{R}} = \mathbf{0}$ *in an interval of collision, then it results* $(\Delta \mathbf{H})_0 = \mathbf{0}$ *and* $(\Delta \mathbf{v}_C)_0 = \mathbf{0}$, *hence the momentum of the mechanical system* $\mathcal{S}$ *and the velocity of the mass centre of this system, respectively, are conserved in this interval. As well, if* $\mathbf{M}_O + \overline{\mathbf{M}}_O = \mathbf{0}$ *in an interval of collision, then* $(\Delta \mathbf{K}_O)_0 = \mathbf{0}$, *so that the moment of momentum of the system remains constant in the respective interval.*

Obviously, the above results take place also in the case of a single rigid solid $\mathcal{S}$ subjected to constraints. We mention that the constraints can be of four types: (i) constraints which take place before the collision interval, in this interval or after it; (ii) constraints which take place only in the collision interval or after it; (iii) constraints which take place only before the collision interval or in this interval; (iv) constraints which take place only in the collision interval.

Referring to a non-inertial frame of reference with the pole at the mass centre, the jump relations (17.2.2) becomes

$$\mathbf{I}_C \, (\Delta \boldsymbol{\omega})_0 = \mathbf{M}_C + \overline{\mathbf{M}}_C \,, \tag{17.2.3}$$

where we have put in evidence the jump of the rotation angular velocity vector in the percussion interval. We can write, in components,

$$I_{ij} \left(\Delta \omega_j \right)_0 = M_{Ci} + \overline{M}_{Ci} \,, \quad i = 1,2,3 \,, \tag{17.2.3'}$$

for a rigid solid. If the axes of the considered frame of reference are just the central principal axes of inertia of the rigid solid $\mathcal{S}$, then we have

$$I_1 \, (\Delta \omega_1)_0 = M_{C1} + \overline{M}_{C1} \,, \; I_2 \, (\Delta \omega_2)_0 = M_{C2} + \overline{M}_{C2} \,, \; I_3 \, (\Delta \omega_3)_0 = M_{C3} + \overline{M}_{C3} \,. \tag{17.2.3''}$$

In case of a rigid solid $\mathcal{S}$ with a fixed point O, taken as pole of a non-inertial frame of reference $\mathcal{R}$, we have $\dot{\mathbf{v}}'_C = \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$ with respect to an inertial frame $\mathcal{R}'$; it results that $(\Delta \mathbf{v}'_C)_0 = (\Delta \boldsymbol{\omega})_0 \times \boldsymbol{\rho}$, so that the theorem of motion of the mass centre (17.2.1) takes the form

$$M\,(\Delta\boldsymbol{\omega})_0 \times \boldsymbol{\rho} = \mathbf{R} + \overline{\mathbf{R}}\,. \tag{17.2.4}$$

Analogously, in case of a rigid solid $\mathcal{S}$ with a fixed axis, specified by the fixed points O' and O_1, the equations (14.2.1), (14.2.1') read

$$\begin{aligned} -M\,(\Delta\omega)_0\,\rho_2 &= R_1 + R_1' + R_{11}\,, \\ M\,(\Delta\omega)_0\,\rho_1 &= R_2 + R_2' + R_{12}\,, \\ 0 &= R_3 + R_3' + R_{13}\,, \end{aligned} \tag{17.2.5}$$

$$\begin{aligned} I_{31}\,(\Delta\omega)_0 &= M_{O1} - lR_{12}\,, \\ I_{23}\,(\Delta\omega)_0 &= M_{O2} + lR_{11}\,, \\ I_{33}\,(\Delta\omega)_0 &= M_{O3}\,, \end{aligned} \tag{17.2.5'}$$

where R_j are the components of the given percussions, while R_j', R_{1j}, $j = 1,2,3$, are the components of the constraint percussions at the fixed points O' and O_1, respectively.

17.2.1.2 Centric Collision of Two Spheres

Let be two rigid spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ of masses m_1 and m_2, the centres O_1 and O_2 of which have the velocities $\mathbf{v}_1'$ and $\mathbf{v}_2'$, respectively (we assume that $v_1' > v_2'$, otherwise the spheres can never be in collision), along the line O_1O_2 (Fig. 17.15,a); after collision, the corresponding velocities will be $\mathbf{v}_1''$ and $\mathbf{v}_2''$, respectively, and must be determined (Fig. 17.15b). Because external percussions are not involved, we apply the conservation theorem of momentum (the phenomenon is one-dimensional, so that there intervene only the components of the velocities along the Ox_1-axis, taking the respective fixed pole on the line of the centres)

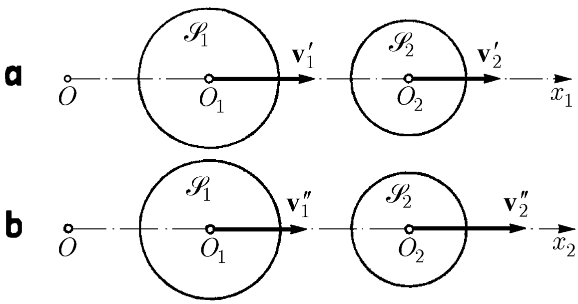

Fig. 17.15 Centric collision of two spheres: (**a**) before and (**b**) after collision

$$m_1 v_1' + m_2 v_2' = m_1 v_1'' + m_2 v_2'' \tag{17.2.6}$$

A second relation necessary to solve the problem (we have two unknowns: $\mathbf{v}_1''$ and $\mathbf{v}_2''$) is obtained by a mathematical modelling corresponding to the mechanical phenomenon. We consider thus a first phase of compression ($t \in [t', t_0)$) which takes place as long as the velocity of the centre O_1 is greater than the velocity of the centre O_2, lasting till the equalization of the two velocities ($t = t_0$). The corresponding percussions of this phase is $P_c = m_1 (v_1' - v^0) = m_2 (v^0 - v_2')$, where $\mathbf{v}^0$ is the common velocity; it results

$$v^0 = \frac{m_1 v_1' + m_2 v_2'}{m_1 + m_2}, \quad P_c = \frac{m_1 m_2 (v_2' - v_1')}{m_1 + m_2}. \tag{17.2.7}$$

In the phase of relaxation ($t \in (t_0, t'']$), the velocity of the centre O_1 continues to decrease till v_1'', and the velocity of the centre O_2 increases till v_2''; the percussion corresponding to this phase is $P_r = m_1 (v^0 - v_1'') = m_2 (v_2'' - v^0)$, so that

$$v^0 = \frac{m_1 v_1'' + m_2 v_2''}{m_1 + m_2}, \quad P_r = \frac{m_1 m_2 (v_2'' - v_1'')}{m_1 + m_2}. \tag{17.2.7'}$$

Introducing also the *coefficient of restitution* (coefficient of elasticity by collision)

$$k = \frac{P_r}{P_c} = \frac{v_2'' - v_1''}{v_1' - v_2'}, \tag{17.2.8}$$

we obtain the second relation, which is added to the relation (17.2.6). Finally, we can write

$$\begin{aligned} v_1'' &= v_1' - (1+k)\frac{m_2}{m_1 + m_2}(v_1' - v_2'), \\ v_2'' &= v_2' + (1+k)\frac{m_1}{m_1 + m_2}(v_1' - v_2') \end{aligned} \tag{17.2.9}$$

and one observe that $v_1'' < v_2''$, the spheres moving away one from the other.

In the case in which $k = 1$ (hence, $P_r = P_c$), the two spheres are compressed in the first phase, returning then to the initial form; there corresponds a phenomenon of *elastic collision*, the velocities after it being given by

$$\begin{aligned} v_1'' &= \frac{1}{m_1 + m_2}[2 m_2 v_2' + (m_1 - m_2) v_1'], \\ v_2'' &= \frac{1}{m_1 + m_2}[2 m_1 v_1' - (m_1 - m_2) v_2']. \end{aligned} \tag{17.2.9'}$$

If $k = 0$ (hence, $P_r = 0$), then the relaxation phase takes no more place; the spheres adhere to each other and remain glued together, so that

$$v_1'' = v_2'' = \frac{1}{m_1 + m_2}(m_1 v_1' + m_2 v_2'). \tag{17.2.9''}$$

The respective phenomenon is called *plastic collision*. Between these two limit cases (for $0 < k < 1$) the spheres return practically to their initial forms, the phenomenon being a *natural collision* (an *elastic-plastic collision*).

Taking into account (17.2.9), *the loss of kinetic energy* ($(\Delta T)^0 = T' - T''$ $= (m_1 v_1'^2 + m_2 v_2'^2)/2 - (m_1 v_1''^2 + m_2 v_2''^2)/2$) is given by

$$(\Delta T)^0 = \frac{1}{2}(1 - k^2)\, m\, (v_1' - v_2')^2, \quad \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{17.2.10}$$

In case of an elastic collision, we have $(\Delta T)^0 = 0$, the phenomenon taking place without loss of kinetic energy; in case of a plastic collision, we find again the formula (13.1.77). The lost kinetic energy is transformed in work of deformation, in caloric or luminous energy etc.

If $m_1 = m_2$ (it is not necessary that the spheres be identical), we get

$$v_1'' = \frac{1}{2}[(1 - k)v_1' + (1 + k)v_2'], \quad v_2'' = \frac{1}{2}[(1 - k)v_2' + (1 + k)v_1']. \tag{17.2.9'''}$$

In case of an elastic collision, it results $v_1'' = v_2'$ and $v_2'' = v_1'$, the spheres transmitting the energy each other. In particular, if $v_2' = 0$, then we have $v_1'' = 0$ too; hence, *if an elastic sphere $\mathscr{S}_1$ strikes, with a velocity v_1', another elastic sphere $\mathscr{S}_2$, having the same mass and being at rest with respect to a given inertial frame of reference, then it transmits to the latter one its velocity v_1' and then stops.*

17.2.1.3 Technical Applications

A particular case of the problem considered at the preceding subsection, interesting for technical applications, is that of the plastic collision of a sphere $\mathscr{S}_1$ of mass m_1 and velocity v_1 with another sphere $\mathscr{S}_2$ of mass m_2, at rest with respect to a given frame of reference. If we make $v_2' = 0$ in (17.2.9"), then we obtain

$$v_1'' = v_2'' = \frac{m_1}{m_1 + m_2} v_1'. \tag{17.2.11}$$

As well, the relation (17.2.10) reads

$$(\Delta T)^0 = \frac{1}{2} m v_1'^2. \tag{17.2.11'}$$

The relative loss of kinetic energy with respect to the kinetic energy before collision ($T = (1/2) m_1 v_1'^2$) will be

$$\frac{(\Delta T)^0}{T} = \frac{m_2}{m_1 + m_2} = \frac{1}{1 + m_1 / m_2}. \tag{17.2.11''}$$

These results of theoretical nature (the considered mathematical modelling) can be successfully used for different technical applications, e.g., in case of beating a nail with a hammer (m_1 is the mass of the hammer, m_2 is the mass of the nail, while v_1' is the velocity with which the hammer beats the nail) or in case of beating a pilot with a rammer (m_1 is the mass of the rammer, m_2 is the mass of the pilot, while v_1' is the velocity with which the rammer beats the pilot). It is important, in both cases, that the relative loss of kinetic energy be as small as possible; one observes, from (17.2.11"), that the ratio m_1 / m_2 must be as great as possible, hence the mass of the hammer (or the rammer) must be much more greater than the mass of the nail (pilot).

In case of a process of working up a piece (bending, adjustment, rivetting etc.) it is useful a greater loss of kinetic energy, which is transformed in work of deformation. The ratio $(\varDelta T)^0 / T$ is as greater as the ratio m_1 / m_2 is smaller, so that the hammer must have a small mass m_1, while the piece to be worked up must have a great mass m_2. Thus, the piece to be worked up is put on a bench (e.g., an anvil), increasing thus the mass m_2. As well, in case of rivetting, the rivet head bears on a special metallic piece (a rivetting knob), increasing much its mass m_2.

17.2.1.4 Oblique Collision of Two Spheres

Let be once more the two spheres $\mathscr{S}_1$ and $\mathscr{S}_2$, of centres O_1 and O_2 and masses m_1 and m_2, the centres of which have the velocities $\mathbf{V}_1'$ and $\mathbf{V}_2'$, respectively (with respect to a given fixed frame of reference); the supports of these velocities are no more directed along the line of the centres and have the components $\mathbf{v}_1'$ and $\mathbf{v}_2'$ ($v_1' > v_2'$) along this line and the components $\mathbf{u}_1'$ and $\mathbf{u}_2'$ normal to it (Fig. 17.16a). After collision, these velocities become $\mathbf{V}_1'' = \mathbf{v}_1'' + \mathbf{u}_1''$, $\mathbf{V}_2'' = \mathbf{v}_2'' + \mathbf{u}_2''$, the notations corresponding to the precedent ones (Fig. 17.16b). If α_1 and α_2 are the angles made by the velocities $\mathbf{V}_1'$ and $\mathbf{V}_2'$, respectively, with the line of centres, there results

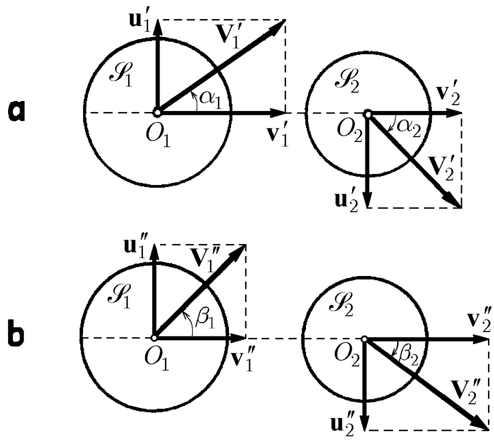

Fig. 17.16 Oblique collision of two spheres: (**a**) before and (**b**) after collision

$$v_1' = V_1' \cos\alpha_1, \quad u_1' = V_1' \sin\alpha_1, \quad v_2' = V_2' \cos\alpha_2, \quad u_2' = V_2' \sin\alpha_2.$$

Assuming that at the contact of the two spheres intervene internal forces, hence internal percussions, only along the direction of the centres (we neglect the friction forces and the corresponding percussions), we can use the results in Sect.. 2.1.2 (especially the formulae (17.2.9) (for v_1' and v_2'), to which we associate the relations $u_1'' = u_1'$, $u_2'' = u_2'$. In this case,

$$V_1'' = \sqrt{v_1''^2 + u_1'^2}\,, \quad V_2'' = \sqrt{v_2''^2 + u_2'^2}\,, \tag{17.2.12}$$

the angles made by the velocities $\mathbf{V}_1''$ and $\mathbf{V}_2''$ with the line of centres being given by

$$\begin{aligned} \tan\beta_1 &= \frac{u_1'}{v_1''} = \frac{(m_1 + m_2)V_1'\sin\alpha_1}{(m_1 - km_2)V_1'\cos\alpha_1 + (1+k)m_2V_2'\cos\alpha_2}\,, \\ \tan\beta_2 &= \frac{u_2'}{v_2''} = \frac{(m_1 + m_2)V_2'\sin\alpha_2}{(1+k)m_1V_1'\cos\alpha_1 + (m_2 - km_1)V_2'\cos\alpha_2}\,, \end{aligned} \tag{17.2.12'}$$

respectively.

As well, the formulae (17.2.9) lead to

$$\begin{aligned} V_1'' &= \frac{(m_1 - km_2)V_1'\cos\alpha_1 + (1+k)m_2V_2'\cos\alpha_2}{(m_1 + m_2)\cos\beta_1}\,, \\ V_2'' &= \frac{(1+k)m_1V_1'\cos\alpha_1 + (m_2 - km_1)V_2'\cos\alpha_2}{(m_1 + m_2)\cos\beta_2}\,; \end{aligned} \tag{17.2.12''}$$

using (17.2.10), the loss of kinetic energy reads

$$(\Delta T)^0 = \frac{1}{2}(1 - k^2)\,m\,(V_1'\cos\alpha_1 - V_2'\cos\alpha_2)^2\,, \quad \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}\,. \tag{17.2.12'''}$$

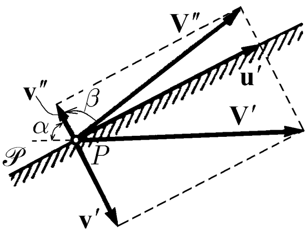

Fig. 17.17 Collision of a sphere with a fixed wall

In particular, let us consider a limit case in which one of the spheres is replaced by a fixed wall $\mathscr{P}$, to which a sphere $\mathscr{S}$ strikes at the point P with a velocity $\mathbf{V}' = \mathbf{v}' + \mathbf{u}'$, which makes an angle α with the normal component $\mathbf{v}'$ (Fig. 17.17).

Neglecting the friction, the component $\mathbf{u}'$ of magnitude $u' = V'\sin\alpha$ is not modified; the component $\mathbf{v}'$ of magnitude $v' = V'\cos\alpha$ changes of direction after collision, becoming $\mathbf{v}''$ of magnitude $v'' = kv'$ (corresponding to the relation (17.2.8), where we make $\mathbf{v}_2' = \mathbf{v}_2'' = \mathbf{0}$, the wall being fixed). We find thus

$$V'' = V'\sqrt{\sin^2\alpha + k^2\cos^2\alpha} = kV'\frac{\cos\alpha}{\cos\beta} \le V', \quad \tan\beta = \frac{1}{k}\tan\alpha, \tag{17.2.13}$$

where β is the angle made by the velocity $\mathbf{V}''$ with the normal to the plane $\mathscr{P}$ (see the analogous relations (13.1.21'), (13.1.21") too; because $k < 1$, it results $\beta \ge \alpha$. Hence, *by the collision of a sphere $\mathscr{S}$ with a fixed wall $\mathscr{P}$, the velocity decreases in magnitude, while the velocity vector moves away from the normal to the wall.* In case of an elastic collision, we have $k = 1$; one obtains $V'' = V'$ and $\beta = \alpha$, finding again *Huygens's laws of reflection of the light photon* which hits obliquely a mirror (the velocity maintains its magnitude, while the angle of reflection is equal to the angle of incidence).

Another mathematical model has been built up by I. Țăposu in 1991; he supposed that, besides (17.2.8), one has

$$k = \frac{u_2'' - u_1''}{u_2' - u_1'} \tag{17.2.8'}$$

too. He obtained thus

$$\begin{aligned} \tan\beta_1 &= \frac{(m_1 + km_2)V_1'\sin\alpha_1 + (1-k)m_2V_2'\sin\alpha_2}{(m_1 - km_2)V_1'\cos\alpha_1 + (1+k)m_2V_2'\cos\alpha_2}, \\ \tan\beta_2 &= \frac{(1-k)m_1V_1'\sin\alpha_1 + (m_2 + km_1)V_2'\sin\alpha_2}{(1+k)m_1V_1'\cos\alpha_1 + (m_2 - km_1)V_2'\cos\alpha_2}, \end{aligned} \tag{17.2.14}$$

results identical with the classical ones for $k = 1$ (elastic collision). The velocities after collision are

$$\begin{aligned} V_1'' &= \frac{(m_1 - km_2)V_1'\cos\alpha_1 + (1+k)m_2V_2'\cos\alpha_2}{(m_1 + m_2)\cos\beta_1} \\ &= \frac{(m_1 + km_2)V_1'\sin\alpha_1 + (1-k)m_2V_2'\sin\alpha_2}{(m_1 + m_2)\sin\beta_1}, \\ V_2'' &= \frac{(1+k)m_1V_1'\cos\alpha_1 + (m_2 - km_1)V_2'\cos\alpha_2}{(m_1 + m_2)\cos\beta_2} \\ &= \frac{(1-k)m_1V_1'\sin\alpha_1 + (m_2 + km_1)V_2'\sin\alpha_2}{(m_1 + m_2)\sin\beta_2}, \end{aligned} \tag{17.2.14'}$$

their components being given by $v_1'' = V_1''\cos\beta_1$, $v_2'' = V_2''\cos\beta_2$, $u_1'' = V_1''\sin\beta_1$, $u_2'' = V_2''\sin\beta_2$. The loss of kinetic energy becomes

$$(\Delta T)^0 = \frac{1}{2}(1-k)^2\, m\left[v_1'^2 + v_2'^2 - 2v_1'v_2'\cos(\alpha_1 - \alpha_2)\right], \tag{17.2.14"}$$

vanishing in the elastic case ($k = 1$), as in the classical model.

One can show that the components of the velocities normal to the line of centres remain constant, as in the classical case, if $u_1' = u_2'$, that is if the velocities of the two spheres have the same inclination on the above mentioned line.

17.2.1.5 Collision of a Sphere with a Rigid Solid in Rotation About a Fixed Axis

Let be a rigid solid $\mathscr{S}$, which is in rotation about a fixed axis, having a moment of inertia I with respect to it; we suppose that this solid is hit by a sphere $\overline{\mathscr{S}}$ of mass m, in a plane perpendicular to the axis (which contains the pole O of the axis and the centre $\bar{O}$ of the sphere), along the common normal at the point P (Fig. 17.18). We denote by $\mathbf{v}'$ and $\mathbf{v}''$ the velocities of the point P before and after collision, respectively, with respect to a fixed frame of reference with the pole at O; the magnitudes of these velocities, which make the angle φ with the normal at the point P, will be $v' = \omega' r$, $v'' = \omega'' r$, with $r = \overline{OP}$, where $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ are the angular velocity vectors, before and after collision, respectively.

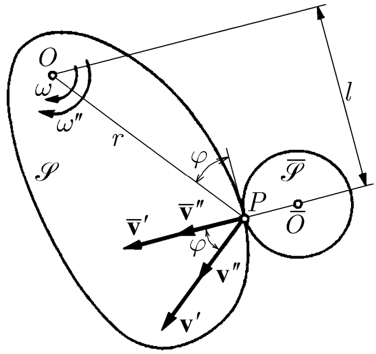

Fig. 17.18 Collision of a sphere with a rigid solid in rotation about a fixed axis

In case of the mechanical system formed by $\mathscr{S}$ and $\overline{\mathscr{S}}$ appear only the external constraint percussions at the points of the rotation axis; the moment of the percussions with respect to this axis vanishes, so that we can write a conservative theorem of the moment of momentum in the form

$$I\omega' + m\bar{v}'l = I\omega'' + m\bar{v}''l\,, \tag{17.2.15}$$

where l is the distance from the point O to the common normal at the point P. We notice that

$$v'\cos\varphi = \omega' r\cos\varphi = \omega' l\,, \quad v''\cos\varphi = \omega'' r\cos\varphi = \omega'' l$$

are the components of the velocities $\mathbf{v}'$ and $\mathbf{v}''$, respectively, along this normal. Taking into account the study made in Sect. 17.2.1.2, we introduce the coefficient of restitution (to fix the ideas, we assume that $\overline{v}' > v' \cos\varphi$; otherwise, both the signs of the denominator and of the numerator change)

$$k = \frac{v'' \cos\varphi - \overline{v}''}{\overline{v}' - v' \cos\varphi} = \frac{\omega'' l - \overline{v}''}{\overline{v}' - \omega' l}. \tag{17.2.16}$$

The relations (17.2.15), (17.2.16) lead to

$$\overline{v}'' = \overline{v}' - (1+k)\frac{I}{I+ml^2}(\overline{v}' - \omega' l), \quad \omega'' = \omega' + (1+k)\frac{ml}{I+ml^2}(\overline{v}' - \omega' l). \tag{17.2.17}$$

As well, the loss of the kinetic energy will be given by

$$(\varDelta T)^0 = \frac{1}{2}(1-k^2)\frac{Im}{I+ml^2}(\overline{v}' - \omega' l)^2. \tag{17.2.18}$$

The elastic collision ($k = 1$) takes place without loss of kinetic energy; in exchange, the natural collision ($0 < k < 1$) takes place with loss of kinetic energy, which is maximal in case of a plastic collision ($k = 0$), being given by

$$(\varDelta T)^0 = \frac{Im}{2(I+ml^2)}(\overline{v}' - \omega' l)^2. \tag{17.2.18'}$$

17.2.1.6 Collision of Two Arbitrary Rigid Solids

Let us consider now two rigid solids $\mathscr{S}_1$ and $\mathscr{S}_2$ of masses M_1 and M_2 and mass centres C_1 and C_2, respectively, which are in collision at the moment t_0, the point of impact being P (Fig. 17.19). We denote by $\mathbf{v}'_{C_1}$ and $\mathbf{v}'_{C_2}$ and $\mathbf{v}''_{C_1}$ and $\mathbf{v}''_{C_2}$, the velocities of the mass centres before and after collision, respectively, with respect to a given fixed frame of reference $\mathscr{R}'$; in this case, the theorem of motion of the mass centre (the formula (13.1.24")), applied to each rigid solid, gives

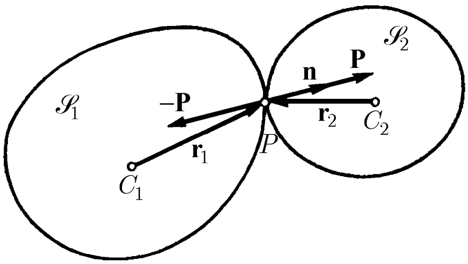

Fig. 17.19 Collision of two arbitrary rigid solids

$$M_1(\mathbf{v}''_{C_1} - \mathbf{v}'_{C_1}) = -\mathbf{P}, \quad M_2(\mathbf{v}''_{C_2} - \mathbf{v}'_{C_2}) = \mathbf{P}, \tag{17.2.19}$$

where $\mathbf{P}$ is the percussion, applied at the point P, by which the rigid solid $\mathcal{S}_1$ acts upon the rigid solid $\mathcal{S}_2$. As well, if $\mathbf{K}'_{C_1}$, $\mathbf{K}'_{C_2}$ and $\mathbf{K}''_{C_1}$, $\mathbf{K}''_{C_2}$, are the angular momenta of the two solids with respect to the corresponding mass centres, in the movable frames $\mathcal{R}_1$ and $\mathcal{R}_2$, rigidly linked to the respective solids, with the poles at these centres, before and after collision, respectively, then we can write the theorem of moment of momentum (the formula (13.1.25)) with respect to each centre of mass, in the mentioned frames, in the form

$$\mathbf{K}''_{C_1} - \mathbf{K}'_{C_1} = -\mathbf{r}_1 \times \mathbf{P}, \quad \mathbf{K}''_{C_2} - \mathbf{K}'_{C_2} = \mathbf{r}_2 \times \mathbf{P}, \tag{17.2.19'}$$

where $\mathbf{r}_1 = \overrightarrow{C_1P}$, $\mathbf{r}_2 = \overrightarrow{C_2P}$; we obtain thus four vector equations (17.1.25), (17.1.25') (12 scalar equations) for the five vector unknowns $\mathbf{v}''_1$, $\mathbf{v}''_2$, $\mathbf{K}''_{C_1}$, $\mathbf{K}''_{C_2}$ and $\mathbf{P}$ (15 scalar unknowns).

We assume that at the contact point P do not appear constraint percussions, the solids $\mathcal{S}_1$ and $\mathcal{S}_2$ being perfectly smooth, and we can write

$$\mathbf{P} = P\mathbf{n}, \quad P > 0, \tag{17.2.20}$$

where $\mathbf{n}$ is the unit vector of the common normal at the point P, with the sense from the solid $\mathcal{S}_1$ to the solid $\mathcal{S}_2$; thus, the number of the scalar unknowns is reduced to 13. Otherwise, it is necessary to introduce supplementary hypotheses concerning the phenomenon of friction (we can introduce, e.g., a Coulombian sliding friction).

The velocities of the points P_1 and P_2 of the solids $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, which coincide with the point P at the theoretic moment of impact, are expressed, with respect to the fixed frame of reference $\mathcal{R}'$, in the form

$$\mathbf{v}_{P_1} = \mathbf{v}_{C_1} + \boldsymbol{\omega}_1 \times \mathbf{r}_1, \quad \mathbf{v}_{P_2} = \mathbf{v}_{C_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_2, \tag{17.2.21}$$

where $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are the angular velocities of the frames $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, with respect to the frame $\mathcal{R}'$. The relative velocity of the point P_1 with respect to the point P_2 will, obviously, be $\left(\mathbf{v}_{P_1} - \mathbf{v}_{P_2}\right) \cdot \mathbf{n}$, hence a velocity of compression, corresponding to the respective phase (the interval of time $[t', t_0)$); the respective velocity vanishes at the end of the phase and we may write $\left(\mathbf{v}^0_{P_1} - \mathbf{v}^0_{P_2}\right) \cdot \mathbf{n} = 0$, so that the common velocity at the theoretic moment t_0 will be

$$\mathbf{v}^0 = \mathbf{v}^0_{P_1} \cdot \mathbf{n} = \mathbf{v}^0_{P_2} \cdot \mathbf{n}. \tag{17.2.22}$$

As well, the equations (17.2.19), (17.2.19') read

$$M_1\left(\mathbf{v}^0_{C_1} - \mathbf{v}'_{C_1}\right) = -P_c\mathbf{n}, \quad M_2\left(\mathbf{v}^0_{C_2} - \mathbf{v}'_{C_2}\right) = P_c\mathbf{n}, \tag{17.2.23}$$

$$\mathbf{K}^0_{C_1} - \mathbf{K}'_{C_1} = -\mathbf{r}_1 \times (P_c\mathbf{n}), \quad \mathbf{K}^0_{C_2} - \mathbf{K}'_{C_2} = \mathbf{r}_2 \times (P_c\mathbf{n}). \tag{17.2.23'}$$

These four vector equations, together with the scalar equation (17.2.22), where we take into account (17.2.21), allow to determine the velocities $\mathbf{v}_{C_1}^0$, $\mathbf{v}_{C_2}^0$ and the angular momenta $\mathbf{K}_{C_1}^0$, $\mathbf{K}_{C_2}^0$, at the end of the phase of compression; in these relations, we take into account that the angular momenta $\mathbf{K}'_{C_1}$, and $\mathbf{K}'_{C_2}$ are expressed as functions of the rotation angular velocity vectors $\boldsymbol{\omega}'_1$ and $\boldsymbol{\omega}'_2$, respectively, so that the number of the unknown vectors is not great.

The velocity $\mathbf{v}^0$ of the point P at the end of the compression phase (the moment t_0) is given by (13.2.22), the corresponding percussion being

$$P_c = M_1 \left(\mathbf{v}'_{C_1} - \mathbf{v}_{C_1}^0 \right) \cdot \mathbf{n} = M_2 \left(\mathbf{v}'_{C_2} - \mathbf{v}_{C_2}^0 \right) \cdot \mathbf{n} . \tag{17.2.24}$$

By introducing the coefficient of restitution k, the percussion corresponding to the restitution phase will be given by $\mathbf{P}_r = k\mathbf{P}_c$, so that $\mathbf{P} = \mathbf{P}_c + \mathbf{P}_r = (1 + k)\mathbf{P}_c$ or

$$\mathbf{P} = (1 + k) P_{\mathbf{c}} \mathbf{n} , \tag{17.2.25}$$

the percussion $\mathbf{P}$ being entirely determined. Replacing in (17.2.19), (17.2.19'), the problem can be completely solved.

One can show that, in case of an elastic collision ($k = 1$), the kinetic energy is conserved ($(\Delta T)^0 = 0$).

17.2.1.7 Theorems of Extremum

Let be a discrete system $\mathscr{S}$ of particles P_i, of masses m_i, driven by the velocities $\mathbf{v}'_i$ and $\mathbf{v}''_i$ before and after collision, respectively, and acted upon by the given and constraint percussions $\mathbf{P}_i$ and $\mathbf{P}_{Ri}$, $i = 1,2,...,p$, respectively. These particles can be rigid solids too, modelled as particles, for which we assume that the rotation angular velocities have not jumps in the interval of percussion. We can write the theorem of momentum in the form

$$m_i (\mathbf{v}''_i - \mathbf{v}'_i) = \mathbf{P}_i + \mathbf{P}_{Ri} , \quad i = 1,2,...,p , \tag{17.2.26}$$

for each particle; effecting a scalar product of these equations by the arbitrary vectors $\mathbf{w}_i$, $i = 1,2,...,p$, and summing, we get

$$\sum_{i=1}^{p} m_i (\mathbf{v}''_i - \mathbf{v}'_i) \cdot \mathbf{w}_i = \sum_{i=1}^{p} (\mathbf{P}_i + \mathbf{P}_{Ri}) \cdot \mathbf{w}_i . \tag{17.2.26'}$$

We notice that this relation is of the form of the principle of virtual velocities (the formula (13.1.57')).

The mechanical system $\mathscr{S}$ can be subjected to constraints which are maintained during the application of the percussions, can appear suddenly or can disappear in this interval of time (as it has been shown in Sect. 17.2.1.1). If $\mathbf{w}_i$ are velocities which satisfy the constraint relations, then the relation

$$\sum_{i=1}^{p} \mathbf{P}_{Ri} \cdot \mathbf{w}_i = 0 \tag{17.2.27}$$

corresponding to the relation which defines the ideal constraints, takes place.

If we take into account (17.2.27) and assume that

$$\sum_{i=1}^{p} \mathbf{P}_i \cdot \mathbf{v}_i'' = 0 \tag{17.2.28}$$

for $\mathbf{w}_i = \mathbf{v}_i''$, then it results

$$\sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right) \cdot \mathbf{v}_i'' = 0 \,.$$

We can write

$$T' - T'' = \frac{1}{2} \sum_{i=1}^{p} m_i \left(v_i'^2 - v_i''^2 \right) = \frac{1}{2} \sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right)^2 - \sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right) \cdot \mathbf{v}_i'' \,.$$

Introducing the kinetic energy of the lost velocities, we get

$$\left(\varDelta T \right)^0 = T_0 > 0 \tag{17.2.28'}$$

and we can state

Theorem 17.2.1 (*Carnot*, I) *If upon a discrete mechanical system $\mathcal{S}$ does not act any given percussive force, then the sudden apparition of a constraint leads to a loss of kinetic energy.*

This theorem can be put in connection with the Theorem 13.1.5.

If we assume that $\mathbf{w}_i = \mathbf{v}_i'$ and that

$$\sum_{i=1}^{p} \mathbf{P}_i \cdot \mathbf{v}_i' = 0 \,, \tag{17.2.29}$$

then it results

$$\sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right) \cdot \mathbf{v}_i' = 0 \,,$$

where we took into account (17.2.27). We can write

$$T'' - T' = \frac{1}{2} \sum_{i=1}^{p} m_i \left(v_i''^2 - v_i'^2 \right) = \frac{1}{2} \sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right)^2 + \sum_{i=1}^{p} m_i \left(\mathbf{v}_i'' - \mathbf{v}_i' \right) \cdot \mathbf{v}_i' \,,$$

so that

$$(\varDelta T)_0 = T_0 > 0\,, \tag{17.2.29'}$$

and we may state

Theorem 17.2.2 (*Carnot*, II) *An internal "explosion" in a non-deformable discrete mechanical system $\mathcal{S}$ leads to a "tearing" of its rigidity, with an increase of the kinetic energy.*

As a matter of fact, by "explosion" we intend the action of an internal percussive force, which leads to an increase of the distance between the particles, the mechanical system becoming deformable (the distances between two arbitrary particles do no more remain constant in time). This result can be put in connection with the Theorem 13.1.6.

Let be a discrete mechanical system $\mathcal{S}$ at rest with an inertial frame of reference; we assume that one applies percussive forces upon some of the particles P_i, so that these particles have prescribed velocities $\mathbf{v}_i''$. Let us consider, as well, a possible motion, which satisfies the constraints of the system, the considered particles having the velocities $\mathbf{v}_i$. Let us denote $\mathbf{w}_i = \mathbf{v}_i'' - \overline{\mathbf{v}}_i$ and let us suppose that $\overline{\mathbf{v}}_i = \mathbf{v}_i''$; in this case

$$\sum_{i=1}^{p} \mathbf{P}_i \cdot \mathbf{w}_i = 0\,, \tag{17.2.30}$$

because $\mathbf{w}_i = \mathbf{0}$ for the particles mentioned above, while $\mathbf{P}_i = \mathbf{0}$ for the other particles. Observing that the relation (17.2.27) takes, as well, place and assuming that $\mathbf{v}_i' = \mathbf{0}$, the relation (17.2.26') reads

$$\sum_{i=1}^{p} m_i \mathbf{v}_i'' \cdot (\mathbf{v}_i'' - \overline{\mathbf{v}}_i) = 0$$

and we have

$$\overline{T} - T'' = \frac{1}{2}\sum_{i=1}^{p} m_i \left(\overline{v}_i^2 - v_i''^2\right) = \frac{1}{2}\sum_{i=1}^{p} m_i (\mathbf{v}_i'' - \overline{\mathbf{v}}_i)^2 - \sum_{i=1}^{p} m_i \mathbf{v}_i'' \cdot (\mathbf{v}_i'' - \mathbf{v}_i')\,,$$

so that

$$(\Delta \overline{T})^0 = \overline{T}_0 > 0\,,\quad (\Delta \overline{T})^0 = \overline{T} - \overline{T}''\,,\quad \overline{T}_0 = \frac{1}{2}\sum_{i=1}^{p} m_i (\mathbf{v}_i'' - \overline{\mathbf{v}}_i)^2\,; \tag{17.2.30'}$$

we can thus state

Theorem 17.2.3 (*Kelvin*) *If a discrete mechanical system $\mathcal{S}$, at rest with respect to an inertial frame of reference, at the initial moment, is put in motion by percussive forces which act upon some of its particles, so that to these particles are impressed prescribed velocities, then the corresponding kinetic energy is smaller than the kinetic energy of any possible motion which satisfies the constraints of the system, the considered particles having the same prescribed velocities.*

Let $\mathbf{v}'_i$ an $\mathbf{v}''_i$ be the velocities of the particles of a mechanical system $\overline{\mathscr{S}}$, with the general significance previously given (the velocities before and after the application of the percussive forces, respectively, assuming that the system is subjected to certain constraints); in case of the existence of supplementary constraints, consistent with the motion of the system $\mathscr{S}$ before the collision, the application of the same percussive forces leads to the velocities $\overline{\mathbf{v}}_i$ after the percussion interval. For $\mathbf{w}_i = \overline{\mathbf{v}}_i$ takes place a relation of the form (17.2.27) in both situations; indeed, in the second case intervene only supplementary constraint forces. The relation (17.2.26') leads to

$$\sum_{i=1}^{p} m_i \left(\mathbf{v}''_i - \mathbf{v}'_i \right) \cdot \overline{\mathbf{v}}_i = \sum_{i=1}^{p} \mathbf{P}_i \cdot \overline{\mathbf{v}}_i \,, \quad \sum_{i=1}^{p} m_i \left(\overline{\mathbf{v}}_i - \mathbf{v}'_i \right) \cdot \overline{\mathbf{v}}_i = \sum_{i=1}^{p} \mathbf{P}_i \cdot \overline{\mathbf{v}}_i \,;$$

subtracting the two relations one from the other, we obtain

$$\sum_{i=1}^{p} m_i \left(\mathbf{v}''_i - \overline{\mathbf{v}}_i \right) \cdot \overline{\mathbf{v}}_i = 0 \,. \tag{17.2.31}$$

We can write

$$T'' - \overline{T} = \frac{1}{2} \sum_{i=1}^{p} m_i \left(v''^2_i - \overline{v}^2_i \right) = \frac{1}{2} \sum_{i=1}^{p} m_i \left(\mathbf{v}''_i - \overline{\mathbf{v}}_i \right)^2 + \sum_{i=1}^{p} m_i \left(\mathbf{v}''_i - \overline{\mathbf{v}}_i \right) \cdot \overline{\mathbf{v}}_i \,,$$

so that

$$\left(\Delta \overline{T} \right)_0 = \overline{T}_0 > 0 \,, \quad \left(\Delta \overline{T} \right)_0 = - \left(\Delta \overline{T} \right)^0 \,, \tag{17.2.31'}$$

with the notations in (17.2.30'); we state

Theorem 17.2.4 (*Bertrand*) *The kinetic energy corresponding to the application of some percussive forces upon a discrete mechanical system $\mathscr{S}$, in motion with respect to an inertial frame of reference, is greater than the kinetic energy corresponding to the application of the same percussive forces upon the same initial motion, assuming that supplementary constraints, consistent with the mentioned motion, have been introduced.*

17.2.2 Motion of a Rigid Solid Subjected to the Action of a Percussive Force

In what follows we consider the motion of a free or constraint rigid solid (with a fixed axis or with a fixed point) subjected to the action of a percussive force or suddenly fixed; the results thus obtained will be applied to the ballistic pendulum.

17.2.2.1 Motion of a Rigid Solid with a Fixed Axis Subjected to the Action of a Percussive Force. Centre of Percussion

Let us consider a rigid solid $\mathscr{S}$ with a fixed axis Δ (specified by the fixed points O' and O_1, situated at a distance l one of the other). We choose a fixed frame of reference

$\mathscr{R}'$ with the $O'x_3'$-axis along $O'O_1$ and a movable frame $\mathscr{R}$ with the pole O on the fixed axis (in general, O distinct from O') and with the Ox_3-axis along OO_1 too (Fig. 17.20). We obtain thus the jump relations (17.2.5), (17.2.5'), corresponding to the theorems of momentum and moment of momentum. We find thus the jump of the angular velocity

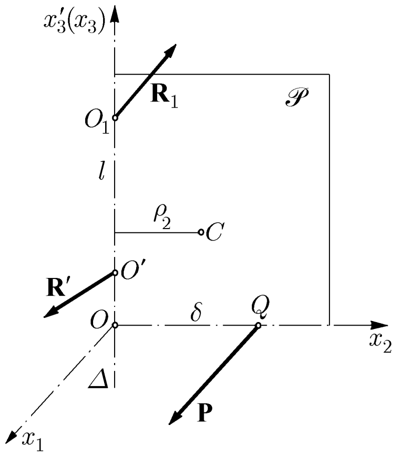

Fig. 17.20 Motion of a rigid solid with a fixed axis subjected to the action of a percussive force

$$(\Delta\omega)_0 = \frac{M_{O3}}{I_{33}}, \tag{17.2.32}$$

as well as the components of the constraint percussions

$$R_{11} = -\frac{1}{l}\left(M_{O2} - \frac{I_{23}}{I_{33}} M_{O3} \right), \quad R_1' = -R_1 + \frac{1}{l}\left(M_{O2} - \frac{I_{23}}{I_{33}} M_{O3} \right) - \frac{M_{O3}}{I_{33}} M\rho_2,$$
$$R_{12} = \frac{1}{l}\left(M_{O1} - \frac{I_{31}}{I_{33}} M_{O3} \right), \quad R_2' = -R_2 - \frac{1}{l}\left(M_{O1} - \frac{I_{31}}{I_{33}} M_{O3} \right) + \frac{M_{O3}}{I_{33}} M\rho_1 .$$
(17.2.32')

The other two components remain non-determinate, because one can know only their sum ($R_3' + R_{13} = R_3$); the rigid solid is statically indeterminate from this point of view.

Let be the case in which upon the rigid solid $\mathscr{S}$ acts only one percussive force, which leads to the percussion $\mathbf{P}$ ($R_i = P_i$, $i = 1,2,3$). The problem is put to find the conditions in which the constraint percussions at O and O_1 vanish (jerks do not appear in the axis of rotation); we must have $R_{1i} = R_i' = 0$, $i = 1,2,3$. In the first case, it results $P_3 = 0$, so that the percussion must be normal to the axis of rotation. We choose the Ox_1-axis parallel to $\mathbf{P}$, so that to have $P_2 = 0$, hence $P_1 = P$. To simplify the calculation, we assume that the percussion **P** is applied at the point Q on the Ox_2-axis; it results $M_{O1} = M_{O2} = 0$ too. From (17.2.32') we find the conditions $I_{31} = I_{23} = 0$, $\rho_1 = 0$. Hence, the percussion **P** must be normal to the Ox_2x_3-plane, which contains the mass centre C. As well, the axis of rotation must be a principal axis of inertia at the point O at which the plane which contains the percussion and is normal to the axis is pierced by it.

We notice that

$$(\Delta\omega)_0 = -\frac{P}{M\rho_2}, \tag{17.2.33}$$

corresponding a loss or an increase of angular velocity, as P_1 and ρ_2 have the same sign or are of opposite signs. Because $M_{O3} = -P\delta$ (in the hypothesis $P_1 > 0$), where $\delta = \overline{OQ}$, Q being on the half-straight line Ox_2 (Fig. 17.20), the relations (17.2.32), (17.2.33) lead to (i_3 is the gyration radius corresponding to the axis of rotation)

$$\delta = \frac{I_{33}}{M\rho_2} = \frac{i_3^2}{\rho_2}, \tag{17.2.34}$$

the position of the point Q, called *centre of percussion*, being thus specified. Using the Huygens-Steiner theorem, the formula (3.1.113') allows to write $i_3^2 = i_C^2 + \rho_2^2$, where i_C is the gyration radius corresponding to an axis which passes through the centre of mass C and is parallel to the rotation axis; in this case,

$$\delta = \rho_2 + l > \rho_2, \quad l = \frac{i_C^2}{\rho_2}, \tag{17.2.34'}$$

the centre of percussion being farther from the axis of rotation than the centre of mass. In fact, δ is *the length of the mathematical pendulum synchronous with the physical pendulum*, formed by the rigid solid which is in rotation about the fixed axis Ox_3'. All the results obtained in Sect. 14.2.1.2 for the physical pendulum can be adapted to the problem considered above. For instance, to a fixed axis $Q\overline{x}_3$ (parallel to the Ox_3-axis), which is the principal axis of inertia for the rigid solid $\mathscr{S}$, corresponds the percussion centre O. We can thus state

Theorem 17.2.5 *If the fixed axis Δ of a rigid solid $\mathscr{S}$, which is in rotation, is a principal axis of inertia for a point O of it, then any percussion normal to the meridian plane $\mathscr{P}$ of the mass centre C (plane determined by the axis Δ and the centre C) at a point of it, situated on a normal at O to the axis Δ, at the distance δ of it and on the same part with the centre C, does not give constraint percussions on the axis Δ.*

Hence, *to any straight line Δ, principal axis of inertia at a point O of a rigid solid, corresponds a centre of percussion Q, so that on the axis Δ, as axis of rotation, do not arise any constraint percussion*. In particular, in case of a plate, to any straight line in the plane of the plate corresponds a centre of percussion.

One can put also the inverse problem: The centre of percussion Q being given, it is asked to determine the position of the axis Δ (of rotation) which is not acted on by the phenomenon of collision. For instance, a blacksmith which strikes the anvil with the hammer (at the point Q) knows, instinctively, how to take the hammer so that not to feel at the palm (at the point Q) a too great shock (the point O is the point at which the axis Δ pierces the plane in which the hammer is rotating).

17.2.2.2 The Ballistic Pendulum

The ballistic pendulum is a device intended to measure the velocity of the projectiles, formed by a metallic cylinder (e.g., of pig iron) filled up with a soft and viscous material (e.g., earth); it is fixed at a point O' and oscillates about a horizontal axis $O'x_3'$, its position being specified by an angle θ made with the descendent vertical $O'x_1'$. The $O'x_1'x_2'$-plane is a plane of symmetry of the mechanical system $\mathcal{S}$, passing through the mass centre C, situated at the distance ρ_1' from the pole O'. At the initial moment, the mechanical system $\mathcal{S}$ is at rest with respect to the frame of reference $\mathcal{R}'$. A projectile of mass m, launched horizontally, at the distance $\delta = \rho_1' + l$ from the fixed point O', in the plane of symmetry, with the velocity $\mathbf{v}$, strikes the pendulum and remains fixed at the point Q' (specified by the angle α made by $O'Q'$ with the $O'x_1'$-axis and by $\overline{O'Q'} = l'$), on the horizontal of the point Q of the $O'x_1'$-axis (Fig. 17.21). The problem is put to determine the velocity $\mathbf{v}$ if the angle $\theta^0 = \theta_{\max}$, corresponding to the oscillations of the physical pendulum, is known.

The percussions between the projectile and the pendulum are internal and the moment at O' of the percussions of the $O'x_3'$-axis vanishes, so that we can apply a conservation theorem of moment of momentum of the mechanical system $\mathcal{S}$ formed by the pendulum and the projectile. The moment of momentum before the interval of collision is $mv\delta$, after collision being equal to $\left(I + ml'^2\right)\omega_0$, where I is the moment of inertia with respect to the $O'x_3'$-axis, while ω_0 is the angular velocity of the mechanical system $\mathcal{S}$; equating these angular momenta, we get

$$\omega_0 = \frac{mv\delta}{I + ml'^2}. \qquad (17.2.35)$$

To the mechanical system $\mathcal{S}$ one can apply the theorem of moment of momentum with respect to the $O'x_3'$-axis in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(I + ml'^2\right)\omega\right] = -Mg\rho_1' \sin\theta - mgl' \sin(\theta + \alpha),$$

M being the mass of the pendulum; multiplying by $\omega = \dot{\theta}$ and integrating, there results the relation

$$\left(I + ml'^2\right)\left(\omega^2 - \omega_0^2\right) + 2g\{M\rho_1'(1 - \cos\theta) + ml'[\cos\alpha - \cos(\theta + \alpha)]\} = 0, \tag{17.2.36}$$

corresponding to the theorem of kinetic energy, where, at the initial moment of this motion (which coincides to the end of the interval of percussion), we imposed the conditions $\theta = 0$ and $\omega = \omega_0$.

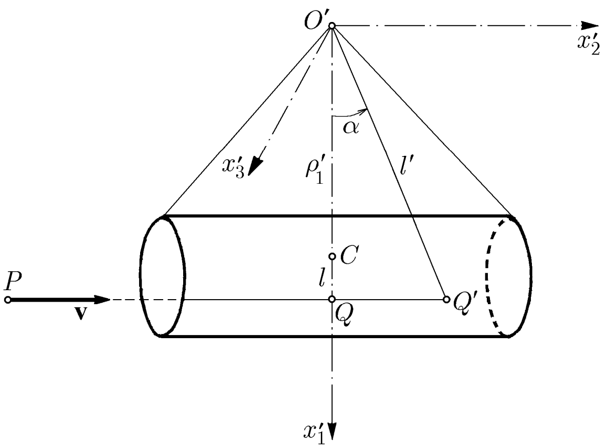

Fig. 17.21 The ballistic pendulum

Making $\theta = \theta^0$ and $\omega = 0$, we obtain

$$\omega_0^2 = \frac{4g}{I + ml'^2}\left[M\rho_1' \sin\frac{\theta^0}{2} + ml' \sin\left(\frac{\theta^0}{2} + \alpha\right)\right]\sin\frac{\theta^0}{2}, \tag{17.2.37}$$

wherefrom

$$v^2 = \frac{4g\left(I + ml'^2\right)}{m^2\delta^2}\left[M\rho_1' \sin\frac{\theta^0}{2} + ml' \sin\left(\frac{\theta^0}{2} + \alpha\right)\right]\sin\frac{\theta^0}{2}. \tag{17.2.38}$$

We notice that $l'\cos\alpha = \delta$.

If the projectile stops at the point Q on the $O'x_1'$-axis, then we have $\alpha = 0$, so that

$$\omega_0 = 2\sqrt{\frac{M\rho_1' + m\delta}{I + m\delta^2}}\sin\frac{\theta^0}{2}, \tag{17.2.37'}$$

the velocity of the projectile being

$$v = \frac{2}{m\delta}\sqrt{g(M\rho_1' + m\delta)(I + m\delta^2)}\sin\frac{\theta^0}{2}. \tag{17.2.38'}$$

The angle θ^0 is measured experimentally on an indicator dial.

We observe that, in the given conditions, the $O'x_3'$-axis is a principal axis of inertia. The pendulum is not acted upon by percussions on its axis of suspension (to have an as small as possible wear of the device in running) if the projectile remains fixed at the centre of percussion, hence at the point *Q*, so that $\rho_1'\delta = i^2 = I/M$. We find thus

$$v = \frac{2}{m}(M\rho_1' + m\delta)\sqrt{\frac{g}{\delta}}\sin\frac{\theta^0}{2}. \tag{17.2.39}$$

17.2.2.3 Motion of a Rigid Solid with a Fixed Point Subjected to the Action of a Percussive Force

Let be a rigid solid with a fixed point $O \equiv O'$, subjected to the action of a percussive force, which leads to the percussion **P** applied at the point *Q*; at the fixed point appears a constraint percussion $\mathbf{P}_R$. We denote by $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}''$ the rotation angular velocities before and after the interval of collision, respectively. The theorem of moment of momentum with respect to the frame of reference $\mathscr{R}$ reads

$$\mathbf{I}_O(\Delta\boldsymbol{\omega})_0 = \mathbf{M}_O, \quad (\Delta\boldsymbol{\omega})_0 = \boldsymbol{\omega}'' - \boldsymbol{\omega}', \tag{17.2.40}$$

the moment of the percussion $\mathbf{P}_R$ vanishing, while $\mathbf{M}_O$ is the moment of the percussion **P** (eventually, the resultant moment of a certain number of given external percussions). Projecting on the principal axes of inertia corresponding to the point *O*, it results

$$(\Delta\omega_1)_0 = \frac{M_{O1}}{I_1}, \quad (\Delta\omega_2)_0 = \frac{M_{O2}}{I_2}, \quad (\Delta\omega_3)_0 = \frac{M_{O3}}{I_3}, \tag{17.2.40'}$$

obtaining thus the jumps of the angular velocity. The equation of motion of the mass centre (17.2.4) allows to express the components of the constraint percussion in the form

$$\begin{aligned} P_{R1} &= -P_1 + M\left(\frac{M_{O2}}{I_2}\rho_3 - \frac{M_{O3}}{I_3}\rho_2\right), \\ P_{R2} &= -P_2 + M\left(\frac{M_{O3}}{I_3}\rho_1 - \frac{M_{O1}}{I_1}\rho_3\right), \\ P_{R3} &= -P_3 + M\left(\frac{M_{O1}}{I_1}\rho_2 - \frac{M_{O2}}{I_2}\rho_1\right). \end{aligned} \tag{17.2.41}$$

Introducing the ellipsoid of inertia $\mathscr{E}$, given by the equation (15.1.63), we can consider the plane which passes through the fixed point O and the moment $\mathbf{M}_O$, of equation

$$M_{O1}x_1 + M_{O2}x_2 + M_{O3}x_3 = 0\,,$$

which can be written in the form

$$I_1(\Delta\omega_1)_0\,x_1 + I_2(\Delta\omega_2)_0\,x_2 + I_3(\Delta\omega_3)_0\,x_3 = 0 \tag{17.2.42}$$

too. The diameter conjugate to this plane, with respect to the ellipsoid $\mathscr{E}$, will be of equations

$$\frac{x_1}{(\Delta\omega_1)_0} = \frac{x_2}{(\Delta\omega_2)_0} = \frac{x_3}{(\Delta\omega_3)_0}\,, \tag{17.2.42'}$$

representing just the support of the vector $(\Delta\boldsymbol{\omega})_0$.

Hence, *if we draw a plane tangent to the ellipsoid of inertia* $\mathscr{E}$, *normal to the direction of the resultant moment* $\mathbf{M}_O$ *of the given percussions* (*taken with respect to the fixed point* O), *the jump of the angular velocity vector* $(\Delta\boldsymbol{\omega})_0$ *will be situated along the position vector of the tangent point* (*conjugate diameter of the considered plane*). If, before the interval of collision, the rigid solid $\mathscr{S}$ is at rest ($\boldsymbol{\omega}' = \mathbf{0}$), then, after the action of the percussive forces, this solid begins to rotate about the support of the position vector of that point of the ellipsoid $\mathscr{E}$ at which the tangent plane is normal to the resultant moment of the percussions, with respect to the fixed point O. The connection between the considered mechanical phenomenon and the geometric aspect given by Poinsot to the Eulerian case of motion of a rigid solid with a fixed point is thus put in evidence.

17.2.2.4 Motion of a Free Rigid Solid Subjected to the Action of a Percussive Force

Let us consider a free rigid solid $\mathscr{S}$, subjected to the action of a percussive force, which leads to the percussion $\mathbf{P}$ applied at the point Q. We denote by $\mathbf{v}'_C$, $\mathbf{v}''_C$ and $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$ the velocities of the mass centre C and the rotation angular velocities of the rigid solid about this point, with respect to an inertial frame of reference $\mathscr{R}'$, before and after the interval of percussion, respectively. The equations (17.2.1), (17.2.3) read

$$M(\Delta\mathbf{v}_C)_0 = \mathbf{R}\,, \tag{17.2.43}$$

$$\mathbf{I}_C(\Delta\boldsymbol{\omega})_0 = \mathbf{M}_C\,; \tag{17.2.43'}$$

we can thus calculate easily the jumps $(\Delta\mathbf{v}_C)_0 = \mathbf{v}''_C - \mathbf{v}'_C$ and $(\Delta\boldsymbol{\omega})_0 = \boldsymbol{\omega}'' - \boldsymbol{\omega}'$. We get

$$(\Delta\mathbf{v}_C)_0 = \frac{1}{M}\mathbf{R}\,, \tag{17.2.44}$$

as well as (we use the central principal axes of inertia of the rigid solid $\mathscr{S}$)

$$(\Delta\omega_1)_0 = \frac{M_{C1}}{I_1}, \quad (\Delta\omega_2)_0 = \frac{M_{C2}}{I_2}, \quad (\Delta\omega_3)_0 = \frac{M_{C3}}{I_3}. \tag{17.2.44'}$$

Obviously, these results can be applied in case of an arbitrary number of external percussions too.

17.2.2.5 Motion of a Rigid Solid Subjected Suddenly to a Fixation

One can study an inverse problem too: a rigid solid $\mathscr{S}$ has an arbitrary motion; if, at a given moment, one of its points is suddenly fixed, then the velocities of all points will have jumps, intervening also the effect of unknown constraint forces. In this case too, one must determine the velocities after fixing.

For instance, let us consider a rigid solid with a fixed point O' and let us assume that, at a given moment t_0, a second point O_1 is fixed, so that the rigid solid $\mathscr{S}$ moves now as a solid with a fixed axis. If $\mathbf{u} = \operatorname{vers}\overrightarrow{O'O_1}$, then we can write $(\Delta\mathbf{K}_{O'})_0 \cdot \mathbf{u} = 0$, $(\Delta\mathbf{K}_{O'})_0 = \mathbf{K}''_{O'} - \mathbf{K}'_{O'}$ (putting in evidence the moment of momentum after and before the collision, respectively), because the moment of the constraint percussion which arises at O_1 vanishes in projection on the fixed axis $O'O_1$. After the fixation of the point O_1, we will have $\mathbf{K}''_{O'} \cdot \mathbf{u} = I\omega''$, where I is the moment of inertia with respect to the new axis of rotation, while ω'' is the corresponding angular velocity. We can write $\mathbf{K}'_{O'} = \mathbf{I}_O\boldsymbol{\omega}'$, where $\boldsymbol{\omega}'$ is the angular velocity before fixation, with respect to an inertial frame of reference $\mathscr{R}'$ with the pole at $O \equiv O'$.

Thus, takes place the relation

$$(\mathbf{I}_O\boldsymbol{\omega}') \cdot \mathbf{u} = I\omega'', \tag{17.2.45}$$

which determines the angular velocity $\boldsymbol{\omega}''$ (the initial velocity for the motion after fixation); in a developed form, after the principal axes of inertia at O, we have

$$I_1\omega'_1u_1 + I_2\omega'_2u_2 + I_3\omega'_3u_3 = I\omega''. \tag{17.2.45'}$$

The motion is studied further using the results obtained in case of the rigid solid with a fixed axis.

17.3 Applications in Dynamics of Engines

After some general results with a theoretical character concerning the dynamics of engines, we deal with some applications concerning their running; we mention, especially, the equilibration of the movable masses and the regulation of the working of engines.

17.3.1 General Results

We call *engine* (*machine*) a set of mechanisms (a mechanical system $\mathcal{S}$, formed by n rigid solids, eventually deformable, $\mathcal{S}_j$, $j = 1,2,...,n$), which perform prescribed motions, with the goal to realize a useful work or to transform a mechanical energy. A machine is formed, in general, by three parts: the driving mechanism, the transmission gear and the mechanism of execution.

In the following, we present firstly some elements of kineto-statics, putting in evidence the forces of inertia which act upon the engines. As well, we introduce some mechanical quantities (*reduced mechanical quantities*), which can characterize the motion of a machine in its totality (mass, moment of inertia, force, power).

17.3.1.1 Forces of Inertia which Act Upon the Machines. Elements of Kineto-Statics

Upon a machine act: (i) *given forces* (which are external and known), as driving forces, forces of technological resistance, forces of weight, forces of inertia, resistance of the medium etc. and (ii) *unknown forces*, e.g., constraint forces in kinematic couples, forces of friction, forces of balancing etc. They can depend on position, velocity and time or only on some of these factors.

At the small devices, with reduced velocities, the forces of inertia can be – in general– neglected, unlike the big devices (with great velocities), to which they can have values of the same order of magnitude as that of the external forces. Let be $\tau_{O'}\{\mathbf{F}\} = \{\mathbf{R}, \mathbf{M}_{O'}\}$ and $\tau_{O'}\{\mathbf{F}^{\mathrm{i}}\} = \{\mathbf{R}^{\mathrm{i}}, \mathbf{M}^{\mathrm{i}}_{O'}\}$ *the torsors of the given and inertia forces*, respectively, with respect to a fixed pole O', for an element of the device (of the mechanical system $\mathcal{S}$). Corresponding to d'Alembert's theorem 11.1.26, we can write

$$\tau_{O'}\{\mathbf{F}^{\mathrm{i}}\} + \tau_{O'}\{\mathbf{F}\} = \mathbf{0}, \quad \mathbf{R}^{\mathrm{i}} + \mathbf{R} = \mathbf{0}, \quad \mathbf{M}^{\mathrm{i}}_{O'} + \mathbf{M}_{O'} = \mathbf{0}. \tag{17.3.1}$$

Taking into account the universal theorems of mechanics, it results

$$\mathbf{R}^{\mathrm{i}} = -\frac{\mathrm{d}\mathbf{H}'}{\mathrm{d}t}, \quad \mathbf{M}^{\mathrm{i}}_{O'} = -\frac{\mathrm{d}\mathbf{K}'_{O'}}{\mathrm{d}t}, \tag{17.3.2}$$

the differentiation taking place with respect to the inertial frame of reference $\mathcal{R}'$. Choosing a non-inertial frame $\mathcal{R}$ having the pole at the mass centre C, the torsor of the forces of inertia at this point is given by (corresponding to the equations (14.1.47), (14.1.48))

$$\mathbf{R}^{\mathrm{i}} = -M\frac{\mathrm{d}\mathbf{v}'_C}{\mathrm{d}t}, \tag{17.3.3}$$

$$\mathbf{M}^{\mathrm{i}}_C = -\mathbf{I}_C\dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times (\mathbf{I}_C\boldsymbol{\omega}), \tag{17.3.4}$$

where M is the mass of the considered element, $\mathbf{v}'_C$ is the velocity of the centre of mass, $\boldsymbol{\omega}$ is the rotation angular velocity vector, while $\mathbf{I}_C$ is the central tensor of inertia. Denoting by $\boldsymbol{\rho}'$ the position vector of the mass centre, we can write

$$R_j^{\mathrm{i}} = -M\frac{\mathrm{d}^2\rho'_j}{\mathrm{d}t^2}, \quad j = 1,2,3. \tag{17.3.3'}$$

Analogously, we can write

$$\begin{aligned} M_{C1}^{\mathrm{i}} &= -[I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3], \\ M_{C2}^{\mathrm{i}} &= -[I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1], \\ M_{C3}^{\mathrm{i}} &= -[I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2], \end{aligned} \tag{17.3.4'}$$

where we used the central principal axes of inertia, $I_1 \geq I_2 \geq I_3$ being the corresponding principal moments of inertia.

In case of a *motion of translation* ($\boldsymbol{\omega} = \mathbf{0}$), it results

$$\mathbf{R}^{\mathrm{i}} = -M\frac{\mathrm{d}\mathbf{v}'_C}{\mathrm{d}t}, \quad \mathbf{M}_C^{\mathrm{i}} = \mathbf{0}, \tag{17.3.5}$$

the system of inertia forces of the element being thus reduced to a unique force of inertia applied at the centre of mass and equal to the product, with a changed sign, of the mass of the element by the acceleration of its centre of mass. We notice thus that, by the starting of the motor, when the element has an accelerated motion, the force of inertia is a *resistant force*; instead, at the stopping of the motor, the force of inertia becomes a *driving force*. In particular, if $\mathbf{a}'_C = \mathrm{d}\mathbf{v}'_C / \mathrm{d}t$, then the element has a rectilinear and uniform motion and is no more acted upon by a force of inertia.

In the hypothesis of a motion of rotation about a central principal axis of inertia (let be $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$), we get

$$\mathbf{R}^{\mathrm{i}} = \mathbf{0}, \quad \mathbf{M}_C^{\mathrm{i}} = -I_3\dot{\omega}\mathbf{i}_3 = -I_3\dot{\boldsymbol{\omega}}. \tag{17.3.6}$$

Hence, in case of the rotation of an element of the engine about one of these axes, the system of forces of inertia is reduced to a couple, the moment of which is situated along the axis of rotation, having its sense opposite to that of the angular velocity vector. As above, there appear the notion of *resistant couple* and the notion of *driving couple* (due to which the great and heavy runners (I_3 is greater) are rotating for a long time after the ceasing of the action of the couple which produced the motion). If, in particular, $\omega = \mathrm{const}$, then the element has a motion of uniform rotation, non-being acted by any force of inertia.

Let be now the case of rotation of the considered element about a principal axis of inertia (the $O'x'_3$-axis), parallel to a central one (the Cx_3-axis); hence, $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$ and we have

$$\mathbf{R}^{\mathrm{i}} = -M\mathbf{a}'_C\,, \quad \mathbf{M}^{\mathrm{i}}_C = -I_3\dot{\boldsymbol{\omega}}\,. \tag{17.3.7}$$

Observing that $\mathbf{a}'_C \perp O'x'_3$ (the centre C has a motion of rotation in a plane normal to $O'x'_3$), hence $\mathbf{a}'_C \perp \dot{\boldsymbol{\omega}}$, it results that the scalar of the torsor of the forces of inertia vanishes ($\mathbf{R}^{\mathrm{i}} \cdot \mathbf{M}^{\mathrm{i}}_C = 0$, so that this system of forces is reduced to a resultant applied at the point O_1, on the straight line $O'C$ (obviously, we choose the point O_1 conventionally, the force $\mathbf{R}^{\mathrm{i}}$, applied at O_1, sliding along its support) (Fig. 17.22). We choose the Cx_2-axis along $O'C$ and the Cx_1-axis so that the frames $\mathscr{R}'$ and $\mathscr{R}$ be right-handed ones. Projecting on the axes of the frame $\mathscr{R}$, we obtain (we take into account the intrinsic components of the acceleration $\mathbf{a}'_C$, that is $a'_{C1} = -\rho'\dot{\omega}$, $a'_{C2} = -\rho'\omega^2$)

$$R^{\mathrm{i}}_1 = M\rho'\dot{\omega}\,, \quad R^{\mathrm{i}}_2 = -M\rho'\omega^2\,, \quad R^{\mathrm{i}}_3 = 0\,, \quad M^{\mathrm{i}}_{C1} = M^{\mathrm{i}}_{C2} = 0\,, \quad M^{\mathrm{i}}_{C3} = -I_3\dot{\omega}\,. \tag{17.3.7'}$$

The position of the point O_1 will be specified by $l = \overline{CO_1} = -M^{\mathrm{i}}_{C3} / R^{\mathrm{i}}_1 = I_3 / M\rho'$, so that ($l' = \overline{O'O_1}$)

$$l' = \rho' + l\,, \quad l = \frac{i_C^2}{\rho'}\,, \tag{17.3.8}$$

where i_C is the central radius of gyration with respect to the Cx_3-axis. We notice that the point O' is a centre of suspension, while the point O_1 is a centre of oscillation of the element, considered as a physical pendulum, l' being the length of the mathematical pendulum synchronous with this one.

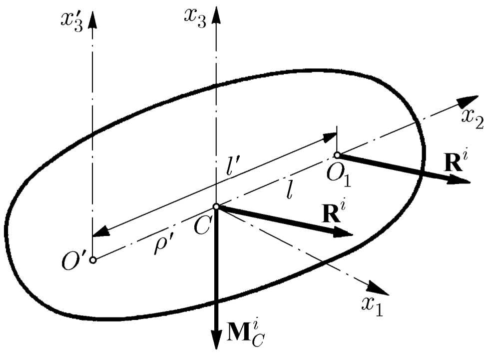

Fig. 17.22 Rotation of an element of the engine about a principal axis of inertia

In case of a plane-parallel motion we have $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$ (we choose the Cx_1x_2-plane as fixed plane), hence the torsor of the forces of inertia will be of the form (17.3.7) too. The scalar of the torsor will be also equal to zero, so that the system of the forces of inertia is reduced to a resultant along the central axis, contained in the fixed plane of equation (see the equation (2.2.31"))

$$M_{C3}^{\mathrm{i}} + x_2 R_1^{\mathrm{i}} - x_1 R_2^{\mathrm{i}} = 0\,; \tag{17.3.9}$$

we associate the relations

$$R_1^{\mathrm{i}} = -M a'_{C1}\,, \quad R_2^{\mathrm{i}} = -M a'_{C2}\,, \quad M_{C3}^{\mathrm{i}} = -I_3 \dot{\omega}\,. \tag{17.3.9'}$$

The distribution of the accelerations is, in general, unknown; it can be obtained starting from the accelerations of two points of the element and using the polygon of accelerations and the method of the pole of accelerations (see Chap. 5, Sect.. 2.3.4 too).

The problem becomes complicated in case of elements of arbitrary shape, having a complex motion; in this case, one can use *the method of concentration of masses*, replacing the mass M of the considered element by a system of masses M_i, concentrated at *the concentration points* P_i, $i = 1,2,...,n$, chosen so that the action of the new system of masses upon the other elements of the system be equivalent with the action of the mass of this element. In this case, the sum of the torsors of the forces of inertia corresponding to the concentrated masses, with respect to the mass centre, must be equal to the torsor of the forces of inertia of the studied element, with respect to the same centre. It is necessary that: (i) the sum of the concentrated masses be equal to the mass of the element; (ii) the mass centre of the system of concentrated masses does coincide with the mass centre of the element, their accelerations being equal. This corresponds to a *static repartition* of the mass of the element. To obtain a *dynamic repartition* of this mass, it is also necessary that: (iii) the sum of the kinetic energies of the concentrated masses be equal to the kinetic energy of the element (hence, the sum of the moments of inertia of the concentrated masses with respect to the mass centre be equal to the moment of inertia of the element with respect to the same centre). It is convenient that the points P_i be on the axes of the articulations, being possible that one of them does coincide with the centre of mass C of the element. Practically, the mass of an element is replaced by two or three concentrated masses.

Once the forces of inertia determined, the constraint forces of the kinematic couples can be calculated by usual methods from the static study of the mechanical system (analytical methods, grapho-analytical methods and graphic methods).

17.3.1.2 Kinetic Energy of an Engine

The kinetic energy of an engine can be calculated starting from the kinetic energy of each element $\mathscr{S}_i$; the quantities which intervene will be calculated with respect to a fixed frame of reference $\mathscr{R}'$. In case of an element individualized by the index *j*, in motion of translation, we have

$$T'_j = \frac{1}{2} M_j v'^2_{C_j}\,, \tag{17.3.10}$$

where M_j is the mass of the element, while $\mathbf{v}'_{C_j}$ is the velocity of its centre of mass C_j, the same for all the points of the element. If the element is in motion of finite rotation about an axis which passes through its mass centre, we can write

$$T'_j = \frac{1}{2} I_{C_j} \omega_j^2 , \tag{17.3.11}$$

where I_{C_j} is the moment of inertia of the element with respect to the central axis of rotation, $\boldsymbol{\omega}_j$ being the corresponding rotation angular velocity. In case of a general motion of the element, the formulae (14.1.29'), (14.1.35) allow to write

$$T'_j = \frac{1}{2} M_j v'^2_{C_j} + \frac{1}{2} \boldsymbol{\omega}_j \cdot \left(\mathbf{I}_{C_j} \boldsymbol{\omega}_j \right) , \tag{17.3.12}$$

where $\mathbf{I}_{C_j}$ is the corresponding central tensor of inertia. If the motion is a plane-parallel one, as – in general – in case of an engine, it results

$$T'_j = \frac{1}{2} M_j v'^2_{C_j} + \frac{1}{2} I_{C_j} \omega_j^2 , \tag{17.3.12'}$$

where I_{C_j} is the moment of inertia with respect to the axis of rotation (axis normal to the fixed plane and fixed with respect to the considered element) which passes through the mass centre C_j. Finally, we can assume that the kinetic energy of the engine (of the mechanical system $\mathscr{S}$) with respect to the frame $\mathscr{R}'$ is given by

$$T' = \frac{1}{2} \sum_{j=1}^{n} M_j v'^2_{C_j} + \frac{1}{2} \sum_{j=1}^{n} I_{C_j} \omega_j^2 . \tag{17.3.13}$$

The linear and angular velocities of the elements vary during a cycle of running of the engine; hence, the kinetic energy of the motion has a cyclic variation during its running too. Corresponding to the Theorem 11.1.10", the variation of the kinetic energy in a finite interval of time is equal to the work of the given external forces which act upon the engine, in that interval of time. If, in an interval of time, the work of the external forces is positive (negative), then the kinetic energy increases (decreases); if this work vanishes, then the kinetic energy is conserved. In the latter case, the increases of kinetic energy of some elements take place on account of the decreases of kinetic energy of other elements, because the kinetic energy of each element has a cyclic variation.

17.3.1.3 Reduced Mechanical Quantities

Because an engine has only one degree of mobility (degree of freedom), it is sufficient to know the law of motion of a single element of it (which is, usually, the initial element of the engine); one arrives thus to the idea of replacing the study of the engine as a mechanical system by the study of a fictitious element, having some of the properties of the engine, necessary to the search one wishes to make.

Thus, we can replace the mass of the engine by a conventional one M_{red} (*reduced mass*), situated at a part of the engine called *reduction centre*; as well, we can introduce a *reduction element* (even an element of the engine), having a *reduced moment of inertia* I_{red}. The reduction of the masses of the engine to a reduction centre or to a reduction element is made so that their kinetic energy be, at any moment, equal to the kinetic energy of the engine; obviously, the law of motion of this conventional mechanical system $\overline{\mathscr{S}}$ (of translation velocity $\overline{\mathbf{v}}'$ or rotation angular velocity about its mass centre $\overline{\boldsymbol{\omega}}$) is the same to that corresponding to the situation in which the respective system is a part of the considered engine. Equating the kinetic energies, we obtain

$$M_{\mathrm{red}} = \sum_{j=1}^{n} M_j \left(\frac{v'_{C_j}}{\overline{v}'} \right)^2 + \sum_{j=1}^{n} I_{C_j} \left(\frac{\omega_j}{\overline{v}'} \right)^2 , \tag{17.3.14}$$

the reduced mass of the engine being thus a non-negative quantity (it vanishes if the engine is not in motion), which has a periodical variation, function of the linear and angular velocities of its elements. Analogously, it results

$$I_{\mathrm{red}} = \sum_{j=1}^{n} M_j \left(\frac{v'_{C_j}}{\overline{\omega}} \right)^2 + \sum_{j=1}^{n} I_{C_j} \left(\frac{\omega_j}{\overline{\omega}} \right)^2 ; \tag{17.3.15}$$

this quantity has the same properties as M_{red}. We notice that a relation of the form

$$I_{\mathrm{red}} = M_{\mathrm{red}} \overline{l}^2 \tag{17.3.16}$$

takes place, where $\overline{l} = \overline{v}' / \overline{\omega}$ is the length of the reduction element. If we denote by φ the rotation angle of the reduction element, then it results $I_{\mathrm{red}} = I_{\mathrm{red}}(\varphi)$; as well, $T' = T'(I_{\mathrm{red}})$, wherefrom $T' = T'(\varphi)$, so that we can draw both diagrams.

Analogously, we can reduce the given external forces and their moments with respect to a pole. For this, we calculate the power produced by the forces and the moments which act upon the element of engine in the form

$$P = \sum_{j=1}^{n} F_j v'_{C_j} + \sum_{j=1}^{n} M_{C_j} \omega_j , \tag{17.3.17}$$

where $\mathbf{F}_j$ and $\mathbf{M}_{C_j}$ represent the resultant and the moment resultant, respectively, which act upon the element of index j at its mass centre, $\mathbf{v}'_{C_j}$ and $\boldsymbol{\omega}_j$ being the corresponding linear and angular velocities, respectively (along the directions of the corresponding forces and moments, respectively). Equating to the power $P_{\mathrm{red}} = F_{\mathrm{red}} \overline{v}' = M_{\overline{C}\mathrm{red}} \overline{\omega}$, we get

$$F_{\mathrm{red}} = \sum_{j=1}^{n} F_j \frac{v'_{C_j}}{\overline{v}'} + \sum_{j=1}^{n} M_{C_j} \frac{\omega_j}{\overline{v}'}, \tag{17.3.18}$$

as well as

$$M_{\overline{C}\mathrm{red}} = \sum_{j=1}^{n} F_j \frac{v'_{C_j}}{\overline{\omega}} + \sum_{j=1}^{n} M_{C_j} \frac{\omega_j}{\overline{\omega}}. \tag{17.3.19}$$

The obvious relation

$$M_{\overline{C}\mathrm{red}} = F_{\mathrm{red}} \overline{l} \tag{17.3.20}$$

takes place. The reduced force and the reduced moment are variable quantities. In case of an element of reduction $\mathbf{F}_{\mathrm{red}} = \mathbf{F}_{\mathrm{red}}(\varphi)$ and $\mathbf{M}_{\overline{C}\mathrm{red}} = \mathbf{M}_{\overline{C}\mathrm{red}}(\varphi)$, we can draw the corresponding diagrams.

If $\mathbf{F}_c$ and $\mathbf{M}_c$ are *the counter-balance force and moment*, respectively, components of the counter-balance torsor, applied on the same reduction element, then we can write

$$\mathbf{F}_c = -\mathbf{F}_{\mathrm{red}}, \quad \mathbf{M}_c = -\mathbf{M}_{\overline{C}\mathrm{red}}. \tag{17.3.21}$$

17.3.2 Applications

In what follows, we present some interesting applications of the theoretical results obtained above. We consider thus the problem of the equilibration of the mobile masses and the problem of work of engines (including their settlement).

17.3.2.1 Equilibration of the Mobile Masses

In case of engines which have some elements deficiently assembled (e.g., a runner deficiently centred), which are not perfectly homogeneous or which are performed with some tolerance, appear non-equilibrated forces of inertia, as well as their non-equilibrated moments, yielding supplementary dynamic efforts in the kinematic couples. We say that these elements are *out-of-balance*; the elimination of the corresponding injuries is obtained by *the balance of the inertia forces*, hence by *the balance of the mobile masses*.

Let be an element of mass M, which is uniformly rotating with an angular velocity $\boldsymbol{\omega}$ about an axis Δ, which does not pass through the centre of mass C. Thus arises a radial (centrifugal) non-equilibrated force of inertia $\mathbf{F}^{\mathrm{i}}$, applied at C and normal to the axis Δ, as well as a non-balanced moment of this force $\mathbf{M}^{\mathrm{i}}$, normal to the same axis and applied at a point of it (if the motion would be varied, then there would appear also a tangential force of inertia, contained in the plane of rotation of the centre C too). We choose the axis Δ as $O'x'_3$-axis, the $O'x'_1x'_2$-plane passing through $C(\rho'_1, \rho'_2, 0)$ ($\overline{O'C} = \rho'$). The centrifugal force is given by (the formulae (14.2.1))

$$F_1^{\mathrm{i}} = M\omega^2\rho_1 , \quad F_2^{\mathrm{i}} = M\omega^2\rho_2 , \quad F_3^{\mathrm{i}} = 0 , \tag{17.3.22}$$

the moment of the forces of inertia being specified by (the formulae (14.2.1'))

$$M_1^{\mathrm{i}} = \omega^2 I_{23} , \quad M_2^{\mathrm{i}} = -\omega^2 I_{31} , \quad M_3^{\mathrm{i}} = 0 . \tag{17.3.22'}$$

To equilibrate this element, we must have $\mathbf{F}^{\mathrm{i}} = \mathbf{0}$ (hence, $\rho_1 = \rho_2 = 0$, the mass centre C being thus on the rotation axis Δ) and $\mathbf{M}^{\mathrm{i}} = \mathbf{0}$ (hence, $I_{23} = I_{31} = 0$, the rotation axis Δ being a principal axis of inertia for the point of application of the moment $\mathbf{M}^{\mathrm{i}}$). Hence, for its equilibration, the element must rotate about a principal axis of its central ellipsoid of inertia.

If $\mathbf{F}^{\mathrm{i}} \neq \mathbf{0}$, $\mathbf{M}^{\mathrm{i}} = \mathbf{0}$, then the torsor of the forces of inertia is reduced to a centrifugal unbalanced force, the support of which passes through the mass centre C, eccentrically situated with respect to the rotation axis Δ; there results a *static out-of-balance* (which is put in evidence even at rest), which can be eliminated by an operation of *static balance*. If $\mathbf{F}^{\mathrm{i}} = \mathbf{0}$, $\mathbf{M}^{\mathrm{i}} \neq \mathbf{0}$, then the mass centre C is on the axis of rotation Δ, the element being balanced at rest; during the motion of rotation appears a *moment of deviation* of the axis of rotation (unbalanced moment of the forces of inertia), leading to a *dynamic out-of-balance*, which can be eliminated by a *dynamic balance*. In general, if $\mathbf{F}^{\mathrm{i}} \neq \mathbf{0}$, $\mathbf{M}^{\mathrm{i}} \neq \mathbf{0}$, then takes place a general (complete) out-of-balance, which can be eliminated by an operation of general (complete) balance. These operations are realized by eliminating some masses (from those parts of the element which are heavier) or by adding some supplementary masses (to those parts of the element which are lighter), placed conveniently, so that the quantities $\mathbf{F}^{\mathrm{i}}$ and $\mathbf{M}^{\mathrm{i}}$ do vanish.

In case of masses in rotation, the forces of inertia are decomposed in components contained in two parallel planes (*equilibration planes*); the resultants of these forces in the two planes can be annulled by the application of a supplementary mass in each of them, along the direction of the corresponding force. Obviously, this balance can be realized in various modes, because the equilibration planes can be chosen arbitrarily, as well as the position and magnitude of the supplementary masses.

In case of a mechanism, the relations (17.3.2) lead to the components of the torsor of inertia forces of the form

$$R_j^{\mathrm{i}} = -\sum m\ddot{x}_j' , \quad M_{O'j}^{\mathrm{i}} = -\in_{jkl} \sum m x_k' \ddot{x}_l' , \quad j = 1,2,3 , \tag{17.3.23}$$

where the sum (which can be an integral too) is referring to an arbitrary point of the mechanism. If the mechanism is plane (a plane-parallel motion), then the equations (17.3.23) read

$$\begin{gathered} R_1^{\mathrm{i}} = -\sum m\ddot{x}_1' , \quad R_2^{\mathrm{i}} = -\sum m\ddot{x}_2' , \quad R_3^{\mathrm{i}} = 0 , \\ M_{O'1}^{\mathrm{i}} = \sum m x_3' \ddot{x}_2' , M_{O'2}^{\mathrm{i}} = -\sum m x_3' \ddot{x}_1' , M_{O'3}^{\mathrm{i}} = -\sum m (x_1' \ddot{x}_2' - x_2' \ddot{x}_1') . \end{gathered} \tag{17.3.24}$$

Because $x'_\alpha = x'_\alpha(\varphi)$, $\alpha = 1,2$, where $\varphi = \varphi(t)$ is the angle of rotation of the leading element, it results

$$\ddot{x}'_\alpha = \frac{\mathrm{d}^2 x'_\alpha}{\mathrm{d}\varphi^2}\dot{\varphi}^2 + \frac{\mathrm{d}x'_\alpha}{\mathrm{d}\varphi}\ddot{\varphi}, \quad \alpha = 1,2,$$

so that

$$R^{\mathrm{i}}_\alpha = -\left(\dot{\varphi}^2 \sum m \frac{\mathrm{d}^2 x'_\alpha}{\mathrm{d}\varphi^2} + \ddot{\varphi} \sum m \frac{\mathrm{d}x'_\alpha}{\mathrm{d}\varphi}\right), \quad \alpha = 1,2, \tag{17.3.25}$$

$$M^{\mathrm{i}}_{O'1} = \dot{\varphi}^2 \sum m x'_3 \frac{\mathrm{d}^2 x'_2}{\mathrm{d}\varphi^2} + \ddot{\varphi} \sum m x'_3 \frac{\mathrm{d}x'_2}{\mathrm{d}\varphi},$$

$$M^{\mathrm{i}}_{O'2} = -\left(\dot{\varphi}^2 \sum m x'_3 \frac{\mathrm{d}^2 x'_1}{\mathrm{d}\varphi^2} + \ddot{\varphi} \sum m x'_3 \frac{\mathrm{d}x'_1}{\mathrm{d}\varphi}\right), \tag{17.3.25'}$$

$$M^{\mathrm{i}}_{O'3} = \dot{\varphi}^2 \left(\sum m x'_2 \frac{\mathrm{d}^2 x'_1}{\mathrm{d}\varphi^2} - \sum m x'_1 \frac{\mathrm{d}^2 x'_2}{\mathrm{d}\varphi^2}\right) + \ddot{\varphi}\left(\sum m x'_2 \frac{\mathrm{d}x'_1}{\mathrm{d}\varphi} - \sum m x'_1 \frac{\mathrm{d}x'_2}{\mathrm{d}\varphi}\right). \tag{17.3.25''}$$

We notice that the moment $M^{\mathrm{i}}_{O'3}$ is contained in the moment of the given and constraint forces, so that only the relations (17.3.25), (17.3.25') remain to be considered. To balance the forces of inertia, the quantities R^{i}_α and $M^{\mathrm{i}}_{O'\alpha}$, $\alpha = 1,2$, must also vanish for any $\dot{\varphi}$ and $\ddot{\varphi}$; to do this, the relations

$$\sum m \frac{\mathrm{d}x'_\alpha}{\mathrm{d}\varphi} = 0, \quad \sum m x'_3 \frac{\mathrm{d}x'_\alpha}{\mathrm{d}\varphi} = 0, \quad \alpha = 1,2, \tag{17.3.26}$$

are sufficient. Indeed, in this case we have

$$\sum m \frac{\mathrm{d}^2 x'_\alpha}{\mathrm{d}\varphi^2} = 0, \quad \sum m x'_3 \frac{\mathrm{d}^2 x'_\alpha}{\mathrm{d}\varphi^2} = 0, \quad \alpha = 1,2$$

too. Taking into account $\sum m x'_\alpha = M\rho'_\alpha$, $\alpha = 1,2$, where M is the mass of the mechanism, the first relations (17.3.26) lead to $\rho'_\alpha = \mathrm{const}$, $\alpha = 1,2$. As well, observing that $I_{\alpha 3} = -\sum m x'_3 x'_\alpha$, the last relations (17.3.26) allow to state that $I_{\alpha 3} = \mathrm{const}$, $\alpha = 1,2$. Hence, to equilibrate the resultant force $\mathbf{R}^{\mathrm{i}}$, the masses of the mechanism must be distributed so that their mass centre be fixed during the motion. To balance the resultant moment $\mathbf{M}^{\mathrm{i}}_{O'}$, it is necessary that the $O'x'_3$-axis be a principal axis of inertia at the point O'.

Often, the moment of the inertia forces is neglected, this one being sufficiently small, remaining to be fulfilled the condition concerning the centre of mass. For instance, in case of an articulated quadrangle, the centre of mass will have a fixed position on the straight line joining the two fixed articulations if two masses of balance are introduced.

17.3.2.2 Work of Engines

In the work of an engine, we mark out – in general – three phases: *the phase of starting* (of duration t_s), *the phase of regime* (of duration $t_r = kt_c$, assuming the existence of k *cycles of motion*, of duration t_c) and *the phase of stopping* (of duration t_{st}); the total duration of motion of the engine will be, obviously, $T = t_s + t_r + t_{st}$. The diagram $\omega = \omega(t)$ of variation of the angular velocity of the leading element (*the tachogram of the machine*) will be represented in Fig. 17.23. In the phase of regime, the mean angular velocity $\omega_m = (\omega_{\max} + \omega_{\min})/2$ is put in evidence.

The kinetic energy of the engine is given by the relation (17.3.13) or, using the reduced mass or the reduced moment of inertia, by one of the relations

$$T' = \frac{1}{2} M_{\mathrm{red}} \overline{v}'^2 = \frac{1}{2} I_{\mathrm{red}} \overline{\omega}^2 . \tag{17.3.27}$$

The work effected by all the external forces which act upon the elements of the engine in the whole duration of running of it is of the form $W' = W'_m - W'_r$, where W'_m is the motive work, while W'_r is the resistant work. Applying the theorem of kinetic energy, we may write

$$\mathrm{d}T' = \mathrm{d}W'_m - \mathrm{d}W'_r . \tag{17.3.28}$$

Hence, the kinetic energy of the engine increases or decreases as $\mathrm{d}W'_m > \mathrm{d}W'_r$ or $\mathrm{d}W'_m < \mathrm{d}W'_r$; if $\mathrm{d}W'_m = \mathrm{d}W'_r$, then the kinetic energy passes through an extremum. The theorem of kinetic energy in finite form leads to (between the limits t_0 and t for the time)

$$T' - T'_0 = \frac{1}{2} I_{\mathrm{red}} \left(\overline{\omega}^2 - \overline{\omega}_0^2 \right) = \frac{1}{2} M_{\mathrm{red}} \left(\overline{v}'^2 - \overline{v}_0'^2 \right) = W'_m - W'_r . \tag{17.3.28'}$$

The expressions at the left represent the work W^{i} of the forces of inertia, which is positive if the angular velocity increases ($\overline{\omega} > \overline{\omega}_0$) and negative otherwise ($\overline{\omega} < \overline{\omega}_0$); as well, the difference $W'_m - W'_r$ is the work of the reduced moment (at the reduction element) or of the reduced force (at the reduction centre).

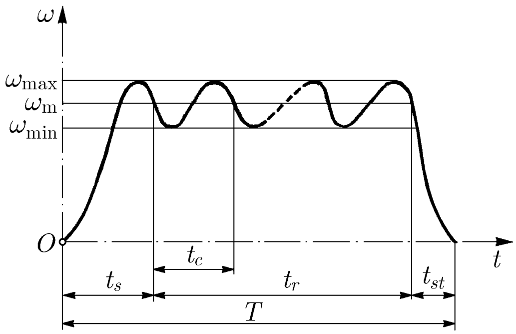

Fig. 17.23 Diagram of variation of the angular velocity of the leading element (the tachogram of the machine)

The phase of starting is characterized by $\overline{\omega}_0 = 0$ and $\overline{\omega} = \omega_r$, where ω_r is the angular velocity of regime (of mean value ω_m); it results $W_m' > W_r'$. To diminish the starting time t_s, *the no-load start* of the engine, by uncoupling the useful load, is recommended. In the phase of regime, the angular velocity has a cyclic variation ($\overline{\omega}^0 = \overline{\omega}_0$) and we have $W_m' = W_r'$ (the motive work during a cycle is consumed by the respective resistant work). In the stopping phase, the angular velocity decreases from ω_r to zero; we have $W_m' < W_r'$. Practically, the motive source is uncoupled ($W_m' = 0$), and, to reduce the time of stopping t_{st}, the work W_r' is increased, by *braking*.

The resistant work can be expressed, in general, in the form

$$W_r' = W_u' + W_p' \pm W_g' , \tag{17.3.29}$$

where W_u' is *the useful work* consumed by the engine to overcome the technological resistances, W_p' is *the passive* (*lost*) *work* (by friction, resistance of the medium etc.), while W_g' is *the work of the gravity forces* (one takes the sign + or the sign – as the gravity centre of the elements in motion moves downwards or upwards). Taking into account (17.3.28'), we get

$$W_m' = W_u' + W_p' \pm W_g' \pm W'^{\mathrm{i}} , \tag{17.3.30}$$

putting thus in evidence the equilibrium of the motive work and of the other works consumed or produced during its running (W'^i is the *inertial work*). Hence, it results $\mathrm{d}W_m' = \mathrm{d}W_u' + \mathrm{d}W_p' \pm \mathrm{d}W_g' \pm \mathrm{d}W'^{\mathrm{i}}$; dividing by $\mathrm{d}t$, we obtain

$$P_m' = P_u' + P_p' \pm P_g' \pm P'^{\mathrm{i}} , \tag{17.3.30'}$$

where P_m' is *the power due to the motive forces*, P_u' is *the useful power*, P_p' is *the positive* (*lost*) *power*, P_g' is *the gravitational power* (the power necessary to balance the forces of weight if the centre of gravity of the elements moves up or the power produced by these forces if the centre of gravity moves down), while P'^{i} is *the inertial power* (the power necessary to increase the kinetic energy of the engine or the power obtained by decreasing the kinetic energy). The equation (17.3.30') represents *the equation of energetic balance of the engine*, specifying the power necessary to the engine and allowing an estimation of it from an economical point of view.

Taking into account Chap. 6, Sect. 1.1.3, we will use the notion of *the mechanical efficiency* $0 \leq \eta < 1$, given by (6.1.17), as well as that of *coefficient of loss* $\varphi = W_p' / W_m'$; it results $\eta = 1 - \varphi$. As the coefficient of loss is smaller, so the mechanical efficiency is greater (the work is better used by the engine); if the engine is in no-load moving ($W_p' = W_m'$), then we have $\eta = 0$. In case of *self-braking of the engine* ($W_p' > W_m'$), the mechanical efficiency would become negative (the engine cannot work in regime).

Let be n mechanisms or engines $\mathscr{M}_j$, $j = 1,2,...,n$, grouped together in series. We assume that W_m' is the work of the setting by means of the mechanism $\mathscr{M}_1$ and that W_u' is the useful work of it by means of the mechanism $\mathscr{M}_n$. We may write $\eta_1 = W_1' / W_m'$, $\eta_2 = W_2' / W_1'$,..., $\eta_n = W_u' / W_{n-1}'$; we obtain thus

$$\eta_{1,n} = \eta_1 \eta_2 \ldots \eta_n\,, \tag{17.3.31}$$

with $\eta_{1,n} = W_u' / W_m'$. Hence, *the mechanical efficiency of a setting composed by mechanisms* (or *engines*) *coupled in series is equal to the product of the mechanical efficiencies of the mechanisms* (or *engines*) *which compose this setting*. The mechanical efficiency of the setting is smaller than each of the component mechanical efficiencies; this mechanical efficiency is as smaller as the setting is more complex. As well, the formula (17.3.31) puts in evidence the impossibility of existence of a perpetuum mobile (which works without loss of energy, having the mechanical efficiency $\eta_{1,n} = 1$).

17.3.2.3 Adjustment of the Work of Engines

The angular velocity $\boldsymbol{\omega}$ of the leading element of the engine varies *periodically* or *non-periodically* during the motion of rotation. As it is seen in Fig. 17.23, the periodic variations take place between $\omega_{\min}$ and $\omega_{\max}$; from the equation $I_{\mathrm{red}}\dot{\omega} = M_{\bar{C}\mathrm{red}}$ it results that for the extreme values we have $M_{\bar{C}\mathrm{red}} = 0$. The non-periodic variations, which can appear because of various causes (coupling or non-coupling of supplementary charges, accidental charges, random deficiencies etc.), must be limited too. *The problem of adjustment of the work of engines* is thus put (maintaining the angular velocity ω between some limits or even constant). From the relation $\mathrm{d}\omega = M_{\bar{C}\mathrm{red}}\mathrm{d}t / I_{\mathrm{red}}$ one observes that the variations of the angular velocity are as smaller as $M_{\bar{C}\mathrm{red}}$ and their duration are smaller or as I_{red} is greater.

The increasing of the reduced moment of inertia I_{red} can be obtained with the aid of a fly wheel (a great and heavy wheel) fixed on the element in motion of rotation; hence,

$I_{\text{red}} = I_{m\text{red}} + I_f$, where $I_{m\text{red}}$ is the reduced moment of inertia without fly wheel, while I_f is the moment of inertia of the fly wheel. As a matter of fact, the fly wheel is an *accumulator of kinetic energy*. Indeed, during an energetic cycle of an engine, the fly wheel accumulates kinetic energy in excess, when the angular velocity is increasing, handing it back when the angular velocity decreases; thus, the angular velocity $\boldsymbol{\omega}$ is close to the mean velocity (of regime). But the fly wheel cannot impede an engine to pass from a state of regime to another one; it makes only easier the passing (which takes thus place less sudden) and is not efficacious in case of non-periodical variations of the velocity.

To maintain the mean angular velocity ω_m one uses particular mechanisms, called *regulators* (which act, especially, on the admission of the motive energy). Hence, *the regulator establishes and maintains the value of the mean velocity of the engine*, while *the fly wheel diminishes the periodic variations of the velocity around this value*.

The regulator is a mechanism which controls automatically the work of the motive force, decreasing it when the angular velocity is increasing and increasing it when the angular velocity decreases, so that the equilibrium between the motive couple and the resistant one is re-established. *The regulators* can be *centrifugal* (influenced by the centrifugal force), *of inertia* (influenced by the tangential inertia forces) etc. We mention also *the power controllers* (e.g., for lifting pumps). Between the centrifugal regulators, one can mention: *the Watt regulator* (which has the advantage to be *stable*) and *the Rankine regulator* (which is isochronous, but has the disadvantage to be *labile*) etc.

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